Q.4 Properties of Quadratic Function and Optimization Problems

In the previous section, we examined how to graph and read the characteristics of the graph of a quadratic function given in vertex form, $f(x) = a(x-p)^2 + q$. In this section, we discuss the ways of **graphing** and reading the **characteristics** of the graph of a quadratic function given in **standard form**, $\boldsymbol{f(x) = ax^2 + bx + c}$. One of these ways is to convert standard form of the function to vertex form by **completing the square** so that the information from the vertex form may be used for graphing. The other handy way of graphing and reading properties of a quadratic function is to **factor** the defining trinomial and use the **symmetry** of a parabolic function.

At the end of this section, we apply properties of quadratic functions to solve certain **optimization problems**. To solve these problems, we look for the **maximum** or **minimum** of a particular quadratic function satisfying specified conditions called **constraints**. Optimization problems often appear in geometry, calculus, business, computer science, etc.

Graphing Quadratic Functions Given in the Standard Form $\boldsymbol{f(x) = ax^2 + bx + c}$

To graph a quadratic function given in standard form, $f(x) = ax^2 + bx + c$, we can use one of the following methods:

1. constructing a **table of values** (this would always work, but it could be cumbersome);
2. converting to **vertex form** by using the technique of completing the square (see *Example 1-3*);
3. **factoring** and employing the properties of a parabolic function. (this is a handy method if the function can be easily factored – see *Example 3 and 4*)

The table of values approach can be used for any function, and it was already discussed on various occasions throughout this textbook.

Converting to **vertex form** involves completing the square. For example, to convert the function $f(x) = 2x^2 + x - 5$ to its vertex form, we might want to start by dividing both sides of the equation by the leading coefficient 2, and then complete the square for the polynomial on the right side of the equation, as below.

$$\frac{f(x)}{2} = x^2 + \frac{1}{2}x - \frac{5}{2}$$

$$\frac{f(x)}{2} = \left(x + \frac{1}{4}\right)^2 - \frac{1}{16} - \frac{5 \cdot 8}{2 \cdot 8}$$

$$\frac{f(x)}{2} = \left(x + \frac{1}{4}\right)^2 - \frac{41}{16}$$

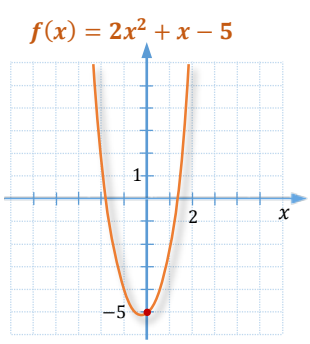


Figure 4.1

Finally, the vertex form is obtained by multiplying both sides of the equation back by 2. So, we have

$$f(x) = 2\left(x + \frac{1}{4}\right)^2 - \frac{41}{8}$$

This form lets us identify the vertex, $\left(-\frac{1}{4}, -\frac{41}{8}\right)$, and the shape, $y = 2x^2$, of the parabola, which is essential for graphing it. To create an approximate graph of

function f, we may want to round the vertex to approximately $(-0.25, -5.1)$ and evaluate $f(0) = 2 \cdot 0^2 + 0 - 5 = -5$. So, the graph is as in *Figure 4.1*.

Example 1 ► **Converting the Standard Form of a Quadratic Function to the Vertex Form**

Rewrite each function in its vertex form. Then, identify the vertex.

a. $f(x) = -3x^2 + 2x$ **b.** $g(x) = \frac{1}{2}x^2 + x + 3$

Solution ►

a. To convert f to its vertex form, we follow the completing the square procedure. After dividing the equation by the leading coefficient,

$$f(x) = -3x^2 + 2x, \qquad / \div (-3)$$

we have

$$\frac{f(x)}{-3} = x^2 - \frac{2}{3}x$$

Then, we complete the square for the right side of the equation,

$$\frac{f(x)}{-3} = \left(x - \frac{1}{3}\right)^2 - \frac{1}{9}, \qquad / \cdot (-3)$$

and finally, multiply back by the leading coefficient,

$$f(x) = -3\left(x - \frac{1}{3}\right)^2 + \frac{1}{3}.$$

Therefore, the vertex of this parabola is at the point $\left(\mathbf{\frac{1}{3}}, \mathbf{\frac{1}{3}}\right)$.

b. As in the previous example, to convert g to its vertex form, we first wish to get rid of the leading coefficient. This can be achieved by multiplying both sides of the equation $g(x) = \frac{1}{2}x^2 - x + 3$ by 2. So, we obtain

$$2g(x) = x^2 + 2x + 6$$

$$2g(x) = (x + 1)^2 - 1 + 6$$

$$2g(x) = (x + 1)^2 + 5, \qquad / \div 2$$

which can be solved back for g,

$$g(x) = \frac{1}{2}(x + 1)^2 + \frac{5}{2}.$$

Therefore, the vertex of this parabola is at the point $\left(\mathbf{-1}, \mathbf{\frac{5}{2}}\right)$.

Completing the square allows us to derive a formula for the vertex of the graph of any quadratic function given in its standard form, $f(x) = ax^2 + bx + c$, where $a \neq 0$. Applying the same procedure as in *Example 1*, we calculate

$$f(x) = ax^2 + bx + c \qquad / \div a$$

$$\frac{f(x)}{a} = x^2 + \frac{b}{a}x + \frac{c}{a}$$

$$\frac{f(x)}{a} = \left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}$$

$$\frac{f(x)}{a} = \left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2} \qquad / \cdot a$$

Recall: This is the **discriminant** **Δ!**

$$f(x) = a\left(x - \left(-\frac{b}{2a}\right)\right)^2 + \frac{-(b^2 - 4ac)}{4a}$$

Thus, the coordinates of the vertex $(\boldsymbol{p}, \boldsymbol{q})$ are $\boldsymbol{p} = -\frac{\boldsymbol{b}}{\boldsymbol{2a}}$ and $\boldsymbol{q} = \frac{-(\boldsymbol{b}^2 - \boldsymbol{4ac})}{\boldsymbol{4a}} = \frac{-\Delta}{\boldsymbol{4a}}$.

Observation: Notice that the expression for q can also be found by evaluating f at $x = -\frac{\boldsymbol{b}}{\boldsymbol{2a}}$.

So, the vertex of the parabola can also be expressed as $\left(-\frac{\boldsymbol{b}}{\boldsymbol{2a}}, \boldsymbol{f}\left(-\frac{\boldsymbol{b}}{\boldsymbol{2a}}\right)\right)$.

VERTEX FORMULA

Summarizing, the **vertex** of a parabola defined by $f(x) = \boldsymbol{a}x^2 + \boldsymbol{b}x + \boldsymbol{c}$, where $\boldsymbol{a} \neq 0$, can be calculated by following one of the formulas:

$$\left(-\frac{\boldsymbol{b}}{\boldsymbol{2a}}, \frac{-(\boldsymbol{b}^2 - \boldsymbol{4ac})}{\boldsymbol{4a}}\right) = \left(-\frac{\boldsymbol{b}}{\boldsymbol{2a}}, \frac{-\Delta}{\boldsymbol{4a}}\right) = \left(-\frac{\boldsymbol{b}}{\boldsymbol{2a}}, \boldsymbol{f}\left(-\frac{\boldsymbol{b}}{\boldsymbol{2a}}\right)\right)$$

Example 2 ► **Using the Vertex Formula to Find the Vertex of a Parabola**

Use the vertex formula to find the vertex of the graph of $f(x) = -x^2 - x + 1$.

Solution ► The first coordinate of the vertex is equal to $-\frac{b}{2a} = -\frac{-1}{2\cdot(-1)} = -\frac{\boldsymbol{1}}{\boldsymbol{2}}$.

The second coordinate can be calculated by following the formula

$$\frac{-\Delta}{4a} = \frac{-\left((-1)^2 - 4\cdot(-1)\cdot 1\right)}{4\cdot(-1)} = \frac{\boldsymbol{5}}{\boldsymbol{4}},$$

or by evaluating $f\left(-\frac{1}{2}\right) = -\left(-\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right) + 1 = -\frac{1}{4} + \frac{1}{2} + 1 = \frac{\boldsymbol{5}}{\boldsymbol{4}}$.

So, the vertex is $\left(-\frac{\boldsymbol{1}}{\boldsymbol{2}}, \frac{\boldsymbol{5}}{\boldsymbol{4}}\right)$.

Example 3 ► **Graphing a Quadratic Function Given in the Standard Form**

Graph each function.

a. $g(x) = \frac{1}{2}x^2 + x + 3$ **b.** $f(x) = -x^2 - x + 1$

Solution ►

a. The shape of the graph of function g is the same as this of $y = \frac{1}{2}x^2$. Since the leading coefficient is positive, the arms of the parabola **open up**.

The **vertex**, $\left(\mathbf{-1}, \frac{\mathbf{5}}{\mathbf{2}}\right)$, was found in *Example 1b* as a result of completing the square. Since the vertex is in quadrant II and the graph opens up, we do not expect any x-intercepts. However, without much effort, we can find the y-intercept by evaluating $g(0) = \mathbf{3}$. Furthermore, since $(0, 3)$ belongs to the graph, then by symmetry, $(-2, 3)$ must also belong to the graph. So, we graph function g is as in *Figure 4.2*.

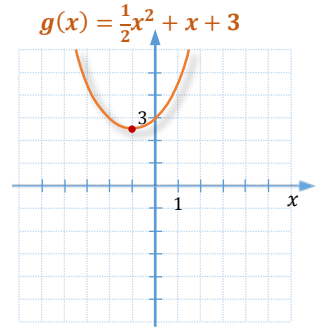


Figure 4.2

When plotting points with fractional coordinates, round the values to one place value.

b. The graph of function f has the shape of the basic parabola. Since the leading coefficient is negative, the arms of the parabola **open down**.

The **vertex**, $\left(-\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{5}}{\mathbf{4}}\right)$, was found in *Example 2* by using the vertex formula. Since the vertex is in quadrant II and the graph opens down, we expect two x-intercepts. Their values can be found via the quadratic formula applied to the equation $-x^2 - x + 1 = 0$. So, the x-intercepts are $x_{1,2} = \frac{1 \pm \sqrt{5}}{-2} \approx -\mathbf{1.6}$ *or* $\mathbf{0.6}$. In addition, the y-intercept of the graph is $f(0) = \mathbf{1}$.

Using all this information, we graph function f, as in *Figure 4.3*.

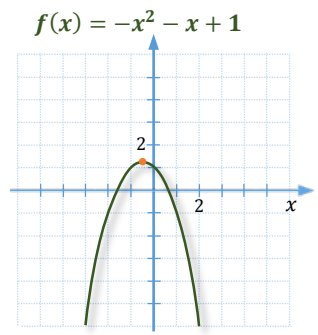


Figure 4.3

Graphing Quadratic Functions Given in the Factored Form $f(x) = a(x - r_1)(x - r_2)$

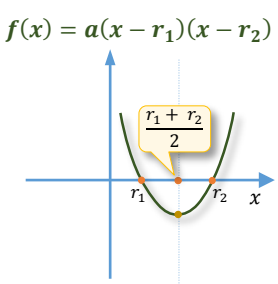


Figure 4.4

What if a quadratic function is given in factored form? Do we have to change it to vertex or standard form in order to find the vertex and graph it?

The factored form, $f(x) = a(x - r_1)(x - r_2)$, allows us to find the roots (or x-intercepts) of such a function. These are r_1 and r_2. A parabola is symmetrical about the axis of symmetry, which is the vertical line passing through its vertex. So, the first coordinate of the vertex is the same as the first coordinate of the midpoint of the line segment connecting the roots, r_1 with r_2, as indicated in *Figure 4.4*. Thus, the first coordinate of the vertex is the average of the two roots, $\frac{r_1 + r_2}{2}$. Then, the second coordinate of the vertex can be found by evaluating $f\left(\frac{r_1 + r_2}{2}\right)$.

Example 4 ► **Graphing a Quadratic Function Given in a Factored Form**

Graph function $g(x) = -(x-2)(x+1)$.

Solution ►

First, observe that the graph of function g has the same shape as the graph of the basic parabola, $f(x) = x^2$. Since the leading coefficient is negative, the arms of the parabola **open down**. Also, the graph intersects the x-axis at 2 and -1. So, the first coordinate of the vertex is the average of 2 and -1, which is $\frac{1}{2}$. The second coordinate is

$$g\left(\frac{1}{2}\right) = -\left(\frac{1}{2}-2\right)\left(\frac{1}{2}+1\right) = -\left(-\frac{3}{2}\right)\left(\frac{3}{2}\right) = \frac{9}{4}$$

Therefore, function g can be graphed by connecting the vertex, $\left(\frac{1}{2}, \frac{9}{4}\right)$, and the x-intercepts, $(-1,0)$ and $(-1,0)$, with a parabolic curve, as in *Figure 4.5*. For a more precise graph, we may additionally plot the y-intercept, $g(0) = 2$, and the symmetrical point $g(1) = 2$.

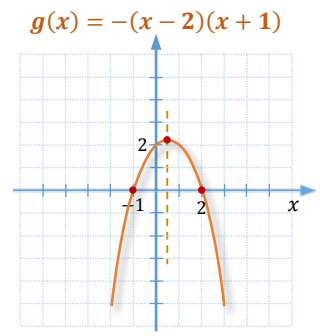


Figure 4.5

Example 5 ► **Using Complete Factorization to Graph a Quadratic Function**

Graph function $f(x) = 4x^2 - 2x - 6$.

Solution ►

Since the discriminant $\Delta = (-2)^2 - 4 \cdot 4 \cdot (-6) = 4 + 96 = 100$ is a perfect square number, the defining trinomial is factorable. So, to graph function f, we may want to factor it first. Notice that the GCF of all the terms is 2. So, $f(x) = 2(2x^2 - x - 3)$. Then, using factoring techniques discussed in *section F2*, we obtain $f(x) = 2(2x-3)(x+1)$. This form allows us to identify the roots (or zeros) of function f, which are $\frac{3}{2}$ and -1. So, the first coordinate of the vertex is the average of $\frac{3}{2} = 1.5$ and -1, which is $\frac{1.5+(-1)}{2} = \frac{0.5}{2} = 0.25$. The second coordinate can be calculated by evaluating

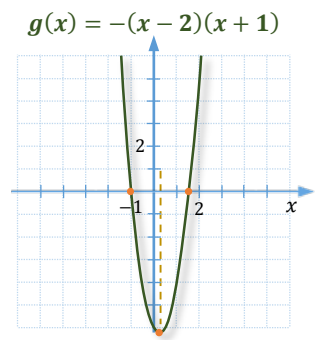


Figure 4.6

$$f(0.25) = 2(2 \cdot 0.25 - 3)(0.25 + 1) = 2(0.5 - 3)(1.25) = 2(-2.5)(1.25) = -6.25$$

So, we can graph function f by connecting its vertex, $(\mathbf{0.25}, \mathbf{-6.25})$, and its x-intercepts, $(-1,0)$ and $(1.5,0)$, with a parabolic curve, as in *Figure 4.6*. For a more precise graph, we may additionally plot the y-intercept, $f(0) = -6$, and by symmetry, $f(0.5) = -6$.

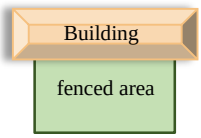
Optimization Problems

In many applied problems we are interested in **maximizing** or **minimizing** some quantity under specific conditions, called **constraints**. For example, we might be interested in

finding the greatest area that can be fenced in by a given length of fence, or minimizing the cost of producing a container of a given shape and volume. These types of problems are called **optimization problems**.
Since the vertex of the graph of a quadratic function is either the highest or the lowest point of the parabola, it can be used in solving optimization problems that can be modeled by a quadratic function.

The vertex of a parabola provides the following information.
- The y-value of the vertex gives the maximum or minimum value of y.
- The x-value tells where the maximum or minimum occurs.

Example 6 ► **Maximizing Area of a Rectangular Region**



A farmer has 60 meters of fencing to enclose a rectangular field next to a building. Assuming that the attached building forms one side of the rectangle, find the maximum area he can enclose and the dimensions of the enclosed field that yields this area.

Solution ►

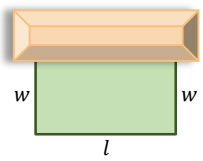


Figure 4.7

Let l and w represent the length and width of the enclosed area correspondingly, as indicated in *Figure 4.7*. The 60 meters of fencing is used to cover the distance of twice along the width and once along the length. So, we can form the constraint equation

$$2W + l = 60 \qquad (1)$$

To analyse the area of the field,

$$A = lw, \qquad (2)$$

we would like to express it as a function of one variable, for example w. To do this, we can solve the constraint equation (1) for l and substitute the obtained expression into the equation of area, (2). Since $l = 60 - 2w$, then

$$A = lw = (60 - 2w)w$$

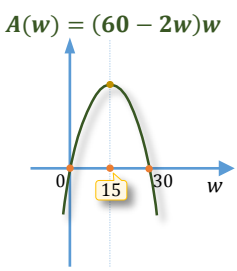


Figure 4.8

Observe that the graph of the function $A(w) = (60 - 2w)w$ is a parabola that opens down and intersects the x-axis at 0 and 30. This is because the leading coefficient of $(60 - 2w)w$ is negative and the roots to the equation $(60 - 2w)w = 0$ are 0 and 30. These roots are symmetrical in the axis of symmetry, which also passes through the vertex of the parabola, as illustrated in *Figure 4.8*. So, the first coordinate of the vertex is the average of the two roots, which is $\frac{0+30}{2} = 15$. Thus, the width that would maximize the enclosed area is $w_{max} = \mathbf{15}$ **meters**. Consequently, the length that would maximize the enclosed area is $l_{max} = 60 - 2w_{max} = 60 - 2 \cdot 15 = \mathbf{30}$ **meters**. The maximum area represented by the second coordinate of the vertex can be obtained by evaluating the function of area at the width of 15 meters.

$$A(15) = (60 - 2 \cdot 15)15 = 30 \cdot 15 = 450 \text{ m}^2$$

So, the maximum area that can be enclosed by 60 meters of fencing is **450 square meters**, and the dimensions of this rectangular area are **15 by 30 meters**.

Example 7 ► **Minimizing Average Unit Cost**

A company producing bicycles has determined that when x hundred bicycles are produced, the average cost of producing one bicycle is given by the function

$$C(x) = 0.1x^2 - 0.7x + 2.425,$$

where $C(x)$ is in hundreds of dollars. How many bicycles should be produced to minimize the average cost of producing one bicycle? What would this cost be?

Solution ►

Since $C(x)$ is a quadratic function, to find its minimum, it is enough to find the vertex of the parabola $C(x) = 0.1x^2 - 0.7x + 2.425$. This can be done either by completing the square method or by using the formula for the vertex, $\left(\frac{-b}{2a}, \frac{-\Delta}{4a}\right)$. We will do the latter. So, the vertex is

$$\left(\frac{-b}{2a}, \frac{-\Delta}{4a}\right) = \left(\frac{0.7}{0.2}, \frac{-(0.7^2 - 4 \cdot 0.1 \cdot 2.425)}{0.4}\right) = \left(3.5, \frac{-(0.49 - 0.97)}{0.4}\right) = \left(3.5, \frac{0.48}{0.4}\right)$$

$$= (\mathbf{3.5}, \mathbf{1.2}).$$

This means that the lowest average unit cost can be achieved when 350 bicycles are produced, and then the average cost of a bicycle would be \$120.

Q.4 Exercises

Vocabulary Check *Complete each blank with the most appropriate term or phrase from the given list:* ***average, completing, down, first, minimum, optimization, second, standard, vertex.***

1. The highest or lowest point of a parabola is called a _____.

2. When a is negative, the parabola given by the function $f(x) = ax^2 + bx + c$ opens _____.

3. When a is positive, the parabola given by the function $f(x) = ax^2 + bx + c$ has a _____ point.

4. When converting the _____ form of a quadratic function $f(x) = ax^2 + bx + c$ to its vertex form $f(x) = a(x - p)^2 + q$, the process of _____ the square is used.

5. The **extreme** value (***minimum*** or ***maximum***) of a quadratic function is the _____ coordinate of the vertex of its graph. This extreme value occurs at the _____ coordinate of the vertex.

6. Application problems that involve finding the minimum or maximum points on a graph of a function are often referred to as _____ problems.

7. The first coordinate of the vertex of the parabola given by the function $f(x) = a(x - r_1)(x - r_2)$ is the _____ of the roots r_1 and r_2.

Concept Check

*Convert each quadratic function to its **vertex form**. Then, state the coordinates of the **vertex**.*

8. $f(x) = x^2 + 6x + 10$

9. $f(x) = x^2 - 4x - 5$

10. $f(x) = x^2 + x - 3$

11. $f(x) = x^2 - x + 7$

12. $f(x) = -x^2 + 7x + 3$

13. $f(x) = 2x^2 - 4x + 1$

14. $f(x) = -3x^2 + 6x + 12$

15. $f(x) = -2x^2 - 8x + 10$

16. $f(x) = \frac{1}{2}x^2 + 3x - 1$

Concept Check

*Use the vertex formula, $\left(-\frac{b}{2a}, \frac{-\Delta}{4a}\right)$, to find the coordinates of the **vertex** of each parabola.*

17. $f(x) = x^2 + 6x + 3$

18. $f(x) = -x^2 + 3x - 5$

19. $f(x) = \frac{1}{2}x^2 - 4x - 7$

20. $f(x) = -3x^2 + 6x + 5$

21. $f(x) = 5x^2 - 7x$

22. $f(x) = 3x^2 + 6x - 20$

*For each parabola, state its **vertex**, **opening** and **shape**. Then **graph** it and state the **domain** and **range**.*

23. $f(x) = x^2 - 5x$

24. $f(x) = x^2 + 3x$

25. $f(x) = x^2 - 2x - 5$

26. $f(x) = -x^2 + 6x - 3$

27. $f(x) = -x^2 - 3x + 2$

28. $f(x) = 2x^2 + 12x + 18$

29. $f(x) = -2x^2 + 3x - 1$

30. $f(x) = -2x^2 + 4x + 1$

31. $f(x) = 3x^2 + 4x + 2$

*For each quadratic function, state its **zeros** (roots), coordinates of the **vertex**, **opening** and **shape**. Then **graph** it and identify its **extreme** (minimum or maximum) **value** as well as where it occurs.*

32. $f(x) = (x - 2)(x + 2)$

33. $f(x) = -(x + 3)(x - 1)$

34. $f(x) = x^2 - 4x$

35. $f(x) = x^2 + 5x$

36. $f(x) = x^2 - 8x + 16$

37. $f(x) = -x^2 - 4x - 4$

38. $f(x) = -3(x^2 - 1)$

39. $f(x) = \frac{1}{2}(x + 3)(x - 4)$

40. $f(x) = -\frac{3}{2}(x - 1)(x - 5)$

Discussion Point

41. Suppose the x-intercepts of the graph of a parabola are $(x_1, 0)$ and $(x_2, 0)$. What is the equation of the axis of symmetry of this graph?

Discussion Point

42. How can we determine the number of x-intercepts of the graph of a quadratic function without graphing the function?

Concept Check *True or false? Explain.*

43. The domain and range of a quadratic function are both the set of real numbers.

44. The graph of every quadratic function has exactly one y-intercept.

45. The graph of $y = -2(x - 1)^2 - 5$ has no x-intercepts.

46. The maximum value of y in the function $y = -4(x-1)^2 + 9$ is 9.

47. The value of the function $f(x) = x^2 - 2x + 1$ is at its minimum when $x = 0$.

48. The graph of $f(x) = 9x^2 + 12x + 4$ has one x-intercept and one y-intercept.

49. If a parabola opens down, it has two x-intercepts.

Analytic Skills *Solve each problem.*

50. If an object on Earth is projected upward with an initial velocity of 32 ft per sec, then its height after t seconds is given by $s(t) = -16t^2 + 32t$. Find the maximum height attained by the object and the number of seconds it takes to hit the ground.

51. A projectile on Earth is fired straight upward so that its distance (in feet) above the ground t seconds after firing is given by $s(t) = -16t^2 + 400t$. Find the maximum height it reaches and the number of seconds it takes to reach that height.

52. John has a frozen yogurt cart. His daily costs are approximated by $C(x) = x^2 - 70x + 1500$, where $C(x)$ is the cost, in dollars, to sell x units of frozen yogurt. Find the number of units of frozen yogurt he must sell to minimize his costs. What is the minimum cost?

53. Chris has a taco stand. He has found that his daily costs are approximated by $C(x) = x^2 - 40x + 610$, where $C(x)$ is the cost, in dollars, to sell x units of tacos. Find the number of units of tacos he should sell to minimize his costs. What is the minimum cost?

54. Find two positive real numbers whose sum is 40 and whose product is a maximum.

55. Find two real numbers whose difference is 40 and whose product is a minimum.

56. Mike wants to enclose a rectangular area for his rabbits alongside his large barn using 30 feet of fencing. What dimensions will maximize the area fenced if the barn is used for one side of the rectangle?

57. Kevin wants to enclose a rectangular garden using 14 eight-foot railroad ties, which he cannot cut. What are the dimensions of the rectangle that maximize the area enclosed?

58. Martin plans to construct a rectangular kennel for two dogs using 120 feet of chain-link fencing. He plans to fence all four sides and down the middle to keep the dogs separate. What overall dimensions will maximize the total area fenced?

59. Mona gives a walking tour of Honolulu to one person for \$49. To increase her business, she advertised that she would lower the price by \$1 per person for each additional person, up to 49 people.

- **a.** Write the price per person p as a function of the number of people n.
- **b.** Write her revenue as a function of the number of people on the tour.
- **c.** How many people on the tour would maximize Mona's revenue and what would this revenue be?

60. At \$10 per ticket, Willie Williams and the Wranglers will fill all 8000 seats in the Assembly Center. The manager knows that for every \$1 increase in the price, 500 tickets will go unsold.

- **a.** Write the number of tickets sold n as a function of ticket price p.
- **b.** Write the total revenue as a function of the ticket price.
- **c.** What ticket price will maximize the revenue and what would this revenue be?