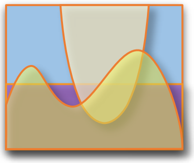
Q.5 Polynomial and Rational Inequalities



In *sections L4* and *L5*, we discussed solving linear inequalities in one variable as well as solving systems of such inequalities. In this section, we examine polynomial and rational inequalities in one variable. Such inequalities can be solved using either graphical or analytic methods. Below, we discuss both types of methods with a particular interest in the analytic one.

Solving Quadratic Inequalities by Graphing

Definition 5.1 ► A **quadratic inequality** is any inequality that can be written in one of the forms

$$\boldsymbol{ax^2 + bx + c > (\geq)\ 0}, \quad \text{or} \quad \boldsymbol{ax^2 + bx + c < (\leq)\ 0}, \quad \text{or} \quad \boldsymbol{ax^2 + bx + c \neq 0}$$

where a, b, and c are real numbers, with $a \neq 0$.

To solve a quadratic inequality, it is useful to solve the related quadratic equation first. For example, to solve $x^2 + 2x - 3 > (\geq)\ 0$, or $x^2 + 2x - 3 < (\leq)\ 0$, we may consider solving the related equation:

$$x^2 + 2x - 3 = 0$$

$$(x + 3)(x - 1) = 0$$

$$x = -3 \text{ or } x = 1.$$

This helps us to sketch an approximate graph of the related function $f(x) = x^2 + x - 2$, as in *Figure 1.1*. The graph of function f is a parabola that crosses the x-axis at $x = -3$ and $x = 1$, and is directed upwards. Observe that the graph extends below the x-axis for x-values from the interval $(-3,1)$ and above the x-axis for x-values from the set $(-\infty, -3) \cup (1, \infty)$. This allows us to read solution sets of several inequalities, as listed below.

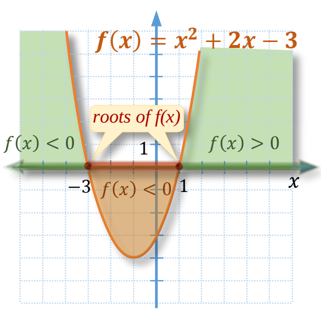


Figure 1.1

Inequality	Solution Set
$\boldsymbol{x^2 + 2x - 3 > 0}$	$(-\infty, -3) \cup (1, \infty)$
$\boldsymbol{x^2 + 2x - 3 \geq 0}$	$(-\infty, -3] \cup [1, \infty)$
$\boldsymbol{x^2 + 2x - 3 < 0}$	$(-3,1)$
$\boldsymbol{x^2 + 2x - 3 \leq 0}$	$[-3,1]$
$\boldsymbol{x^2 + 2x - 3 \neq 0}$	$\mathbb{R} \setminus \{-3,1\}$

Note: If the inequalities contain equations $(\geq, \leq)$, the x-values of the intercepts are included in the solution sets. Otherwise, the x-values of the intercepts are excluded from the solution sets.

Solving Polynomial Inequalities

Definition 5.2 ► A **polynomial inequality** is any inequality that can be written in one of the forms

$$\boldsymbol{P(x) > (\geq)\ 0}, \quad \text{or} \quad \boldsymbol{P(x) < (\leq)\ 0}, \quad \text{or} \quad \boldsymbol{P(x) \neq 0}$$

where $P(x)$ is a polynomial with real coefficients.

Note: Linear or quadratic inequalities are special cases of polynomial inequalities.

Polynomial inequalities can be solved graphically or analytically, without the use of a graph. The analytic method involves determining the sign of the polynomial by analysing signs of the polynomial factors for various x-values, as in the following example.

Example 1 ► **Solving Polynomial Inequalities Using Sign Analysis**

Solve each inequality using sign analysis.

a. $(x+1)(x-2)x > 0$ **b.** $2x^4 + 8 \leq 10x^2$

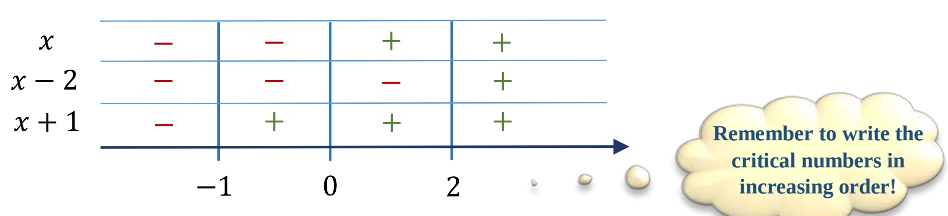
Solution ► **a.** The solution set of $(x+1)(x-2)x > 0$ consists of all x-values that make the product $(x+1)(x-2)x$ positive. To analyse how the sign of this product depends on the x-values, we can visualise the sign behaviour of each factor with respect to the x-value, by recording applicable signs in particular sections of a number line.

For example, the expression $x+1$ changes its sign at $x=-1$.
If $x > -1$, the expression $x+1$ is positive, so we mark "+" in the interval $(-1, \infty)$.
If $x < -1$, the expression $x+1$ is negative, so we mark "−" in the interval $(-\infty, -1)$.

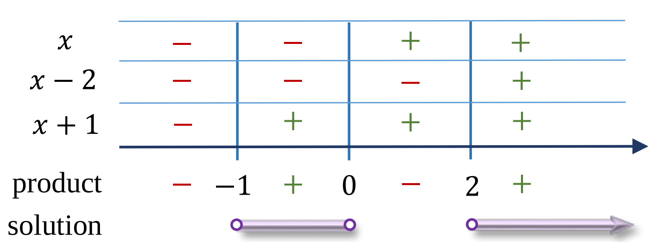
So, the sign behaviour of the expression $x+1$ can be recorded on a number line, as below.

Here, the **zeros** of polynomials are referred to as **critical numbers**. This is because the polynomials may change their signs at these numbers.

A similar analysis can be conducted for the remaining factors, $x-2$ and x. These expressions change their signs at $x=2$ and $x=0$, correspondingly. The sign behaviour of all the factors can be visualised by reserving one line of signs per each factor, as shown below.



The sign of the product $(x+1)(x-2)x$ is obtained by multiplying signs in each column. The result is marked beneath the number line, as below.



The signs in the "product" row give us the grounds to state the solution set for the original inequality, $(x+1)(x-2)x > 0$. Before we write the final answer though, it is helpful to visualize the solution set by graphing it in the "solution" row. To satisfy the original inequality, we need the product to be positive, so we look for the intervals within which the product is positive. Since the inequality does not include an equation, the intervals are open. Therefore, the solution set is $(\mathbf{-1},\mathbf{0}) \cup (\mathbf{2},\infty)$.

b. To solve $2x^4 + 8 \le 10x^2$ by sign analysis, we need to keep one side of the inequality equal to zero and factor the other side so that we may identify the critical numbers. To do this, we may change the inequality as below.

$$2x^4 + 8 \le 10x^2 \qquad / -10x^2$$

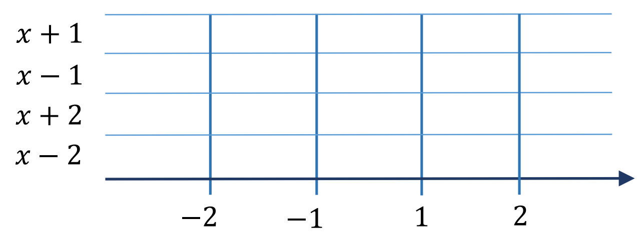
$$2x^4 - 10x^2 + 8 \le 0 \qquad / \div 2$$

$$x^4 - 5x^2 + 4 \le 0$$

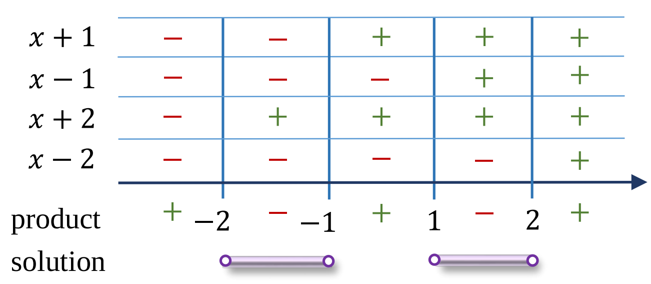
$$(x^2-4)(x^2-1) \le 0$$

$$(x-2)(x+2)(x-1)(x+1) \le 0$$

The critical numbers (the x-values that make the factors equal to zero) are 2, -2, 1, and -1. To create a table of signs, we arrange these numbers on a number line in increasing order and list all the factors in the left column.



Then, we can fill in the table with signs that each factor assumes for the x-values from the corresponding section of the number line.



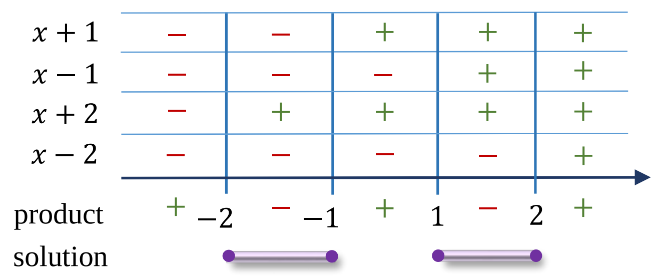
The signs in each part of the number line can be determined either by:

- **analysing** the behaviour of each factor with respect to its critical number (For example, $x-2 > 0$ for $x > 2$. So, the row of signs assumed by $x-2$ consists of negative signs until the critical number 2 and positive signs after this number.)

or

- **testing** an x-value from each section of the number line (For example, since the expression $x-2$ does not change its sign inside the interval $(-1,1)$, to determine its sign, it is enough to evaluate it for an easy to calculate **test number** from this interval. For instance, when $x = 0 \in (-1,1)$, the value of $x - 2$ is negative. This means that all the values of $x - 2$ are negative between -1 and 1.)

Finally, underneath each column, we record the sign of the product and graph the solution set to the inequality $(x-2)(x+2)(x-1)(x+1) \leq 0$



Note: Since the inequality contains an equation, the endpoints of the resulting intervals belong to the solution set as well. Hence, they are marked by **filled-in circles** and notated with **square brackets** in interval notation.

So, the solution set is $[\mathbf{-2}, \mathbf{-1}] \cup [\mathbf{1}, \mathbf{2}]$.

Solving Rational Inequalities

Definition 5.3 ► A **rational inequality** is any inequality that can be written in one of the forms

$$\frac{\boldsymbol{P(x)}}{\boldsymbol{Q(x)}} > (\geq)\ \mathbf{0}, \quad \text{or} \quad \frac{\boldsymbol{P(x)}}{\boldsymbol{Q(x)}} < (\leq)\ \mathbf{0}, \quad \text{or} \quad \frac{\boldsymbol{P(x)}}{\boldsymbol{Q(x)}} \neq \mathbf{0}$$

where $P(x)$ and $Q(x)$ are polynomials with real coefficients.

Rational inequalities can be solved similarly as polynomial inequalities. To solve a rational inequality using the sign analysis method, we need to make sure that one side of the inequality is **zero** and the other side is expressed as a **single algebraic fraction** with a completely factored numerator and denominator.

Example 2 ► **Solving Rational Inequalities Using Sign Analysis**

Solve each inequality using sign analysis.

a. $\frac{(x-2)x}{x+1} \geq 0$ **b.** $\frac{4-x}{x+2} \geq x$

Solution ► **a.** The right side of the inequality $\frac{(x-2)x}{x+1} > 0$ is zero, and the left side is a single fraction with both numerator and denominator in factored form. So, to solve this inequality, it is enough to analyse signs of the expression $\frac{(x-2)x}{x+1}$ at different intervals of the domain.

These intervals are determined by the **critical numbers** (the zeros of the numerator and denominator), which are -1, 0, and 2.

	$(-\infty,-1)$	$(-1,0)$	$(0,2)$	$(2,\infty)$
x	$-$	$-$	$+$	$+$
$x-2$	$-$	$-$	$-$	$+$
$x+1$	$-$	$+$	$+$	$+$
product	$-$	$+$	$-$	$+$
solution		$(-1, 0]$		$[2, \infty)$

As indicated in the above table of signs, the solution set to the inequality $\frac{(x-2)x}{x+1} \geq 0$ contains numbers between -1 and 0 and numbers higher than 2. In addition, since the inequality includes an equation, $x = 0$ and $x = 2$ are also solutions. However, $x = -1$ is not a solution because -1 does not belong to the domain of the expression $\frac{(x-2)x}{x+1}$ since it would make the denominator 0. So, the solution set is $(-\mathbf{1}, \mathbf{0}] \cup [\mathbf{2}, \infty)$.

Attention: **Solutions** to a rational inequality **must belong to the domain** of the inequality. This means that any number that makes the denominator 0 must be excluded from the solution set.

b. To solve $\frac{4-x}{x+2} \geq x$ by the sign analysis method, first, we would like to keep the right side equal to zero. So, we rearrange the inequality as below.

When working with inequalities, **avoid multiplying by the denominator** as it can be positive or negative for different x-values!

$$\frac{4-x}{x+2} \geq x \qquad / -x$$

$$\frac{4-x}{x+2} - x \geq 0$$

$$\frac{4-x-x(x+2)}{x+2} \geq 0$$

$$\frac{4-x-x^2-2x}{x+2} \geq 0$$

$$\frac{-x^2-3x+4}{x+2} \geq 0$$

$$\frac{-(x^2+3x-4)}{x+2} \geq 0$$

$$\frac{-(x+4)(x-1)}{x+2} \geq 0 \qquad / \cdot(-1)$$

$$\frac{(x+4)(x-1)}{x+2} \leq 0$$

When multiplying by a **negative** number, remember to **reverse** the inequality sign!

Now, we can analyse the signs of the expression $\frac{(x+4)(x-1)}{x+2}$, using the table of signs with the critical numbers -4, -2, and 1.

$x+4$	$-$		$+$		$+$		$+$
$x-1$	$-$		$-$		$-$		$+$
$x+2$	$-$		$-$		$+$		$+$
product	$-$	-4	$+$	-2	$-$	1	$+$

solution 

So, the solution set for the inequality $\frac{(x+4)(x-1)}{x+2} \leq 0$, which is equivalent to $\frac{4-x}{x+2} \geq x$, contains numbers lower than -4 and numbers between -2 and 1. Since the inequality includes an equation, $x = -4$ and $x = 1$ are also solutions. However, $x = -2$ is not in the domain of $\frac{(x+4)(x-1)}{x+2}$, and therefore it is not a solution.

Thus, the solution set to the original inequality is $(-\infty, -\mathbf{4}] \cup (-\mathbf{2}, \mathbf{1}]$.

Summary of Solving Polynomial or Rational Inequalities

1. **Write** the inequality so that **one of its sides is zero** and the other side is expressed as the **product or quotient of prime polynomials**.
2. **Determine the critical numbers**, which are the roots of all the prime polynomials appearing in the inequality.
3. **Divide the number line into intervals** formed by the set of critical numbers.
4. **Create a table of signs** for all prime factors in all intervals formed by the set of critical numbers. This can be done by analysing the sign of each factor, or by testing a number from each interval.
5. **Determine the sign of the overall expression** in each of the intervals.
6. **Graph the intervals of numbers that satisfy the inequality**. Make sure to **exclude endpoints that are not in the domain** of the inequality.
7. **State the solution set** to the original inequality **in interval notation**.

Solving Special Cases of Polynomial or Rational Inequalities

Some inequalities can be solved without the use of a graph or a table of signs.

Example 3 ► **Solving Special Cases of Inequalities**

Solve each inequality.

a. $(3x+2)^2 > -1$ **b.** $\frac{(x-4)^2}{x^2} \leq 0$

Solution ► **a.** First, notice that the left side of the inequality $(3x+2)^2 > -1$ is a perfect square and as such, it assumes a nonnegative value for any input x. Since a nonnegative quantity is always bigger than -1, the inequality is satisfied by any real number x. So, the solution set is $\mathbb{R}$.

Note: The solution set of an inequality that is **always true** is the set of all real numbers, $\mathbb{R}$. For example, inequalities that take one of the following forms

$$\textbf{\textit{nonnegative}} > \textbf{\textit{negative}}$$

$$\textbf{\textit{positive}} \geq \textbf{\textit{negative}}$$

$$\textbf{\textit{positive}} > \textbf{\textit{nonpositive}}$$

$$\textbf{\textit{positive}} > \mathbf{0}$$

$$\textbf{\textit{negative}} < \mathbf{0}$$

are **always true**. So their solution set is $\mathbb{R}$.

b. Since the left side of the inequality $\frac{(x-4)^2}{x^2} \leq 0$ is a perfect square, it is bigger or equal to zero for all x-values. So, we have

$$0 \leq \frac{(x-4)^2}{x^2} \leq 0,$$

which can be true only if $\frac{(x-4)^2}{x^2} = 0$. Since a fraction equals to zero only when its numerator equals to zero, the solution to the last equation is $x = 4$. Thus, the solution set for the original inequality is $\{\mathbf{4}\}$.

Observation: Notice that the inequality $\frac{(x-4)^2}{x^2} < 0$ has no solution as a perfect square is never negative.

Note: The solution set of an inequality that is **never true** is the empty set, $\emptyset$. For example, inequalities that take one of the following forms

$$\textbf{\textit{positive}}\ (\textbf{\textit{or nonnegative}}) \leq \textbf{\textit{negative}}$$

$$\textbf{\textit{nonnegative}} < \textbf{\textit{nonpositive}}$$

$$\textbf{\textit{positive}} \leq \textbf{\textit{nonpositive}}$$

$$\textbf{\textit{positive}} \leq \mathbf{0} \quad \text{or} \quad \textbf{\textit{negative}} \geq \mathbf{0}$$

$$\textbf{\textit{nonnegative}} < \mathbf{0} \quad \text{or} \quad \textbf{\textit{nonpositive}} > \mathbf{0}$$

are **never true**. So, their solution sets are $\emptyset$.

Polynomial and Rational Inequalities in Application Problems

Some application problems involve solving polynomial or rational inequalities.

Example 4 ► **Finding the Range of Values Satisfying the Given Condition**

An outerwear manufacturer determines that its weekly revenue, R, for selling a rain parka at a price of p dollars is modeled by the function $R(p) = 150p - p^2$. What range of prices for the parka will generate a weekly revenue of at least \$5000?

Solution ► Since the revenue must be at least \$5000, we can set up the inequality

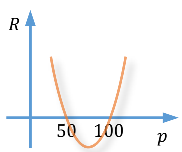
$$R(p) = 150p - p^2 \geq 5000,$$

and solve it for p. So, we have

$$-p^2 + 150p - 5000 \geq 0 \qquad / \cdot (-1)$$

$$p^2 - 150p + 5000 \leq 0$$

$$(p - 50)(p - 100) \leq 0$$



Since the left-hand side expression represents a directed upwards parabola with roots at $p = 50$ and $p = 100$, its graph looks like in the accompanying figure. The graph extends below the p-axis for p-values between 50 and 100. So, to generate weekly revenue of at least \$5000, the price p of a parka must take a value within the interval $[50,100]$.

Q.5 Exercises

Concept Check *True or False.*

1. To determine if the value of an expression is greater than or less than 0 in a given interval, a test number can be used.

2. If the solution to the inequality $P(x) \geq 0$, where $P(x)$ is a polynomial with real coefficients, is $[2, 5)$, then the solution to the inequality $P(x) < 0$ is $(-\infty, 2) \cup [5, \infty)$.

3. The inequalities $(x - 1)(x + 3) \leq 0$ and $\frac{(x-1)}{(x+3)} \leq 0$ have the same solutions.

4. The solution set of the inequality $(x - 1)^2 > 0$ is the set of all real numbers.

5. The inequality $x^2 + 1 \leq 0$ has no solution.

6. The solution set of the inequality $\frac{(x-1)^2}{(x+1)^2} \geq 0$ is the set of all real numbers.

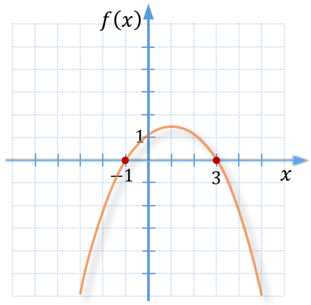
Concept Check

Given the graph of function f, state the solution set for each inequality

a. $f(x) \geq 0$

b. $f(x) < 0$

7.



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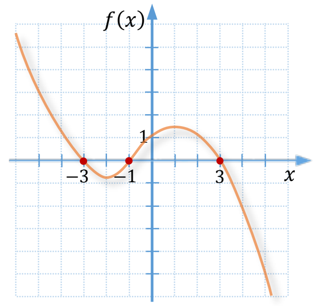

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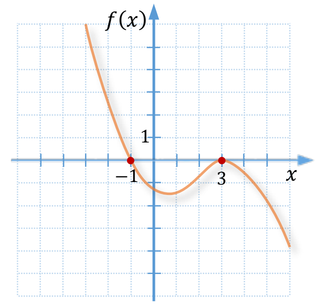

***Solve** each inequality by sketching an approximate graph for the related equation.*

10. $(x+4)(x-2) > 0$

11. $(x+1)(x-2) < 0$

12. $x^2 - 4x + 3 \geq 0$

13. $\frac{1}{2}(x^2 - 3x - 10) \leq 0$

14. $4 - 9x^2 > 0$

15. $-x^2 - 2x < 0$

***Solve** each inequality using **sign analysis**.*

16. $(x-3)(x+2) > 0$

17. $(x+4)(x-5) < 0$

18. $x^2 + 2x - 7 \leq 8$

19. $x^2 - x - 2 \geq 10$

20. $3x^2 + 10x > 8$

21. $2x^2 + 5x < -2$

22. $x^2 + 9 > -6x$

23. $x^2 + 4 \leq 4x$

24. $6 + x - x^2 \leq 0$

25. $20 - x - x^2 < 0$

26. $(x-1)(x+2)(x-3) \geq 0$

27. $(x+3)(x-2)(x-5) \leq 0$

28. $x(x+3)(2x-1) > 0$

29. $x^2(x-2)(2x-1) \geq 0$

30. $x^4 - 13x^2 + 36 \leq 0$

Solve each inequality using sign analysis.

31. $\frac{x}{x+1} > 0$

32. $\frac{x+1}{x-2} < 0$

33. $\frac{2x-1}{x+3} \leq 0$

34. $\frac{2x-3}{x+1} \geq 0$

35. $\frac{3}{y+5} > 1$

36. $\frac{5}{t-1} \leq 2$

37. $\frac{x-1}{x+2} \leq 3$

38. $\frac{a+4}{a+3} \geq 2$

39. $\frac{2t-3}{t+3} < 4$

40. $\frac{3y+9}{2y-3} < 3$

41. $\frac{1-2x}{2x+5} \leq 2$

42. $\frac{2x+3}{1-x} \leq 1$

43. $\frac{4x}{2x-1} \leq x$

44. $\frac{-x}{x+2} > 2x$

45. $\frac{2x-3}{(x+1)^2} \leq 0$

46. $\frac{2x-3}{(x-2)^2} \geq 0$

47. $\frac{x^2+1}{5-x^2} > 0$

48. $x < \frac{3x-8}{5-x}$

49. $\frac{1}{x+2} \geq \frac{1}{x-3}$

50. $\frac{2}{x+3} \leq \frac{1}{x-1}$

51. $\frac{(x-3)(x+1)}{4-x} \geq 0$

52. $\frac{(x+2)(x-1)}{(x+4)^2} \geq 0$

53. $\frac{x^2-2x-8}{x^2+10x+25} > 0$

54. $\frac{x^2-4x}{x^2-x-6} \leq 0$

Solve each inequality.

55. $(4-3x)^2 \geq -2$

56. $(5+2x)^2 < -1$

57. $\frac{(1-2x)^2}{2x^4} \leq 0$

58. $\frac{(1-2x)^2}{(x+2)^2} > -3$

59. $\frac{-2x^2}{(x+2)^2} \geq 0$

60. $\frac{-x^2}{(x-3)^2} < 0$

Analytic Skills *Solve each problem.*

60. A penny is tossed upwards with an initial velocity of 48 ft/sec. The height, h, of the penny relative to the point of release t seconds after it is tossed is modeled by the function $h(t) = -16t^2 + 48t$. For what interval of time is the penny above the point of release?

61. A furniture maker determines that the weekly cost, C, for producing x accent tables is given by the function $C(x) = 2x^2 - 60x + 900$. How many accent tables can be produced to keep the weekly cost under \$500?

62. A gardener wants to enclose a rectangular flower bed using 90 ft of fencing. For what range of lengths will the area exceed 450 ft^2?

63. A company determines that its average cost C, in dollars, for selling x units of a product is modeled by the function $C(x) = \frac{870+2x}{x}$. For what number of units x will the average cost be less than \$8?

64. A publishing company had a revenue of \$18 million last year. The company's financial analyst uses the formula $P(R) = \frac{100R-1800}{R}$ to determine the percent growth, P, in the company's revenue this year over last year, where R is in millions of dollars. For what revenues R will the company's revenue grow by more than 10%?