Quadratic Equations and Functions

In this chapter, we discuss various ways of solving quadratic equations, $ax^2 + bx + c = 0$, including equations quadratic in form, such as $x^{-2} + x^{-1} - 20 = 0$, and solving formulas for a variable that appears in the first and second power, such as k in $k^2 - 3k = 2N$. Frequently used strategies of solving quadratic equations include the **completing the square** procedure and its generalization in the form of the **quadratic formula**. Completing the square allows for rewriting quadratic functions in vertex form, $f(x) = a(x-h)^2 + k$, which is very useful for graphing as it provides information about the location, shape, and direction of the parabola.

In the second part of this chapter, we examine properties and graphs of quadratic functions, including basic transformations of these graphs.

Finally, these properties are used in solving application problems, particularly problems involving **optimization**. In the last section of this chapter, we study how to solve polynomial and rational inequalities using **sign analysis**.

Q1 Methods of Solving Quadratic Equations

As defined in *Section F4*, a quadratic equation is a second-degree polynomial equation in one variable that can be written in standard form as

$$\boldsymbol{ax^2 + bx + c = 0},$$

where a, b, and c are real numbers and $\boldsymbol{a \neq 0}$. Such equations can be solved in many different ways, as presented below.

Solving by Graphing

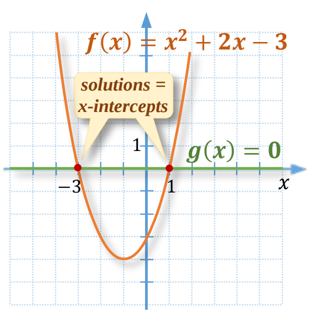


Figure 1.1

To solve a quadratic equation, for example $x^2 + 2x - 3 = 0$, we can consider its left side as a function $f(x) = x^2 + 2x - 3$ and the right side as a function $g(x) = 0$. To satisfy the original equation, both function values must be equal. After graphing both functions on the same grid, one can observe that this happens at points of intersection of the two graphs.

So the **solutions** to the original equation are the x-coordinates of the intersection points of the two graphs. In our example, these are the $\boldsymbol{x}$**-intercepts** or the **roots** of the function $f(x) = x^2 + 2x - 3$, as indicated in *Figure 1.1*.

Thus, the solutions to $x^2 + 2x - 3 = 0$ are $\boldsymbol{x = -3}$ and $\boldsymbol{x = 1}$.

Note: Notice that the graphing method, although visually appealing, is not always reliable. For example, the solutions to the equation $49x^2 - 4 = 0$ are $x = \frac{2}{7}$ and $x = -\frac{2}{7}$. Such numbers would be very hard to read from the graph.

Thus, the graphing method is advisable to use when searching for integral solutions or estimations of solutions.

To find exact solutions, we can use one of the algebraic methods presented below.

Solving by Factoring

Many quadratic equations can be solved by factoring and employing the zero-product property, as in *Section F4*.

For example, the equation $x^2 + 2x - 3 = 0$ can be solved as follows:

$$(x+3)(x-1) = 0$$

so, by zero-product property,

$$x + 3 = 0 \text{ or } x - 1 = 0,$$

which gives us the solutions

$$\boldsymbol{x = -3} \text{ or } \boldsymbol{x = 1}.$$

Solving by Using the Square Root Property

Quadratic equations of the form $\boldsymbol{ax^2 + c = 0}$ can be solved by applying the **square root property**.

Square Root Property:

For any positive real number a, if $\boldsymbol{x^2 = a}$, then $\boldsymbol{x = \pm\sqrt{a}}$.

This is because $\sqrt{x^2} = |x|$. So, after applying the square root operator to both sides of the equation $x^2 = a$, we have

$$\sqrt{x^2} = \sqrt{a}$$
$$|x| = \sqrt{a}$$
$$x = \pm\sqrt{a}$$

The $\pm\sqrt{a}$ is a shorter recording of two solutions: $\sqrt{a}$ and $-\sqrt{a}$.

For example, the equation $49x^2 - 4 = 0$ can be solved as follows:

$$49x^2 - 4 = 0 \qquad / +4$$

$$49x^2 = 4 \qquad / \div 49$$

$$x^2 = \frac{4}{49}$$

apply square root to both sides of the equation

$$\sqrt{x^2} = \sqrt{\frac{4}{49}}$$

Here we use the **square root property**. Remember the $\pm$ sign!

$$x = \pm\sqrt{\frac{4}{49}}$$

$$\boldsymbol{x = \pm\frac{2}{7}}$$

Note: Using the square root property is a common solving strategy for quadratic equations where **one side is a perfect square** of an unknown quantity and the **other side is a constant** number.

Example 1 ► **Solve by the Square Root Property**

Solve each equation using the square root property.

a. $(x-3)^2 = 49$ **b.** $2(3x-6)^2 - 54 = 0$

Solution ► **a.** Applying the square root property, we have

$$\sqrt{(x-3)^2} = \sqrt{49}$$

$$x - 3 = \pm 7 \qquad / +3$$

$$x = 3 \pm 7$$

so

$$\boldsymbol{x = 10} \text{ or } \boldsymbol{x = -4}$$

b. To solve $2(3x-6)^2 - 54 = 0$, we isolate the perfect square first and then apply the square root property. So,

$$2(3x-6)^2 - 54 = 0 \qquad / +54, \div 2$$

$$(3x-6)^2 = \frac{54}{2}$$

$$\sqrt{(3x-6)^2} = \sqrt{27}$$

$$3x - 6 = \pm 3\sqrt{3} \qquad / +6$$

$$3x = 6 \pm 3\sqrt{3} \qquad / \div 3$$

$$x = \frac{6 \pm 3\sqrt{3}}{3}$$

$$x = \frac{\cancel{3}(2 \pm \sqrt{3})}{\cancel{3}}$$

$$x = 2 \pm \sqrt{3}$$

Thus, the solution set is $\{\boldsymbol{2-\sqrt{3}},\ \boldsymbol{2+\sqrt{3}}\}$.

Caution: To simplify expressions such as $\frac{6+3\sqrt{3}}{3}$, we **factor the numerator** first. The common errors to avoid are

incorrect order of operations ← $\frac{6+3\sqrt{3}}{3} = \frac{9\sqrt{3}}{3} = 3\sqrt{3}$

or

incorrect canceling ← $\frac{6+\cancel{3}\sqrt{3}}{\cancel{3}} = 6 + \sqrt{3}$

or

incorrect canceling ← $\frac{\overset{2}{\cancel{6}}+3\sqrt{3}}{\cancel{3}} = 2 + 3\sqrt{3}$

Solving by Completing the Square

So far, we have seen how to solve quadratic equations, $ax^2 + bx + c = 0$, if the expression $ax^2 + bx + c$ is factorable or if the coefficient b is equal to zero. To solve other quadratic equations, we may try to rewrite the variable terms in the form of a perfect square, so that the resulting equation can already be solved by the square root property.

For example, to solve $x^2 + 6x - 3 = 0$, we observe that the variable terms $x^2 + 6x$ could be written in **perfect square** form if we add 9, as illustrated in *Figure 1.2*. This is because

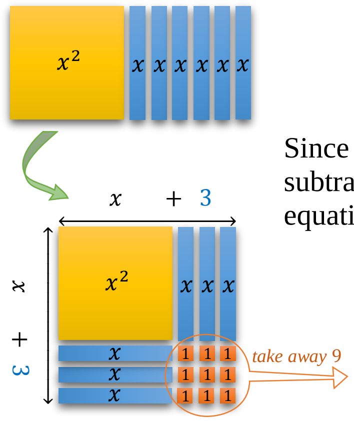


Figure 1.2

$$x^2 + 6x + 9 = (x + 3)^2$$

observe that 3 comes from taking half of 6

Since the original equation can only be changed to an equivalent form, if we add 9, we must subtract 9 as well. *(Alternatively, we could add 9 to both sides of the equation.)* So, the equation can be transformed as follows:

Completing the Square Procedure

$$x^2 + 6x - 3 = 0 \qquad / +12$$

$$\underbrace{x^2 + 6x + 9}_{perfect\ square} - 9 - 3 = 0$$

$$(x + 3)^2 = 12$$

square root property

$$\sqrt{(x + 3)^2} = \sqrt{12}$$

$$x + 3 = \pm 2\sqrt{3}$$

$$x = \mathbf{-3 \pm 2\sqrt{3}}$$

Generally, to **complete the square** for the first two terms of the equation

$$\boldsymbol{x^2 + bx + c = 0},$$

we take half of the x-coefficient, which is $\frac{b}{2}$, and square it. Then, we add and subtract that number, $\left(\frac{b}{2}\right)^2$. *(Alternatively, we could add $\left(\frac{b}{2}\right)^2$ to both sides of the equation.)* This way, we produce an equivalent equation

$$\boldsymbol{x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c = 0},$$

and consequently,

$$\boldsymbol{\left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4} + c = 0}.$$

We can write this equation directly, by following the rule:

Write the sum of $\boldsymbol{x}$ and **half of the middle coefficient**, **square the binomial**, and **subtract the perfect square of the constant** appearing in the bracket.

To **complete the square** for the first two terms of a quadratic equation with a leading coefficient of $a \neq 1$,

$$\boldsymbol{ax^2 + bx + c = 0},$$

we

- divide the equation by $\boldsymbol{a}$ *(alternatively, we could factor a out of the first two terms)* so that the leading coefficient is 1, and then
- complete the square as in the previous case, where $\boldsymbol{a = 1}$.

So, after division by $\boldsymbol{a}$, we obtain

$$\boldsymbol{x^2 + \frac{b}{a}x + \frac{c}{a} = 0}.$$

Since half of $\frac{b}{a}$ is $\frac{b}{2a}$, then we complete the square as follows:

$$\boldsymbol{\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0}.$$

Remember to **subtract the perfect square of the constant** appearing in the bracket!

Example 2 ► **Solve by Completing the Square**

Solve each equation using the completing the square method.

a. $x^2 + 5x - 1 = 0$ **b.** $3x^2 - 12x - 5 = 0$

Solution ► **a.** First, we complete the square for $x^2 + 5x$ by adding and subtracting $\left(\frac{5}{2}\right)^2$ and then we apply the square root property. So, we have

$$\underbrace{x^2 + 5x + \left(\frac{5}{2}\right)^2}_{perfect\ square} - \left(\frac{5}{2}\right)^2 - 1 = 0$$

$$\left(x + \frac{5}{2}\right)^2 - \frac{25}{4} - 1 \cdot \frac{4}{4} = 0$$

$$\left(x + \frac{5}{2}\right)^2 - \frac{29}{4} = 0 \qquad / +\frac{29}{4}$$

$$\left(x + \frac{5}{2}\right)^2 = \frac{29}{4}$$

$$x + \frac{5}{2} = \pm\sqrt{\frac{29}{4}}$$

apply square root to both sides of the equation

remember to use the ± sign!

$$x + \frac{5}{2} = \pm\frac{\sqrt{29}}{2} \qquad / -\frac{5}{2}$$

$$x = \frac{-5 \pm \sqrt{29}}{2}$$

Thus, the solution set is $\left\{\frac{-5-\sqrt{29}}{2}, \frac{-5+\sqrt{29}}{2}\right\}$.

Note: Unless specified otherwise, we are expected to state the **exact solutions** rather than their calculator approximations. Sometimes, however, especially when solving application problems, we may need to use a calculator to approximate the solutions. The reader is encouraged to check that the two decimal **approximations** of the above solutions are

$$\frac{-5-\sqrt{29}}{2} \approx \mathbf{-5.19} \quad \text{and} \quad \frac{-5+\sqrt{29}}{2} \approx \mathbf{0.19}$$

b. In order to apply the strategy as in the previous example, we divide the equation by the leading coefficient, 3. So, we obtain

$$3x^2 - 12x - 5 = 0 \qquad / \div 3$$

$$x^2 - 4x - \frac{5}{3} = 0$$

Then, to complete the square for $x^2 - 4x$, we may add and subtract 4. This allows us to rewrite the equation equivalently, with the variable part in perfect square form.

$$(x-2)^2 - 4 - \frac{5}{3} = 0$$

$$(x-2)^2 = 4 \cdot \frac{3}{3} + \frac{5}{3}$$

$$(x-2)^2 = \frac{17}{3}$$

$$x - 2 = \pm\sqrt{\frac{17}{3}}$$

$$x = \mathbf{2} \pm \frac{\sqrt{\mathbf{17}}}{\sqrt{\mathbf{3}}}$$

Note: The final answer could be written as a single fraction as shown below:

$$x = \frac{2\sqrt{3} \pm \sqrt{17}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\mathbf{6 \pm \sqrt{51}}}{\mathbf{3}}$$

Solving with Quadratic Formula

Applying the completing the square procedure to the quadratic equation

$$ax^2 + bx + c = 0,$$

with real coefficients $a \neq 0$, b, and c, allows us to develop a general formula for finding the solution(s) to any such equation.

Quadratic Formula ► The solution(s) to the equation $ax^2 + bx + c = 0$, where $a \neq 0$, b, c are real coefficients, are given by the formula

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Here $x_{1,2}$ denotes the two solutions, $x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$, and $x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$.

Proof: ► First, since $a \neq 0$, we can divide the equation $ax^2 + bx + c = 0$ by a. So, the equation to solve is

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Then, we complete the square for $x^2 + \frac{b}{a}x$ by adding and subtracting the perfect square of half of the middle coefficient, $\left(\frac{b}{2a}\right)^2$. So, we obtain

$$\underbrace{x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2}_{perfect\ square} - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0 \qquad / +\left(\frac{b}{2a}\right)^2, -\frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} \cdot \frac{4a}{4a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \qquad / -\frac{b}{2a}$$

and finaly,

QUADRATIC FORMULA

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

which concludes the proof.

Example 3 ► **Solving Quadratic Equations with the Use of the Quadratic Formula**

Using the Quadratic Formula, solve each equation, if possible. Then visualize the solutions graphically.

a. $2x^2 + 3x - 20 = 0$ **b.** $3x^2 - 4 = 2x$ **c.** $x^2 - \sqrt{2}x + 3 = 0$

Solution ▶ **a.** To apply the quadratic formula, first, we identify the values of a, b, and c. Since the equation is in standard form, $a = 2$, $b = 3$, and $c = -20$. The solutions are equal to

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{3^2 - 4 \cdot 2(-20)}}{2 \cdot 2} = \frac{-3 \pm \sqrt{9 + 160}}{4}$$

$$= \frac{-3 \pm 13}{4} = \begin{cases} \frac{-3 + 13}{4} = \frac{10}{4} = \mathbf{\frac{5}{2}} \\ \frac{-3 - 13}{4} = \frac{-16}{4} = \mathbf{-4} \end{cases}$$

Thus, the solution set is $\left\{\mathbf{-4}, \mathbf{\frac{5}{2}}\right\}$.

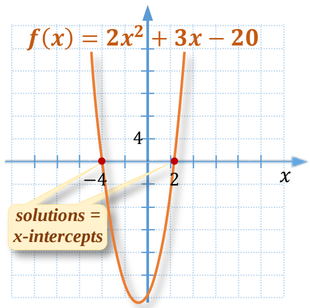


Figure 1.3

These solutions can be seen as x-intercepts of the function $f(x) = 2x^2 + 3x - 20$, as shown in *Figure 1.3.*

b. Before we identify the values of a, b, and c, we need to write the given equation $3x^2 - 4 = 2x$ in standard form. After subtracting $4x$ from both sides of the given equation, we obtain

$$3x^2 - 2x - 4 = 0$$

Since $a = 3$, $b = -2$, and $c = -5$, we evaluate the quadratic formula,

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{(-2)^2 - 4 \cdot 3(-4)}}{2 \cdot 3} = \frac{2 \pm \sqrt{4 + 48}}{6} = \frac{2 \pm \sqrt{52}}{6}$$

$$= \frac{2 \pm \sqrt{4 \cdot 13}}{6} = \frac{2 \pm 2\sqrt{13}}{6} = \frac{2(1 \pm \sqrt{13})}{6} = \mathbf{\frac{1 \pm \sqrt{13}}{3}}$$

simplify by factoring

So, the solution set is $\left\{\mathbf{\frac{1-\sqrt{13}}{3}}, \mathbf{\frac{1+\sqrt{13}}{3}}\right\}$.

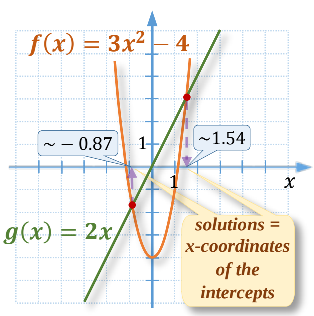


Figure 1.4

We may visualize solutions to the original equation, $3x^2 - 4 = 2x$, by graphing functions $f(x) = 3x^2 - 4$ and $g(x) = 2x$. The x-coordinates of the intersection points are the solutions to the equation $f(x) = g(x)$, and consequently to the original equation. As indicated in *Figure 1.4*, the approximations of these solutions are $\frac{1-\sqrt{13}}{3} \approx -0.87$ and $\frac{1+\sqrt{13}}{3} \approx 1.54$.

c. Substituting $a = 1$, $b = -\sqrt{2}$, and $c = 3$ into the Quadratic Formula, we obtain

$$x_{1,2} = \frac{\sqrt{2} \pm \sqrt{\left(-\sqrt{2}\right)^2 - 4 \cdot 1 \cdot 3}}{2 \cdot 1} = \frac{\sqrt{2} \pm \sqrt{4 - 12}}{2} = \frac{\sqrt{2} \pm \sqrt{-8}}{2}$$

not a real number!

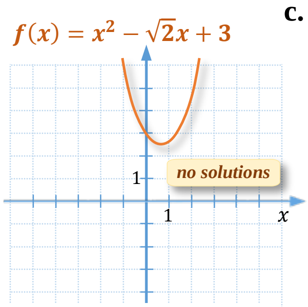


Figure 1.5

Since a square root of a negative number is not a real value, we have **no real solutions**. Thus the solution set to this equation is $\emptyset$. In a graphical representation, this means that the graph of the function $f(x) = x^2 - \sqrt{2}x + 3$ does not cross the x-axis. See *Figure 1.5.*

Observation: Notice that we could find the solution set in *Example 3c* just by evaluating the radicand $b^2 - 4ac$. Since this radicand was negative, we concluded that there is no solution to the given equation as a root of a negative number is not a real number. There was no need to evaluate the whole expression of Quadratic Formula.

So, the radicand in the Quadratic Formula carries important information about the number and nature of roots. Because of it, this radicand earned a special name, the discriminant.

Definition 1.1 ► The radicand $\boldsymbol{b^2 - 4ac}$ in the Quadratic Formula is called the **discriminant** and it is denoted by Δ.

Notice that in terms of Δ, the Quadratic Formula takes the form

$$\boldsymbol{x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a}}$$

Observing the behaviour of the expression $\sqrt{\Delta}$ allows us to classify the number and type of solutions (roots) of a quadratic equation with rational coefficients.

Characteristics of Roots (Solutions) Depending on the Discriminant

Suppose $\boldsymbol{a}x^2 + \boldsymbol{b}x + \boldsymbol{c} = 0$ has **rational** coefficients $\boldsymbol{a} \neq 0$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\Delta = \boldsymbol{b^2 - 4ac}$.

- If $\Delta < \mathbf{0}$, then the equation has **no real solutions**, as $\sqrt{negative}$ is not a real number.
- If $\Delta = \mathbf{0}$, then the equation has **one rational solution**, $\frac{-b}{2a}$.
- If $\Delta > \mathbf{0}$, then the equation has **two solutions**, $\frac{-b-\sqrt{\Delta}}{2a}$ and $\frac{-b+\sqrt{\Delta}}{2a}$.
  These solutions are
  - **irrational**, if Δ is **not** a **perfect square number**
  - **rational**, if Δ is a **perfect square number** (as $\sqrt{perfect\ square} = integer$)

  In addition, if $\Delta \geq \mathbf{0}$ is a **perfect square number**, then the equation could be solved by **factoring**.

Example 4 ► **Determining the Number and Type of Solutions of a Quadratic Equation**

Using the discriminant, determine the number and type of solutions of each equation without solving the equation. If the equation can be solved by factoring, show the factored form of the trinomial.

a. $2x^2 + 7x - 15 = 0$

b. $4x^2 - 12x + 9 = 0$

c. $3x^2 - x + 1 = 0$

d. $2x^2 - 7x + 2 = 0$

Solution ► **a.** $\Delta = 7^2 - 4 \cdot 2 \cdot (-15) = 49 + 120 = 169$

Since 169 is a perfect square number, the equation has **two rational solutions** and it can be solved by factoring. Indeed, $2x^2 + 7x - 15 = (2x - 3)(x + 5)$.

b. $\Delta = (-12)^2 - 4 \cdot 4 \cdot 9 = 144 - 144 = 0$

$\Delta = 0$ indicates that the equation has **one rational solution** and it can be solved by factoring. Indeed, the expression $4x^2 - 12x + 9$ is a perfect square, $(2x - 3)^2$.

c. $\Delta = (-1)^2 - 4 \cdot 3 \cdot 1 = 1 - 12 = -11$

Since $\Delta < 0$, the equation has **no real solutions** and therefore it can not be solved by factoring.

d. $\Delta = (-7)^2 - 4 \cdot 2 \cdot 2 = 49 - 16 = 33$

Since $\Delta > 0$ but it is not a perfect square number, the equation has **two real solutions** but it cannot be solved by factoring.

Example 5 ► **Solving Equations Equivalent to Quadratic**

Solve each equation.

a. $2 + \frac{7}{x} = \frac{5}{x^2}$ **b.** $2x^2 = (x + 2)(x - 1) + 1$

Solution ► **a.** This is a rational equation, with the set of $\mathbb{R} \setminus \{0\}$ as its domain. To solve it, we multiply the equation by the $LCD = x^2$. This brings us to a quadratic equation

$$2x^2 + 7x = 5$$

or equivalently

$$2x^2 + 7x - 5 = 0,$$

which can be solved by following the Quadratic Formula for $a = 2$, $b = 7$, and $c = -5$. So, we have

$$x_{1,2} = \frac{-7 \pm \sqrt{(-7)^2 - 4 \cdot 2(-5)}}{2 \cdot 2} = \frac{-7 \pm \sqrt{49 + 40}}{4} = \frac{\mathbf{-7 \pm \sqrt{89}}}{\mathbf{4}}$$

Since both solutions are in the domain, the solution set is $\left\{\frac{-7-\sqrt{89}}{4}, \frac{-7+\sqrt{89}}{4}\right\}$

b. To solve $1 - 2x^2 = (x + 2)(x - 1)$, we simplify the equation first and rewrite it in standard form. So, we have

$$1 - 2x^2 = x^2 + x - 2 \qquad / -x, +2$$

$$-3x^2 - x + 3 = 0 \qquad / \cdot (-1)$$

$$3x^2 + x - 3 = 0$$

Since the left side of this equation is not factorable, we may use the Quadratic Formula. So, the solutions are

$$x_{1,2} = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 3(-3)}}{2 \cdot 3} = \frac{1 \pm \sqrt{1 + 36}}{6} = \frac{\mathbf{1 \pm \sqrt{37}}}{\mathbf{6}}.$$

Q.1 Exercises

True or False.

1. A quadratic equation is an equation that can be written in the form $ax^2 + bx + c = 0$, where a, b, and c are any real numbers.

2. If the graph of $f(x) = ax^2 + bx + c$ intersects the x-axis twice, the equation $ax^2 + bx + c = 0$ has two solutions.

3. If the equation $ax^2 + bx + c = 0$ has no solution, the graph of $f(x) = ax^2 + bx + c$ does not intersect the x-axis.

4. The Quadratic Formula cannot be used to solve the equation $x^2 - 5 = 0$ because the equation does not contain a linear term.

5. The solution set for the equation $x^2 = 16$ is $\{4\}$.

6. To complete the square for $x^2 + bx$, we add $\left(\frac{b}{2}\right)^2$.

7. If the discriminant is positive, the equation can be solved by factoring.

For each function f,

a) *graph $f(x)$ using a table of values;*

b) *find the x-intercepts of the graph;*

c) *solve the equation $f(x) = 0$ by factoring and compare these solutions to the x-intercepts of the graph.*

8. $f(x) = -x^2 - 3x + 2$

9. $f(x) = x^2 + 2x - 3$

10. $f(x) = 3x + x(x - 2)$

11. $f(x) = 2x - x(x - 3)$

12. $f(x) = 4x^2 - 4x - 3$

13. $f(x) = -\frac{1}{2}(2x^2 + 5x - 12)$

Solve each equation using the ***square root property****.*

14. $x^2 = 49$

15. $x^2 = 32$

16. $a^2 - 50 = 0$

17. $n^2 - 24 = 0$

18. $3x^2 - 72 = 0$

19. $5y^2 - 200 = 0$

20. $(x - 4)^2 = 64$

21. $(x + 3)^2 = 16$

22. $(3n - 1)^2 = 7$

23. $(5t+2)^2 = 12$ **24.** $x^2 - 10x + 25 = 45$ **25.** $y^2 + 8y + 16 = 44$

26. $4a^2 + 12a + 9 = 32$ **27.** $25(y-10)^2 = 36$ **28.** $16(x+4)^2 = 81$

29. $(4x+3)^2 = -25$ **30.** $(3n-2)(3n+2) = -5$ **31.** $2x - 1 = \frac{18}{2x-1}$

Solve each equation using the ***completing the square*** *procedure.*

32. $x^2 + 12x = 0$ **33.** $y^2 - 3y = 0$ **34.** $x^2 - 8x + 2 = 0$

35. $n^2 + 7n = 3n - 4$ **36.** $p^2 - 4p = 4p - 16$ **37.** $y^2 + 7y - 1 = 0$

38. $2x^2 - 8x = -4$ **39.** $3a^2 + 6a = -9$ **40.** $3y^2 - 9y + 15 = 0$

41. $5x^2 - 60x + 80 = 0$ **42.** $2t^2 + 6t - 10 = 0$ **43.** $3x^2 + 2x - 2 = 0$

44. $2x^2 - 16x + 25 = 0$ **45.** $9x^2 - 24x = -13$ **46.** $25n^2 - 20n = 1$

47. $x^2 - \frac{4}{3}x = -\frac{1}{9}$ **48.** $x^2 + \frac{5}{2}x = -1$ **49.** $x^2 - \frac{2}{5}x - 3 = 0$

In problems ***50-51****, find all values of* x *such that* $f(x) = g(x)$ *for the given functions* f *and* g.

50. $f(x) = x^2 - 9$ and $g(x) = 4x - 6$ **51.** $f(x) = 2x^2 - 5x$ and $g(x) = -x + 14$

52. Explain the errors in the following solutions of the equation $5x^2 - 8x + 2 = 0$:

a. $x = \frac{8 \pm \sqrt{-8^2 - 4 \cdot 5 \cdot 2}}{2 \cdot 8} = \frac{8 \pm \sqrt{64-40}}{16} = \frac{8 \pm \sqrt{24}}{16} = \frac{8 \pm 2\sqrt{6}}{16} = \frac{1}{2} \pm 2\sqrt{6}$

b. $x = \frac{8 \pm \sqrt{(-8)^2 - 4 \cdot 5 \cdot 2}}{2 \cdot 8} = \frac{8 \pm \sqrt{64-40}}{16} = \frac{8 \pm \sqrt{24}}{16} = \frac{8 \pm 2\sqrt{6}}{16} = \begin{cases} \frac{10\sqrt{6}}{16} = \frac{5\sqrt{6}}{8} \\ \frac{6\sqrt{6}}{16} = \frac{3\sqrt{6}}{8} \end{cases}$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Solve each equation with the aid of the ***Quadratic Formula****, if possible. Illustrate your solutions graphically, using a table of values or a graphing utility.*

53. $x^2 + 3x + 2 = 0$ **54.** $y^2 - 2 = y$ **55.** $x^2 + x = -3$

56. $2y^2 + 3y = -2$ **57.** $x^2 - 8x + 16 = 0$ **58.** $4n^2 + 1 = 4n$

Solve each equation with the aid of the ***Quadratic Formula****. Give the* ***exact*** *and* ***approximate*** *solutions up to two decimal places.*

59. $a^2 - 4 = 2a$ **60.** $2 - 2x = 3x^2$ **61.** $0.2x^2 + x + 0.7 = 0$

62. $2t^2 - 4t + 2 = 3$ **63.** $y^2 + \frac{y}{3} = \frac{1}{6}$ **64.** $\frac{x^2}{4} - \frac{x}{2} = 1$

65. $5x^2 = 17x - 2$ **66.** $15y = 2y^2 + 16$ **67.** $6x^2 - 8x = 2x - 3$

Use the discriminant to determine the ***number and type of solutions*** *for each equation. Also, without solving, decide whether the equation can be solved by* ***factoring*** *or whether the quadratic formula should be used.*

68. $3x^2 - 5x - 2 = 0$

69. $4x^2 = 4x + 3$

70. $x^2 + 3 = -2\sqrt{3}x$

71. $4y^2 - 28y + 49 = 0$

72. $3y^2 - 10y + 15 = 0$

73. $9x^2 + 6x = -1$

In problems 74-76, find all values of constant k, so that each equation will have ***exactly one*** *rational solution.*

74. $x^2 + ky + 49 = 0$

75. $9y^2 - 30y + k = 0$

76. $kx^2 + 8x + 1 = 0$

77. Suppose that one solution of a quadratic equation with integral coefficients is irrational. Assuming that the equation has two solutions, can the other solution be a rational number? Justify your answer.

Solve each equation using any algebraic method. State the solutions in their exact form.

78. $-2x(x + 2) = -3$

79. $(x + 2)(x - 4) = 1$

80. $(x + 2)(x + 6) = 8$

81. $(2x - 3)^2 = 8(x + 1)$

82. $(3x + 1)^2 = 2(1 - 3x)$

83. $2x^2 - (x + 2)(x - 3) = 12$

84. $(x - 2)^2 + (x + 1)^2 = 0$

85. $1 + \frac{2}{x} + \frac{5}{x^2} = 0$

86. $x = \frac{2(x+3)}{x+5}$

87. $2 + \frac{1}{x} = \frac{3}{x^2}$

88. $\frac{3}{x} + \frac{x}{3} = \frac{5}{2}$

89. $\frac{1}{x} + \frac{1}{x+4} = \frac{1}{7}$

Q2 Applications of Quadratic Equations

Some polynomial, rational or even radical equations are **quadratic in form.** As such, they can be solved using techniques described in the previous section. For instance, the rational equation $\frac{1}{x^2}+\frac{1}{x}-6=0$ is quadratic in form because if we replace $\frac{1}{x}$ with a single variable, say a, then the equation becomes quadratic, $a^2+a-6=0$. In this section, we explore applications of quadratic equations in solving equations quadratic in form as well as solving formulas containing variables in the second power.

We also revisit application problems that involve solving quadratic equations. Some of the application problems that are typically solved with the use of quadratic or polynomial equations were discussed in *Sections F4* and *RT6*. However, in the previous sections, the equations used to solve such problems were all possible to solve by factoring. In this section, we include problems that require the use of methods other than factoring.

Equations Quadratic in Form

Definition 2.1 ► A nonquadratic equation is referred to as **quadratic in form** or **reducible to quadratic** if it can be written in the form

$$a\boldsymbol{u}^2+b\boldsymbol{u}+c=0,$$

where $a\neq 0$ and $\boldsymbol{u}$ represents any *algebraic expression*.

Equations quadratic in form are usually easier to solve by using strategies for solving the related quadratic equation $a\boldsymbol{u}^2+b\boldsymbol{u}+c=0$ for the expression $\boldsymbol{u}$, and then solve for the original variable, as shown in the example below.

Example 1 ► **Solving Equations Quadratic in Form**

Solve each equation.

a. $(x^2-1)^2-(x^2-1)=2$ **b.** $x-3\sqrt{x}=10$

c. $\frac{1}{(a+2)^2}+\frac{1}{a+2}-6=0$

Solution ► **a.** First, observe that the expression x^2-1 appears in the given equation in the first and second power. So, it may be useful to replace x^2-1 with a new variable, for example u. After this substitution, the equation becomes quadratic,

This can be any letter, as long as it is different than the original variable.

$$u^2-u=2, \qquad / -2$$

and can be solved via factoring

$$u^2-u-2=0$$

$$(u-2)(u+1)=0$$

$$u=2 \text{ or } u=-1$$

Since we need to solve the original equation for x, not for u, we replace u back with x^2-1. This gives us

$$x^2 - 1 = 2 \quad \text{or} \quad x^2 - 1 = -1$$

$$x^2 = 3 \quad \text{or} \quad x^2 = 0$$

$$x = \pm\sqrt{3} \quad \text{or} \quad x = 0$$

Thus, the solution set is $\{-\sqrt{\mathbf{3}}, \mathbf{0}, \sqrt{\mathbf{3}}\}$.

b. If we replace $\sqrt{x}$ with, for example, a, then $x = a^2$, and the equation becomes

$$a^2 - 3a = 10, \qquad / \ -10$$

which can be solved by factoring

$$a^2 - 3a - 10 = 0$$

$$(a + 2)(a - 5) = 0$$

$$a = -2 \quad \text{or} \quad a = 5$$

After replacing a back with $\sqrt{x}$, we have

$$\sqrt{x} = -2 \quad \text{or} \quad \sqrt{x} = 5.$$

The first equation, $\sqrt{x} = -2$, does not give us any solution as the square root cannot be negative. After squaring both sides of the second equation, we obtain $x = 25$. So, the solution set is $\{\mathbf{25}\}$.

c. The equation $\frac{1}{(a+2)^2} + \frac{1}{a+2} - 6 = 0$ can be solved as any other rational equation, by clearing the denominators via multiplying by the $LCD = (a + 2)^2$. However, it can also be seen as a quadratic equation as soon as we replace $\frac{1}{a+2}$ with, for example, x. By doing so, we obtain

$$x^2 + x - 6 = 0,$$

> **Remember to use a different letter than the variable in the original equation.**

which after factoring

$$(x + 3)(x - 2) = 0,$$

gives us

$$x = -3 \quad \text{or} \quad x = 2$$

Again, since we need to solve the original equation for a, we replace x back with $\frac{1}{a+2}$. This gives us

$$\frac{1}{a+2} = -3 \quad \text{or} \quad \frac{1}{a+2} = 2 \qquad / \ \textit{take the reciprocal of both sides}$$

$$a + 2 = \frac{1}{-3} \quad \text{or} \quad a + 2 = \frac{1}{2} \qquad / \ -2$$

$$a = -\frac{7}{3} \quad \text{or} \quad a = -\frac{3}{2}$$

Since both values are in the domain of the original equation, which is $\mathbb{R} \setminus \{0\}$, then the solution set is $\left\{-\frac{7}{3}, -\frac{3}{2}\right\}$.

Solving Formulas

When solving formulas for a variable that appears in the second power, we use the same strategies as in solving quadratic equations. For example, we may use the square root property or the quadratic formula.

Example 2 ► **Solving Formulas for a Variable that Appears in the Second Power**

Solve each formula for the given variable.

a. $E = mc^2$, for c **b.** $N = \frac{k^2-3k}{2}$, for k

Solution ► **a.** To solve for c, first, we reverse the multiplication by m via the division by m. Then, we reverse the operation of squaring by taking the square root of both sides of the equation.

$$E = mc^2 \qquad / \div m$$

$$\frac{E}{m} = c^2$$

Then, we reverse the operation of squaring by taking the square root of both sides of the equation. So, we have

$$\sqrt{\frac{E}{m}} = \sqrt{c^2},$$

and therefore

$$\boldsymbol{c = \pm\sqrt{\frac{E}{m}}}$$

Remember that $\sqrt{c^2} = |c|$, so we use the $\pm$ sign in place of $|\ |$.

b. Observe that the variable k appears in the formula $N = \frac{k^2-3k}{2}$ in two places. Once in the first and once in the second power of k. This means that we can treat this formula as a quadratic equation with respect to k and solve it with the aid of the quadratic formula. So, we have

$$N = \frac{k^2 - 3k}{2} \qquad / \cdot 2$$

$$2N = k^2 - 3k \qquad / -2N$$

$$k^2 - 3k - 2N = 0$$

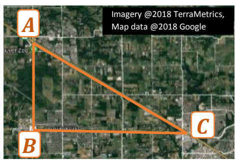
Substituting $a = 1$, $b = -3$, and $c = -2N$ to the quadratic formula, we obtain

$$k_{1,2} = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \cdot 1 \cdot (-2N)}}{2} = \frac{3 \pm \sqrt{9 + 8N}}{2}$$

Application Problems

Many application problems require solving quadratic equations. Sometimes this can be achieved via factoring, but often it is helpful to use the quadratic formula.

Example 3 ► **Solving a Distance Problem with the Aid of the Quadratic Formula**



Three towns A, B, and C are positioned as shown in the accompanying figure. The roads at A form a right angle. The towns A and C are connected by a straight road as well. The distance from A to B is 7 kilometers less than the distance from B to C. The distance from A to C is 20 km. Approximate the remaining distances between the towns up to the tenth of a kilometer.

Solution ► Since the roads between towns form a right triangle, we can employ the Pythagorean equation

$$AC^2 = AB^2 + BC^2$$

Suppose that $BC = x$. Then $AB = x - 7$, and we have

$$20^2 = (x-7)^2 + x^2$$

$$400 = x^2 - 14x + 49 + x^2$$

$$2x^2 - 14x - 351 = 0$$

Applying the quadratic formula, we obtain

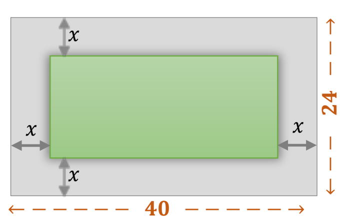
$$x_{1,2} = \frac{14 \pm \sqrt{14^2 + 4 \cdot 2 \cdot 351}}{4} = \frac{14 \pm \sqrt{196 + 2808}}{4} = \frac{14 \pm \sqrt{3004}}{4} \approx 17.2 \text{ or } \cancel{-10.2}$$

Since x represents a distance, it must be positive. So, the only solution is $x \approx 17.2$ km. Thus, the distance $\boldsymbol{BC \approx 17.2}$ **km** and hence $\boldsymbol{AB} \approx 17.2 - 7 = \mathbf{10.2}$ **km**.

Example 4 ► **Solving a Geometry Problem with the Aid of the Quadratic Formula**

A city designated 24 m by 40 m rectangular area for a playground with a sidewalk of uniform width around it. The playground itself is using $\frac{2}{3}$ of the original rectangular area. To the nearest centimeter, what is the width of the sidewalk?

Solution ► To visualize the situation, we may draw a diagram as below.



Suppose x represents the width of the sidewalk. Then, the area of the playground (the green area) can be expressed as $(40-2x)(24-2x)$. Since the green area is $\frac{2}{3}$ of the original rectangular area, we can form the equation

$$(40-2x)(24-2x)=\frac{2}{3}(40\cdot 24)$$

To solve it, we may want to lower the coefficients by dividing both sides of the equation by 4 first. This gives us

$$\frac{2(20-x)\cdot 2(12-x)}{4}=\frac{2}{3}\cdot\frac{(40\cdot 24)}{4}$$

$$(20-x)(12-x)=160$$

$$240-32x+x^2=160 \qquad /-160$$

$$x^2-32x+80=0,$$

which can be solved using the Quadratic Formula:

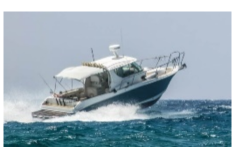
$$x_{1,2}=\frac{32\pm\sqrt{(-32)^2-4\cdot 80}}{2}=\frac{32\pm\sqrt{704}}{2}\approx\frac{32\pm 8\sqrt{11}}{2}$$

$$=16\pm 4\sqrt{11}\approx\begin{cases}29.27\\2.73\end{cases}$$

The width x must be smaller than 12, so this value is too large to be considered.

Thus, the sidewalk is approximately **2.73** meters wide.

Example 5 ► Solving a Motion Problem That Requires the Use of the Quadratic Formula



The Columbia River flows at a rate of 2 mph for the length of a popular boating route. In order for a boat to travel 3 miles upriver and return in a total of 4 hours, how fast must the boat be able to travel in still water?

Solution ►

Suppose the rate of the boat moving in still water is r. Then, $r-2$ represents the rate of the boat moving upriver and $r+2$ represents the rate of the boat moving downriver. We can arrange these data in the table below.

	R ·	T	= D
upriver	$r-2$	$\frac{3}{r-2}$	3
downriver	$r+2$	$\frac{3}{r+2}$	3
total		4	

We fill the time-column by following the formula $T=\frac{D}{R}$.

By adding the times needed for traveling upriver and downriver, we form the rational equation

$$\frac{3}{r-2}+\frac{3}{r+2}=4, \qquad /\cdot(r^2-4)$$

which after multiplying by the $LCD=r^2-4$ becomes a quadratic equation.

$$3(r+2)+3(r-2)=4(r^2-4)$$

$$3r+6+3r-6=4r^2-16 \qquad / \cdot -6r$$

$$0=4r^2-6r-16 \qquad / \div 2$$

$$0=2r^2-3r-8$$

Using the Quadratic Formula, we have

$$r_{1,2}=\frac{3\pm\sqrt{(-3)^2+4\cdot 2\cdot 8}}{2\cdot 2}=\frac{3\pm\sqrt{9+64}}{4}=\frac{3\pm\sqrt{73}}{4}\approx\begin{cases}2.9\\ \cancel{-1.4}\end{cases}$$

Since the rate cannot be negative, the boat moves in still water at approximately **2.9 mph**.

Example 6 ► **Solving a Work Problem That Requires the Use of the Quadratic Formula**

Krista and Joanna work in the same office. Krista can file the daily office documents in 1 hour less time than Joanna can. Working together, they can do the job in 1 hr 45 min. To the nearest minute, how long would it take each person working alone to file these documents?

Solution ► Suppose the time needed for Joanna to complete the job is t, in hours. Then, $t-1$ represents the time needed for Krista to complete the same job. Since we keep time in hours, we need to convert 1 hr 45 min into $1\frac{3}{4}$ hr = $\frac{7}{4}$ hr. Now, we can arrange the given data in a table, as below.

	R ·	T	= Job
Joanna	$\frac{1}{t}$	t	1
Krista	$\frac{1}{t-1}$	$t-1$	1
together	$\frac{4}{7}$	$\frac{7}{4}$	1

We fill the rate-column by following the formula $R=\frac{Job}{T}$.

By adding the rates of work for each person, we form the rational equation

$$\frac{1}{t}+\frac{1}{t-1}=\frac{4}{7}, \qquad / \cdot 7t(t-1)$$

which after multiplying by the $LCD=7t(t-1)$ becomes a quadratic equation.

$$7(t-1)+7t=4(t^2-t)$$

$$7t-7+7t=4t^2-4t \qquad / -6t,+3$$

$$0=4t^2-18t+7$$

Using the Quadratic Formula, we have

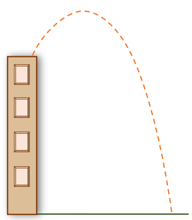
$$t_{1,2}=\frac{18\pm\sqrt{(-18)^2-4\cdot 4\cdot 7}}{2\cdot 4}=\frac{18\pm\sqrt{212}}{8}=\frac{18\pm 2\sqrt{53}}{8}=\frac{9\pm\sqrt{53}}{4}\approx\begin{cases}4.07\\ \cancel{0.43}\end{cases}$$

Since the time needed for Joanna cannot be shorter than 1 hr, we reject the 0.43 possibility. So, working alone, **Joanna** requires approximately 4.07 hours ≈ **4 hours 4 minutes**, while **Krista** can do the same job in approximately 3.07 hours ≈ **3 hours 4 minutes**.

Example 7 ► **Solving a Projectile Problem Using a Quadratic Function**

A ball is projected upward from the top of a 96-ft building at 32 ft/sec. Its height above the ground, h, in feet, can be modelled by the function $h(t) = -16t^2 + 32t + 96$, where t is the time in seconds after the ball was projected. To the nearest tenth of a second, when does the ball hit the ground?

Solution ►



The ball hits the ground when its height h above the ground is equal to zero. So, we look for the solutions to the equation

$$h(t) = 0$$

which is equivalent to

$$-16t^2 + 32t + 96 = 0$$

Before applying the Quadratic Formula, we may want to lower the coefficients by dividing both sides of the equation by -16. So, we have

$$t^2 - 2t - 6 = 0$$

and

$$t_{1,2} = \frac{2 \pm \sqrt{(-2)^2 + 4 \cdot 6}}{2} = \frac{2 \pm \sqrt{28}}{2} = \frac{2 \pm 2\sqrt{7}}{2} = 1 \pm \sqrt{7} \approx \begin{cases} 3.6 \\ \cancel{-1.6} \end{cases}$$

Thus, the ball hits the ground in about **3.6 seconds**.

Q.2 Exercises

1. Discuss the validity of the following solution to the equation $\left(\frac{1}{x-2}\right)^2 - \frac{1}{x-2} - 2 = 0$:

Since this equation is quadratic in form, we solve the related equation $a^2 - a - 2 = 0$ by factoring

$$(a-2)(a+1) = 0.$$

The possible solutions are $a = 2$ and $a = -1$. Since 2 is not in the domain of the original equation, the solution set as $\{-1\}$.

Solve each equation by treating it as a quadratic in form.

2. $x^4 - 6x^2 + 9 = 0$

3. $x^8 - 29x^4 + 100 = 0$

4. $x - 10\sqrt{x} + 9 = 0$

5. $2x - 9\sqrt{x} + 4 = 0$

6. $y^{-2} - 5y^{-1} - 36 = 0$

7. $2a^{-2} + a^{-1} - 1 = 0$

8. $\left(1 + \sqrt{t}\right)^2 + \left(1 + \sqrt{t}\right) - 6 = 0$

9. $\left(2 + \sqrt{x}\right)^2 - 3\left(2 + \sqrt{x}\right) - 10 = 0$

10. $(x^2+5x)^2+2(x^2+5x)-24=0$

11. $(t^2-5t)^2-4(t^2-5t)+3=0$

12. $x^{\frac{2}{3}}-4x^{\frac{1}{3}}-5=0$

13. $x^{\frac{2}{3}}+2x^{\frac{1}{3}}-8=0$

14. $1-\frac{1}{2p+1}-\frac{1}{(2p+1)^2}=0$

15. $\frac{2}{(u+2)^2}+\frac{1}{u+2}=3$

16. $\left(\frac{x+3}{x-3}\right)^2-\left(\frac{x+3}{x-3}\right)=6$

17. $\left(\frac{y^2-1}{y}\right)^2-4\left(\frac{y^2-1}{y}\right)-12=0$

In problems 23-40, solve each formula for the indicated variable.

18. $F=\frac{mv^2}{r}$, for v

19. $V=\pi r^2 h$, for r

20. $A=4\pi r^2$, for r

21. $V=\frac{1}{3}s^2 h$, for s

22. $F=\frac{Gm_1m_2}{r^2}$, for r

23. $N=\frac{kq_1q_2}{s^2}$, for s

24. $a^2+b^2=c^2$, for b

25. $I=\frac{703W}{H^2}$, for H

26. $A=\pi r^2+\pi rs$, for r

27. $A=2\pi r^2+2\pi rh$, for r

28. $s=v_0t+\frac{gt^2}{2}$, for t

29. $t=\frac{a}{\sqrt{a^2+b^2}}$, for a

30. $P=\frac{A}{(1+r)^2}$, for r

31. $P=EI-RI^2$, for I

32. $s(6s-t)=t^2$, for s

33. $m=\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$, for v, assuming that $c>0$ and $m>0$

34. $m=\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$, for c, assuming that $v>0$ and $m>0$

35. $p=\frac{E^2R}{(r+R)^2}$, for E, assuming that $(r+R)>0$

36. The "**golden**" proportions have been considered visually pleasing for the past 2900 years. A rectangle with the width w and length l has "golden" proportions if

$$\frac{\boldsymbol{w}}{\boldsymbol{l}}=\frac{\boldsymbol{l}}{\boldsymbol{w}+\boldsymbol{l}}.$$

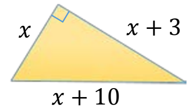
Solve this formula for $\boldsymbol{l}$. Then, find the value of the **golden ratio** $\frac{\boldsymbol{l}}{\boldsymbol{w}}$ up to three decimal places.

Answer each question.

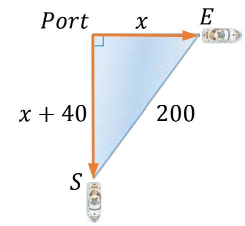
37. A boat moves r km/h in still water. If the rate of the current is c km/h,
- **a.** give an expression for the rate of the boat moving upstream;
- **b.** give an expression for the rate of the boat moving downstream.

38.
- **a.** Vivian marks a class test in n hours. Give an expression representing Vivian's rate of marking, in the number of marked tests per hour.
- **b.** How many tests will she have marked in h hours?

Solve each problem.



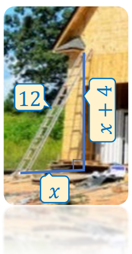
39. Find the exact length of each side of the triangle.



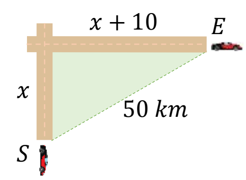
40. Two cruise ships leave a port at the same time, but they move at different rates. The faster ship is heading south, and the slower one is heading east. After a few hours, they are 200 km apart. If the faster ship went 40 km farther than the slower one, how far did each ship travelled?

41. The length of a rectangular area carpet is 2 ft more than twice the width. Diagonally, the carpet measures 13 ft. Find the dimensions of the carpet.

42. The legs of a right triangle with 26 cm long hypotenuse differ by 14 cm. Find the lengths of the legs.



43. A 12-ft ladder is tilting against a house. The top of the ladder is 4 ft further from the ground than the bottom of the ladder is from the house. To the nearest inch, how high does the ladder reach?



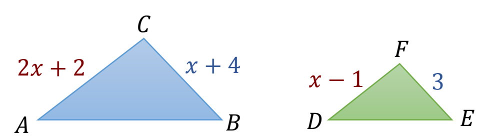
44. Two cars leave an intersection, one heading south and the other heading east. In one hour the cars are 50 kilometers apart. If the faster car went 10 kilometers farther than the slower one, how far did each car travelled?

45. The length and width of a computer screen differ by 4 inches. Find the dimensions of the screen, knowing that its area is 117 square inches.

46. The length of an American flag is 1 inch shorter than twice the width. If the area of this flag is 190 square inches, find the dimensions of the flag.

47. The length of a Canadian flag is twice the width. If the area of this flag is 100 square meters, find the exact dimensions of the flag.

48. ***Thales Theorem*** states that corresponding sides of similar triangles are proportional. The accompanying diagram shows two similar triangles, ΔABC and ΔDEF. Given the information in the diagram, find the length AC.

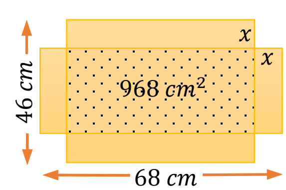


Solve each problem.

49. Sonia bought an area carpet for her 12 ft by 18 ft room. The carpet covers 135 ft^2, and when centered in the room, it leaves a strip of the bare floor of uniform width around the edges of the room. How wide is this strip?

50. Park management plans to create a rectangular 14 m by 20 m flower garden with a sidewalk of uniform width around the perimeter of the garden. There are enough funds to install 152 m^2 of a brick sidewalk. Find the width of the sidewalk.

51. Squares of equal area are cut from each corner of a 46 cm by 68 cm rectangular cardboard. Obtained this way flaps are folded up to create an open box with the area of the base equal to 968 cm^2. What is the height of the box?



52. The outside measurements of a picture frame are 22 cm and 28 cm. If the area of the exposed picture is 301 cm^2, find the width of the frame.

53. The length of a rectangle is one centimeter shorter than twice the width. The rectangle shares its longer side with a square of 169 cm^2 area. What are the dimensions of the rectangle?

54. A rectangular piece of cardboard is 15 centimeters longer than it is wide. 100 cm^2 squares are removed from each corner of the cardboard. Folding up the established flaps creates an open box of 13.5-litre volume. Find the dimensions of the original piece of cardboard. (*Hint:* 1 litre = 1000 cm^2)

55. Karin travelled 420 km by her motorcycle to visit a friend. When planning the return trip by the same road, she calculated that her driving time could be 1 hour shorter if she increases her average speed by 10 km/h. On average, how fast was she driving to her friend?

56. An average, an Airbus A380 flies 80 km/h faster than a Boeing 787 Dreamliner. Suppose an Airbus A380 flew 2600 km in half an hour shorter time than it took a Boeing 787 Dreamliner to fly 2880 mi. Determine the speed of each plane.

57. Two small planes, a Skyhawk and a Mooney Bravo, took off from the same place and at the same time. The Skyhawk flew 500 km. The Mooney Bravo flew 1050 km in one hour longer time and at a 100 km/h faster speed. If the planes fly faster than 200 km/h, find the average rate of each plane.

58. Gina drives 550 km to a conference. Due to heavier traffic, she returns at 10 km/h slower rate. If the round trip took her 10.5 hours, what was Gina's average rate of driving to the workshop?

59. A barge travels 25 km upriver and then returns in a total of 5 hours. If the current in the river is 3 km/hr, approximately how fast would this barge move in still water?

60. A canoeist travels 3 kilometers down a river with a 3 km/h current. For the return trip upriver, the canoeist chose to use a longer branch of the river with a 2 km/hr current. If the return trip is 4 km long and the time needed for travelling both ways is 3 hours, approximate the speed of the canoe in still water.

61. Two planes take off from the same airport and at the same time. The first plane flies with an average speed r km/h and is heading North. The second plane flies faster by 40 km/h and is heading East. In thirty minutes the planes are 580 kilometers apart from each other. Determine the average speed of each plane.

62. Jack flew 650 km to visit his relatives in Alaska. On the way to Alaska, his plane encounter a 40 km/h headwind. On the returning trip, the plane flew with a 20 km/h tailwind. If the total flying time (both ways) was 5 hours 45 minutes, what was the average speed of the plane in still air?

63. Two janitors, an experienced and a newly hired one, need 4 hours to clean a school building. The newly hired worker would need 1.5 hour longer time than the experienced one to clean the school on its own. To the nearest minute, how much time is required for the experienced janitor to clean the school working alone?

64. Two workers can weed out a vegetable garden in 2 hr. On its own, one worker can do the same job in half an hour shorter time than the other. To the nearest minute, how long would it take the faster worker to weed out the garden by himself?

65. Helen and Monica are planting flowers in their garden. On her own, Helen would need an hour longer than Monica to plant all the flowers. Together, they can finish the job in 8 hr. To the nearest minute, how long would it take each person to plant all the flowers if working alone?

66. To prepare the required number of pizza crusts for a day, the owner of Ricardo's Pizza needs 40 minutes shorter time than his worker Sergio. Together, they can make these pizza crusts in 2 hours. To the nearest minute, how long would it take each of them to do this job alone?

67. A fish tank can be filled with water with the use of one of two pipes of different diameters. If only the larger-diameter pipe is used, the tank can be filled in an hour shorter time than if only the smaller-diameter pipe is used. If both pipes are open, the tank can be filled in 1 hr 12 min. How much time is needed for each pipe to fill the tank if working alone?

68. Two roofers, Garry and Larry, can install new asphalt roof shingles in 6 hours 40 min. On his own, Garry can do this job in 3 hours shorter time than Larry can. How much time each or the roofers need to install these shingles alone?

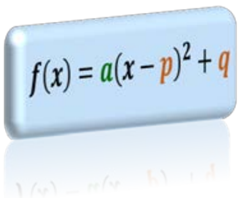
69. A ball is thrown down with the initial velocity of 6 m/sec from a balcony that is 100 m above the ground. Suppose that function $h(t) = -4.9t^2 - 6t + 100$ can be used to determine the height $h(t)$ of the ball t seconds after it was thrown down. Approximately in how many seconds the ball will be 5 meters above the ground?

70. A bakery's weekly profit, P (in dollars), for selling n poppyseed strudels can be modelled by the function $P(n) = -0.05x^2 + 7x - 200$. What is the minimum number of poppyseed strudels that must be sold to make a profit of $200?

71. If P dollars is invested in an account that pays the annual interest rate r (in decimal form), then the amount A of money in the account after 2 years can be determined by the formula $A = P(1+r)^2$. Suppose $3000 invested in this account for 2 years grew to $3257.29. What was the interest rate?

72. To determine the distance, d, of an object to the horizon we can use the equation $d = \sqrt{12800h + h^2}$, where h represents the distance of an object to the Earth's surface, and both, d and h, are in kilometers. To the nearest meter, how far above the Earth's surface is a plane if its distance to the horizon is 400 kilometers?

Q3 Properties and Graphs of Quadratic Functions



In this section, we explore an alternative way of graphing quadratic functions. It turns out that if a quadratic function is given in vertex form, $\boldsymbol{f(x) = a(x-p)^2 + q}$, its graph can be obtained by transforming the shape of the basic **parabola**, $f(x) = x^2$, by applying a **vertical dilation** by the factor of a, as well as a **horizontal translation** by p units and **vertical translation** by q units. This approach makes the graphing process easier than when using a table of values.

In addition, the vertex form allows us to identify the main characteristics of the corresponding graph such as **shape**, **opening**, **vertex**, and **axis of symmetry**. Then, the additional properties of a quadratic function, such as **domain** and **range**, or where the function increases or decreases can be determined by observing the obtained graph.

Properties and Graph of the Basic Parabola $f(x) = x^2$

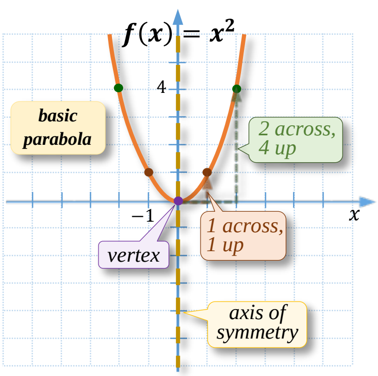


Figure 3.1

Recall the shape of the **basic parabola**, $f(x) = x^2$, as discussed in *Section P4*.

x	x^2
-2	4
-1	1
0	0
1	1
2	4

vertex

symmetry about the y-axis

Observe the relations between the points listed in the table above. If we start with plotting the **vertex** $(\mathbf{0}, \mathbf{0})$, then the next pair of points, $(\mathbf{1}, \mathbf{1})$ and $(-\mathbf{1}, \mathbf{1})$, is plotted **1 unit across** from the vertex (both ways) and **1 unit up**. The following pair, $(\mathbf{2}, \mathbf{4})$ and $(-\mathbf{2}, \mathbf{4})$, is plotted **2 units across** from the vertex and **4 units up**. The graph of the parabola is obtained by connecting these 5 main points by a curve, as illustrated in *Figure 3.1*.

The graph of this parabola is symmetric in the y-axis, so the equation of the **axis of symmetry** is $\boldsymbol{x = 0}$.

The **domain** of the basic parabola is the set of all real numbers, $\mathbb{R}$, as $f(x) = x^2$ is a polynomial, and polynomials can be evaluated for any real x-value.

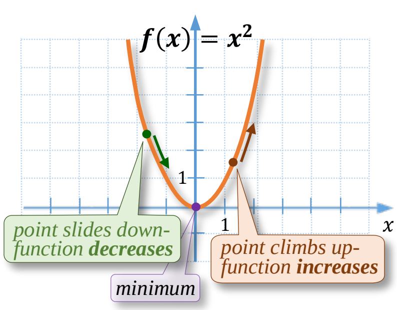


Figure 3.2

The **arms** of the parabola are directed **upwards**, which means that the vertex is the lowest point of the graph. Hence, the **range** of the basic parabola function, $f(x) = x^2$, is the interval $[\mathbf{0}, \infty)$, and the **minimum value** of the function is **0**.

Suppose a point 'lives' on the graph and travels from left to right. Observe that in the case of the basic parabola, if x-coordinates of the 'travelling' point are smaller than 0, the point slides down along the graph. Similarly, if x-coordinates are larger than 0, the point climbs up the graph. (See *Figure 3.2*) To describe this property in mathematical language, we say that the function $f(x) = x^2$ **decreases** in the interval $(-\infty, 0]$ and **increases** in the interval $[0, \infty)$.

Properties and Graphs of a Dilated Parabola $f(x) = ax^2$

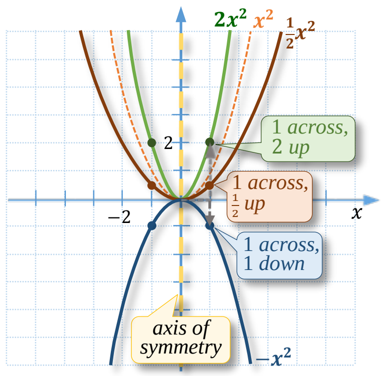


Figure 3.3

Figure 3.3 shows graphs of several functions of the form $f(x) = ax^2$. Observe how the shapes of these parabolas change for various values of a in comparison to the shape of the basic parabola $y = x^2$.

The common point for all of these parabolas is the vertex (0,0). Additional points, essential for graphing such parabolas, are shown in the table below.

x	ax^2
-2	$4a$
-1	a
0	0
1	a
2	$4a$

0 → vertex

1 unit apart from zero, a units up

2 units apart from zero, $4a$ units up

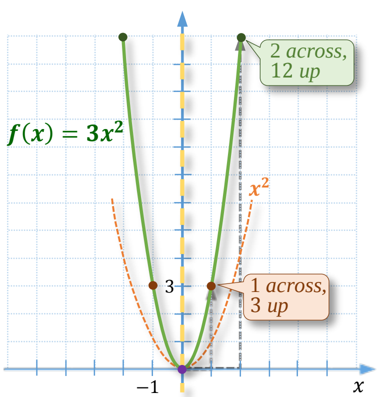


Figure 3.4

For example, to graph $f(x) = 3x^2$, it is convenient to plot the **vertex** first, which is at the point $(\mathbf{0}, \mathbf{0})$. Then, we may move the pen **1** unit **across** from the vertex (either way) and **3** units **up** to plot the points $(-1,3)$ and $(1,3)$. If the grid allows, we might want to plot the next two points, $(-2,12)$ and $(2,12)$, by moving the pen **2** units **across** from the vertex and $4 \cdot 3 = \mathbf{12}$ units **up**, as in *Figure 3.4*.

Notice that the obtained shape (in green) is **narrower** than the shape of the basic parabola (in orange). However, similarly as in the case of the basic parabola, the shape of the dilated function is still **symmetrical about the y-axis, $x = 0$**.

Now, suppose we want to graph the function $f(x) = -\frac{1}{2}x^2$. As before, we may start by plotting the vertex at (0,0). Then, we move the pen **1** unit **across** from the vertex (either way) and **half** a unit **down** to plot the points $\left(-1, -\frac{1}{2}\right)$ and $\left(1, -\frac{1}{2}\right)$, as in *Figure 3.5*. The next pair of points can be plotted by moving the pen **2** units **across** from the vertex and **2** units **down**, as the ordered pairs $(-2,-2)$ and $(2,-2)$ satisfy the equation $f(x) = -\frac{1}{2}x^2$.

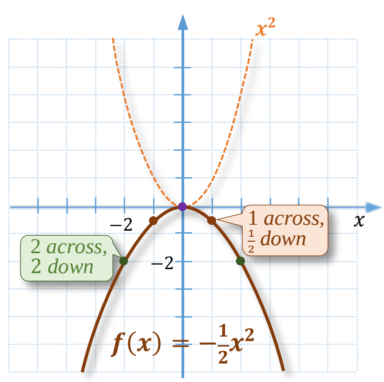


Figure 3.5

Notice that this time the obtained shape (in green) is **wider** than the shape of the basic parabola (in orange). Also, as a result of the **negative a-value**, the parabola opens **down**, and the **range** of this function is $(-\infty, \mathbf{0}]$.

Generally, the **shape** of a quadratic function of the form $f(x) = ax^2$ is
- **narrower** than the shape of the basic parabola, if $|a| > 1$;
- **wider** than the shape of the basic parabola, if $0 < |a| < 1$; and
- **the same** as the shape of the **basic parabola**, $y = x^2$, if $|a| = 1$.

The parabola opens **up**, for $a > 0$, and **down**, for $a < 0$.
Thus the **vertex** becomes the **lowest point** of the graph, if $a > 0$, and the **highest point** of the graph, if $a < 0$.

The **range** of $f(x) = ax^2$ is $[0, \infty)$, if $a > 0$, and $(-\infty, 0]$, if $a < 0$.

The **axis of symmetry** of the dilated parabola $f(x) = ax^2$ remains the same as that of the basic parabola, which is $x = 0$.

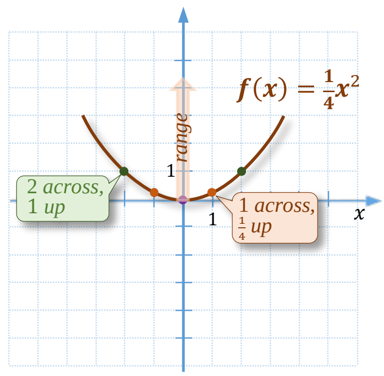
Example 1 ► **Graphing a Dilated Parabola and Describing Its Shape, Opening, and Range**

For each quadratic function, describe its shape and opening. Then graph it and determine its range.

a. $f(x) = \frac{1}{4}x^2$ **b.** $g(x) = -2x^2$

Solution ►

a. Since the leading coefficient of the function $f(x) = \frac{1}{4}x^2$ is positive, the parabola **opens up**. Also, since $0 < \frac{1}{4} < 1$, we expect the shape of the parabola to be **wider** than that of the basic parabola.



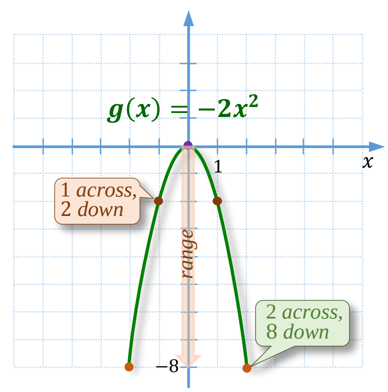
To graph $f(x) = \frac{1}{4}x^2$, first we plot the vertex at $(0,0)$ and then points $\left(\pm 1, \frac{1}{4}\right)$ and $\left(\pm 2, \frac{1}{4} \cdot 4\right) = (\pm 2, 1)$. After connecting these points with a curve, we obtain the graph of the parabola.

By projecting the graph onto the y-axis, we observe that the range of the function is $[0, \infty)$.

b. Since the leading coefficient of the function $g(x) = -2x^2$ is negative, the parabola **opens down**. Also, since $|-2| > 1$, we expect the shape of the parabola to be **narrower** than that of the basic parabola.

To graph $g(x) = -2x^2$, first we plot the vertex at $(0,0)$ and then points $(\pm 1, -2)$ and $(\pm 2, -2 \cdot 4) = (\pm 2, -8)$. After connecting these points with a curve, we obtain the graph of the parabola.

By projecting the graph onto the y-axis, we observe that the range of the function is $(-\infty, 0]$.



Properties and Graphs of the Basic Parabola with Shifts

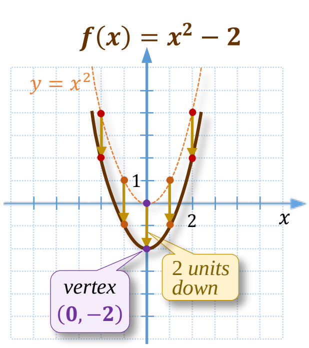


Figure 3.6

Suppose we would like to graph the function $f(x) = x^2 - 2$. We could do this via a table of values, but there is an easier way if we already know the shape of the basic parabola $y = x^2$.

Observe that for every x-value, the value of $x^2 - 2$ is obtained by subtracting 2 from the value of x^2. So, to graph $f(x) = x^2 - 2$, it is enough to **move each point** (x, x^2) of the basic parabola by **two units down**, as indicated in *Figure 3.6*.

The shift of y-values by 2 units down causes the **range** of the new function, $f(x) = x^2 - 2$, to become $[-2, \infty)$. Observe that this vertical shift also changes the minimum value of this function, from 0 to -2.

The **axis of symmetry** remains unchanged, and it is $x = 0$.

Generally, the graph of a quadratic function of the form $\boldsymbol{f(x) = x^2 + q}$ can be obtained by

- **shifting** the graph of the basic parabola $\boldsymbol{q}$ **steps up**, if $\boldsymbol{q > 0}$;
- **shifting** the graph of the basic parabola $\boldsymbol{|q|}$ **steps down**, if $\boldsymbol{q < 0}$.

The **vertex** of such parabola is at $(\boldsymbol{0, q})$. The **range** of it is $[\boldsymbol{q}, \infty)$.
The **minimum** (lowest) **value** of the function is $\boldsymbol{q}$.
The **axis of symmetry** is $\boldsymbol{x = 0}$.

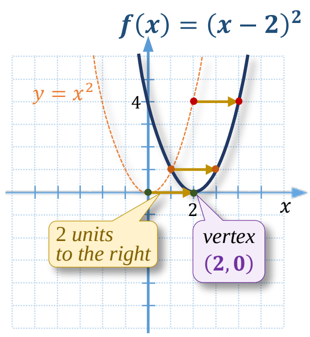


Figure 3.7

Now, suppose we wish to graph the function $f(x) = (x-2)^2$. We can graph it by joining the points calculated in the table below.

x	$(x-2)^2$
0	4
1	1
2	0
3	1
4	4

Observe that the parabola $f(x) = (x-2)^2$ assumes its lowest value at the vertex. The lowest value of the perfect square $(x-2)^2$ is zero, and it is attained at the x-value of 2. Thus, the vertex of this parabola is $(2,0)$.

Notice that the **vertex** $(\boldsymbol{2, 0})$ of $f(x) = (x-2)^2$ is positioned 2 units to the right from the vertex $(0,0)$ of the basic parabola.

This suggests that the graph of the function $f(x) = (x-2)^2$ can be obtained without the aid of a table of values. It is enough to shift the graph of the basic parabola **2 units** to the **right**, as shown in *Figure 3.7*.

Observe that the horizontal shift does not influence the **range** of the new parabola $f(x) = (x-2)^2$. It is still $[\boldsymbol{0}, \infty)$, the same as for the basic parabola. However, the **axis of symmetry** has changed to $\boldsymbol{x = 2}$.

Generally, the graph of a quadratic function of the form $\boldsymbol{f(x) = (x-p)^2}$ can be obtained by

- **shifting** the graph of the basic parabola $\boldsymbol{p}$ **steps** to the **right**, if $\boldsymbol{p > 0}$;
- **shifting** the graph of the basic parabola $\boldsymbol{|p|}$ **steps** to the **left**, if $\boldsymbol{p < 0}$.

The **vertex** of such a parabola is at $(\boldsymbol{p, 0})$. The **range** of it is $[\boldsymbol{0}, \infty)$.

The **minimum value** of the function is $\boldsymbol{0}$.

The **axis of symmetry** is $\boldsymbol{x = p}$.

Example 2 ► **Graphing Parabolas and Observing Transformations of the Basic Parabola**

Graph each parabola by plotting its vertex and following the appropriate opening and shape. Then describe transformations of the basic parabola that would lead to the obtained graph. Finally, state the range and the equation of the axis of symmetry.

a. $f(x) = (x+3)^2$ **b.** $g(x) = -x^2 + 1$

Solution

a. The perfect square $(x+3)^2$ attains its lowest value at $x = -3$. So, the **vertex** of the parabola $f(x) = (x+3)^2$ is $(\mathbf{-3}, \mathbf{0})$. Since the leading coefficient is 1, the parabola takes the shape of $y = x^2$, and its **arms open up**.

The graph of the function f can be obtained by **shifting** the graph of the basic parabola **3 units to the left**, as shown in *Figure 3.8.*

The **range** of function f is $[\mathbf{0}, \infty)$, and the equation of the **axis of symmetry** is $\boldsymbol{x = -3}$.

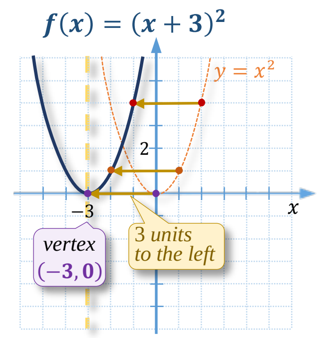


Figure 3.8

b. The expression $-x^2 + 1$ attains its highest value at $x = 0$. So, the **vertex** of the parabola $g(x) = -x^2 + 1$ is $(\mathbf{0}, \mathbf{1})$. Since the leading coefficient is -1, the parabola takes the shape of $y = x^2$, but its **arms open down**.

The graph of the function g can be obtained by:

- first, **flipping the graph** of the basic parabola **over the $\boldsymbol{x}$-axis**, and then
- **shifting** the graph of $y = -x^2$ **1 unit up**, as shown in *Figure 3.9.*

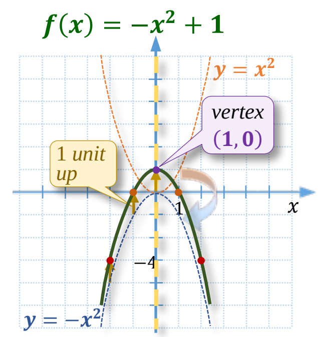


Figure 3.9

The **range** of the function g is $(-\infty, \mathbf{1}]$, and the equation of the **axis of symmetry** is $\boldsymbol{x = 0}$.

Note: The order of transformations in the above example is essential. The reader is encouraged to check that **shifting** the graph of $y = x^2$ by 1 unit up first and then **flipping** it over the x-axis results in a different graph than the one in *Figure 3.9.*

Properties and Graphs of Quadratic Functions Given in the Vertex Form $f(x) = a(x-p)^2 + q$

So far, we have discussed properties and graphs of quadratic functions that can be obtained from the graph of the basic parabola by applying mainly a single transformation. These transformations were: dilations (including flips over the x-axis), and horizontal and vertical shifts. Sometimes, however, we need to apply more than one transformation. We have already encountered such a situation in *Example 2b*, where a flip and a horizontal shift was applied. Now, we will look at properties and graphs of any function of the form $\boldsymbol{f(x) = a(x-p)^2 + q}$, referred to as the **vertex form** of a quadratic function.

Suppose we wish to graph $f(x) = 2(x+1)^2 - 3$. This can be accomplished by connecting the points calculated in a table of values, such as the one below, or by observing the coordinates of the vertex and following the shape of the graph of $y = 2x^2$. Notice that the vertex of our parabola is at $(-1, -3)$. This information can be taken directly from the equation $f(x) = 2(x+1)^2 - 3 = 2(x-(-1))^2 - 3$, without the aid of a table of values.

opposite to the number in the bracket — *the same last number*

x	$2(x+1)^2 - 3$
-3	5
-2	-1
-1	-3
0	-1
1	5

1 unit apart from zero, 2 units up — *vertex*

The rest of the points follow the pattern of the shape for the $y = 2x^2$ parabola: 1 across, 2 up; 2 across, $4 \cdot 2 = 8$ up. So, we connect the points as in *Figure 3.10*.

Notice that the graph of function f could also be obtained as a result of translating the graph of $y = 2x^2$ by 1 unit left and 3 units down, as indicated in *Figure 3.10* by the blue vectors.

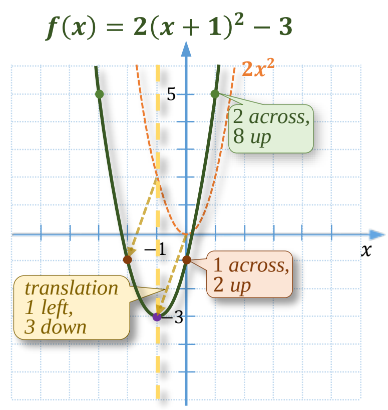


Figure 3.10

Here are the main properties of the graph of function f:
- It has a **shape** of $y = \mathbf{2x^2}$;
- It is a parabola that **opens up**;
- It has a **vertex** at $(\mathbf{-1}, \mathbf{-3})$;
- It is **symmetrical** about the line $\boldsymbol{x = -1}$;
- Its **minimum value** is $\mathbf{-3}$, and this minimum is attained at $x = -1$;
- Its domain is the set of all real numbers, and its **range** is the interval $[\mathbf{-3}, \infty)$;
- It **decreases** for $x \in (-\infty, \mathbf{-1}]$ and **increases** for $x \in [\mathbf{-1}, \infty)$.

The above discussion of properties and graphs of a quadratic function given in vertex form leads us to the following general observations:

Characteristics of Quadratic Functions Given in Vertex Form $f(x) = a(x-p)^2 + q$

1. The graph of a quadratic function given in **vertex form**

$$f(x) = a(x-p)^2 + q, \text{ where } a \neq 0,$$

is a **parabola** with **vertex** (p, q) and **axis of symmetry** $x = p$.

2. The graph **opens up** if a is **positive** and **down** if a is **negative**.

3. If $a > 0$, q is the **minimum value**. If $a < 0$, q is the **maximum value**.

3. The graph is **narrower** than that of $y = x^2$ if $|a| > 1$.
The graph is **wider** than that of $y = x^2$ if $0 < |a| < 1$.

4. The **domain** of function f is the set of real numbers, $\mathbb{R}$.
The **range** of function f is $[q, \infty)$ if a is **positive** and $(-\infty, q]$ if a is **negative**.

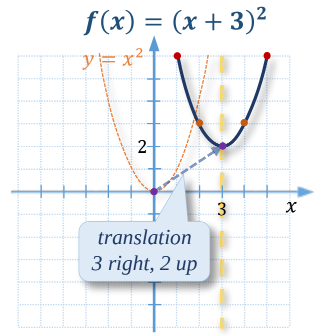
Example 3 ► **Identifying Properties and Graphing Quadratic Functions Given in Vertex Form $f(x) = a(x-p)^2 + q$**

For each function, identify its **vertex**, **opening**, **axis of symmetry**, and **shape**. Then graph the function and state its **domain** and **range**. Finally, describe **transformations** of the basic parabola that would lead to the obtained graph.

a. $f(x) = (x-3)^2 + 2$ **b.** $g(x) = -\frac{1}{2}(x+1)^2 + 3$

Solution ►

a. The vertex of the parabola $f(x) = (x-3)^2 + 2$ is $(\mathbf{3}, \mathbf{2})$; the graph **opens up**, and the equation of the axis of symmetry is $\boldsymbol{x = 3}$. To graph this function, we can plot the vertex first and then follow the shape of the basic parabola $y = \boldsymbol{x^2}$.



The domain of function f is $\mathbb{R}$, and the range is $[\mathbf{2}, \infty)$.

The graph of f can be obtained by shifting the graph of the basic parabola **3 units to the right** and **2 units up**.

b. The vertex of the parabola $g(x) = -\frac{1}{2}(x+1)^2 + 3$ is $(-\mathbf{1}, \mathbf{3})$; the graph **opens down**, and the equation of the axis of symmetry is $\boldsymbol{x = -1}$. To graph this function, we can plot the vertex first and then follow the shape of the parabola $y = -\frac{1}{2}\boldsymbol{x^2}$. This means that starting from the vertex, we move the pen one unit across (both ways) and drop half a unit to plot the next two points, $\left(0, \frac{5}{2}\right)$ and symmetrically $\left(-2, \frac{5}{2}\right)$. To plot the following two points, again, we start from the vertex and move our pen two units across and 2 units down (as $-\frac{1}{2} \cdot 4 = -2$). So, the next two points are $(1,1)$ and symmetrically $(-4,1)$, as indicated in *Figure 3.11*.

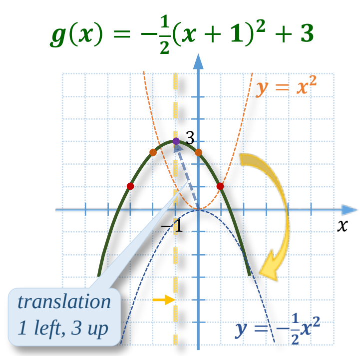


Figure 3.11

The domain of function g is $\mathbb{R}$, and the range is $(-\infty, \mathbf{3}]$.

The graph of g can be obtained from the graph of the basic parabola in two steps:

1. **Dilate** the basic parabola by multiplying its y-values by the factor of $-\frac{1}{2}$.
2. Shift the graph of the dilated parabola $y = -\frac{1}{2}\boldsymbol{x^2}$, **1 unit to the left** and **3 units up**, as indicated in *Figure 3.11*.

Aside from the main properties such as vertex, opening and shape, we are often interested in x- and y-intercepts of the given parabola. The next example illustrates how to find these intercepts from the vertex form of a parabola.

Example 4 ► **Finding the Intercepts from the Vertex Form $f(x) = a(x-p)^2 + q$**

Find the x- and y-intercepts of each parabola.

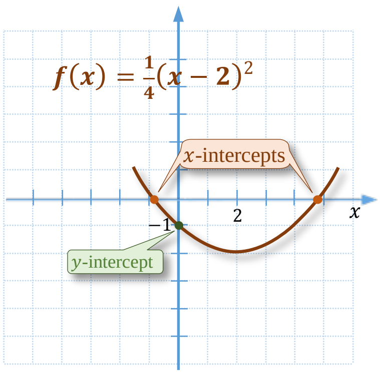
a. $f(x) = \frac{1}{4}(x-2)^2 - 2$ **b.** $g(x) = -2(x+1)^2 - 3$

Solution ► **a.** To find the y-intercept, we evaluate the function at zero. Since

$$f(0) = \frac{1}{4}(-2)^2 - 2 = 1 - 2 = -1,$$

then the y-intercept is $(\mathbf{0}, -\mathbf{1})$.

To find x-intercepts, we set $f(x) = 0$. So, we need to solve the equation



$$\frac{1}{4}(x-2)^2 - 2 = 0 \qquad / +1$$

$$\frac{1}{4}(x-2)^2 = 2 \qquad / \cdot 4$$

$$(x-2)^2 = 8 \qquad / \sqrt{\ }$$

$$\sqrt{(x-2)^2} = \sqrt{8}$$

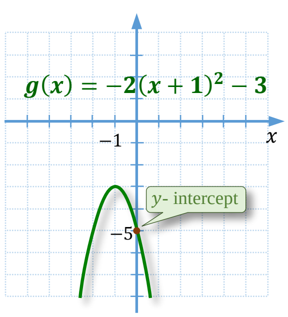
$$|x-2| = 2\sqrt{2}$$

$$x - 2 = \pm 2\sqrt{2} \qquad / +2$$

$$x = 2 \pm 2 = \begin{cases} 2 + 2\sqrt{2} \\ 2 - 2\sqrt{2} \end{cases}$$

Hence, the two x-intercepts are: $\left(\mathbf{2 - 2\sqrt{2}, 0}\right)$ and $\left(\mathbf{2 + 2\sqrt{2}, 0}\right)$.

b. Since $g(0) = -2(1)^2 - 3 = -5$, then the y-intercept is $(\mathbf{0}, -\mathbf{5})$.

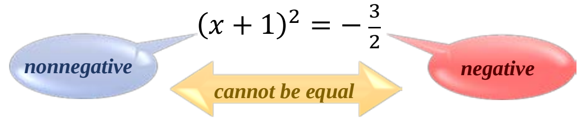


To find x-intercepts, we etempt to solve the equation

$$-2(x+1)^2 - 3 = 0$$

$$-2(x+1)^2 = 3$$

$$(x+1)^2 = -\frac{3}{2}$$



However, since the last equation doesn't have any solution, we conclude that function $g(x)$ has no x-intercepts.

Q.3 Exercises

1. Match each quadratic function **a.-d.** with its graph **I-IV**.

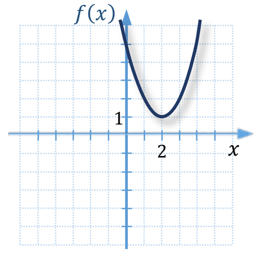
a. $f(x) = (x-2)^2 - 1$

b. $f(x) = (x-2)^2 + 1$

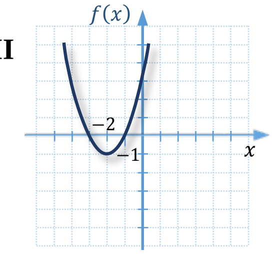
c. $f(x) = (x+2)^2 + 1$

d. $f(x) = (x+2)^2 - 1$

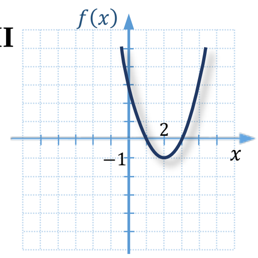
I



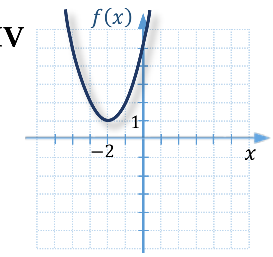
II



III



IV



2. Match each quadratic function **a.-d.** with its graph **I-IV**.

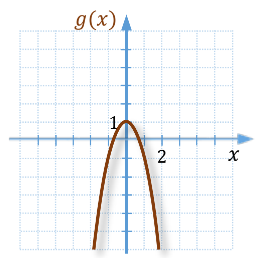
a. $g(x) = -(x-2)^2 + 1$

b. $g(x) = x^2 - 1$

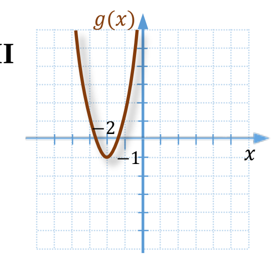
c. $g(x) = -2x^2 + 1$

d. $g(x) = 2(x+2)^2 - 1$

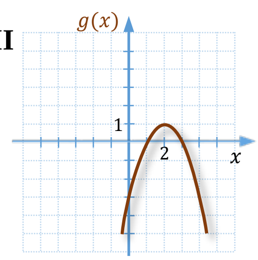
I



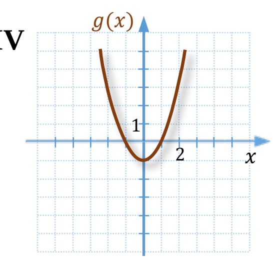
II



III



IV



3. Match each quadratic function with the characteristics of its parabolic graph.

a. $f(x) = 5(x-3)^2 + 2$ **I** vertex $(3,2)$, opens down

b. $f(x) = -4(x+2)^2 - 3$ **II** vertex $(3,2)$, opens up

c. $f(x) = -\frac{1}{2}(x-3)^2 + 2$ **III** vertex $(-2,-3)$, opens down

d. $f(x) = \frac{1}{4}(x+2)^2 - 3$ **IV** vertex $(-2,-3)$, opens up

For each quadratic function, describe the ***shape*** *(as* ***wider***, ***narrower***, *or the* ***same*** *as the shape of $y = x^2$) and* ***opening*** *(****up*** *or* ***down****) of its graph. Then* ***graph it*** *and determine its* ***range****.*

4. $f(x) = 3x^2$

5. $f(x) = -\frac{1}{2}x^2$

6. $f(x) = -\frac{3}{2}x^2$

7. $f(x) = \frac{5}{2}x^2$

8. $f(x) = -x^2$

9. $f(x) = \frac{1}{3}x^2$

Graph *each parabola by plotting its vertex, and following its shape and opening. Then,* ***describe transformations*** *of the basic parabola that would lead to the obtained graph. Finally, state the* ***domain*** *and* ***range****, and the equation of the* ***axis of symmetry****.*

10. $f(x) = (x-3)^2$

11. $f(x) = -x^2 + 2$

12. $f(x) = x^2 - 5$

13. $f(x) = -(x+2)^2$

14. $f(x) = -2x^2 - 1$

15. $f(x) = \frac{1}{2}(x+2)^2$

For each parabola, state its ***vertex****,* ***shape****,* ***opening****, and* ***x- and y-intercepts****. Then,* ***graph*** *the function and describe* ***transformations*** *of the basic parabola that would lead to the obtained graph.*

16. $f(x) = 3x^2 - 1$

17. $f(x) = -\frac{3}{4}x^2 + 3$

18. $f(x) = -\frac{1}{2}(x+4)^2 + 2$

19. $f(x) = \frac{5}{2}(x-2)^2 - 4$

20. $f(x) = 2(x-3)^2 + \frac{3}{2}$

21. $f(x) = -3(x+1)^2 + 5$

22. $f(x) = -\frac{2}{3}(x+2)^2 + 4$

23. $f(x) = \frac{4}{3}(x-3)^2 - 2$

24. Four students, **A**, **B**, **C**, and **D**, tried to graph the function $f(x) = -2(x+1)^2 - 3$ by transforming the graph of the basic parabola, $y = x^2$. Here are the transformations that each student applied

Student ***A****:*
- shift 1 unit left and 3 units down
- dilation of y-values by the factor of -2

Student ***B****:*
- dilation of y-values by the factor of -2
- shift 1 unit left
- shift 3 units down

Student ***C****:*
- flip over the x-axis
- shift 1 unit left and 3 units down
- dilation of y-values by the factor of 2

Student ***D****:*
- shift 1 unit left
- dilation of y-values by the factor of 2
- shift 3 units down
- flip over the x-axis

With the assumption that all transformations were properly applied, discuss whose graph was correct and what went wrong with the rest of the graphs. Is there any other sequence of transformations that would result in a correct graph?

For each parabola, state the coordinates of its ***vertex*** *and then* ***graph*** *it. Finally, state the* ***extreme value*** *(****maximum*** *or* ***minimum****, whichever applies) and the* ***range*** *of the function.*

25. $f(x) = 3(x-1)^2$

26. $f(x) = -\frac{5}{2}(x+3)^2$

27. $f(x) = (x+2)^2 - 3$

28. $f(x) = -3(x+4)^2 + 5$

29. $f(x) = -2(x-5)^2 - 2$

30. $f(x) = 2(x-4)^2 + 1$

31. $f(x) = \frac{1}{2}(x+1)^2 + \frac{3}{2}$

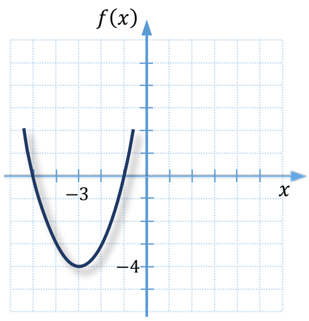
32. $f(x) = -\frac{1}{2}(x-1)^2 - 3$

33. $f(x) = -\frac{1}{4}(x-3)^2 + 4$

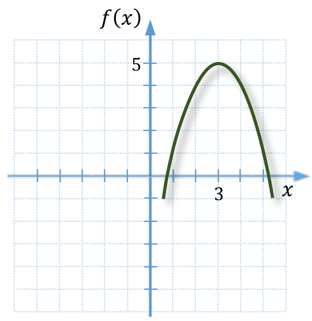
34. $f(x) = \frac{3}{4}\left(x+\frac{5}{2}\right)^2 - \frac{3}{2}$

Given the graph of a parabola, state the most probable ***equation*** *of the corresponding function. Hint: Use the vertex form of a quadratic function.*

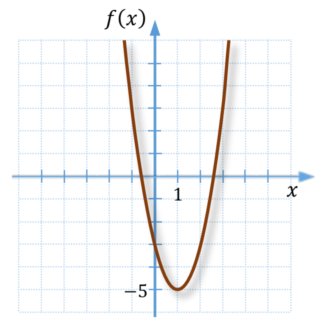
35.



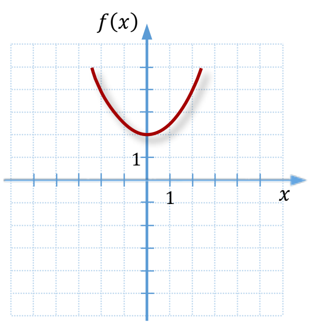
36.



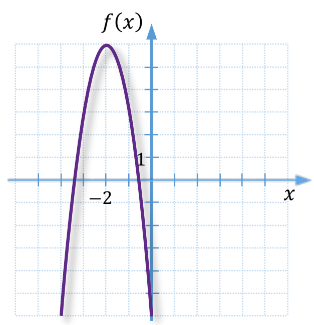
37.



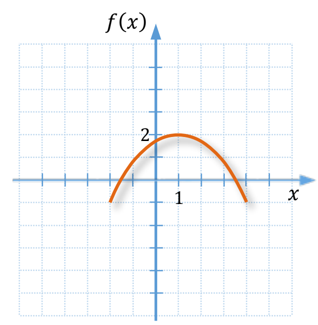
38.



39.



40.



Q4 Properties of Quadratic Function and Optimization Problems

In the previous section, we examined how to graph and read the characteristics of the graph of a quadratic function given in vertex form, $f(x) = a(x-p)^2 + q$. In this section, we discuss the ways of **graphing** and reading the **characteristics** of the graph of a quadratic function given in **standard form**, $\boldsymbol{f(x) = ax^2 + bx + c}$. One of these ways is to convert standard form of the function to vertex form by **completing the square** so that the information from the vertex form may be used for graphing. The other handy way of graphing and reading properties of a quadratic function is to **factor** the defining trinomial and use the **symmetry** of a parabolic function.

At the end of this section, we apply properties of quadratic functions to solve certain **optimization problems**. To solve these problems, we look for the **maximum** or **minimum** of a particular quadratic function satisfying specified conditions called **constraints**. Optimization problems often appear in geometry, calculus, business, computer science, etc.

Graphing Quadratic Functions Given in the Standard Form $\boldsymbol{f(x) = ax^2 + bx + c}$

To graph a quadratic function given in standard form, $f(x) = ax^2 + bx + c$, we can use one of the following methods:

1. constructing a **table of values** (this would always work, but it could be cumbersome);
2. converting to **vertex form** by using the technique of completing the square (see *Example 1-3*);
3. **factoring** and employing the properties of a parabolic function. (this is a handy method if the function can be easily factored – see *Example 3 and 4*)

The table of values approach can be used for any function, and it was already discussed on various occasions throughout this textbook.

Converting to **vertex form** involves completing the square. For example, to convert the function $f(x) = 2x^2 + x - 5$ to its vertex form, we might want to start by dividing both sides of the equation by the leading coefficient 2, and then complete the square for the polynomial on the right side of the equation, as below.

$$\frac{f(x)}{2} = x^2 + \frac{1}{2}x - \frac{5}{2}$$

$$\frac{f(x)}{2} = \left(x + \frac{1}{4}\right)^2 - \frac{1}{16} - \frac{5 \cdot 8}{2 \cdot 8}$$

$$\frac{f(x)}{2} = \left(x + \frac{1}{4}\right)^2 - \frac{41}{16}$$

Finally, the vertex form is obtained by multiplying both sides of the equation back by 2. So, we have

$$f(x) = 2\left(x + \frac{1}{4}\right)^2 - \frac{41}{8}$$

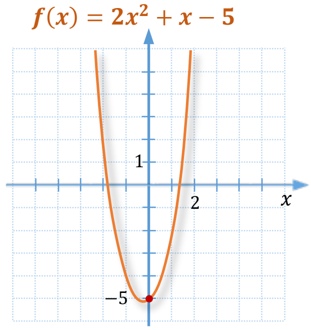


Figure 4.1

This form lets us identify the vertex, $\left(-\frac{1}{4}, -\frac{41}{8}\right)$, and the shape, $y = 2x^2$, of the parabola, which is essential for graphing it. To create an approximate graph of function f, we may want to round the vertex to approximately $(-0.25, -5.1)$ and evaluate $f(0) = 2 \cdot 0^2 + 0 - 5 = -5$. So, the graph is as in *Figure 4.1*.

Example 1 ► **Converting the Standard Form of a Quadratic Function to the Vertex Form**

Rewrite each function in its vertex form. Then, identify the vertex.

a. $f(x) = -3x^2 + 2x$ **b.** $g(x) = \frac{1}{2}x^2 + x + 3$

Solution ► **a.** To convert f to its vertex form, we follow the completing the square procedure. After dividing the equation by the leading coefficient,

$$f(x) = -3x^2 + 2x, \qquad / \div (-3)$$

we have

$$\frac{f(x)}{-3} = x^2 - \frac{2}{3}x$$

Then, we complete the square for the right side of the equation,

$$\frac{f(x)}{-3} = \left(x - \frac{1}{3}\right)^2 - \frac{1}{9}, \qquad / \cdot (-3)$$

and finally, multiply back by the leading coefficient,

$$f(x) = -3\left(x - \frac{1}{3}\right)^2 + \frac{1}{3}.$$

Therefore, the vertex of this parabola is at the point $\boldsymbol{\left(\frac{1}{3}, \frac{1}{3}\right)}$.

b. As in the previous example, to convert g to its vertex form, we first wish to get rid of the leading coefficient. This can be achieved by multiplying both sides of the equation $g(x) = \frac{1}{2}x^2 - x + 3$ by 2. So, we obtain

$$2g(x) = x^2 + 2x + 6$$

$$2g(x) = (x + 1)^2 - 1 + 6$$

$$2g(x) = (x + 1)^2 + 5, \qquad / \div 2$$

which can be solved back for g,

$$g(x) = \frac{1}{2}(x + 1)^2 + \frac{5}{2}.$$

Therefore, the vertex of this parabola is at the point $\boldsymbol{\left(-1, \frac{5}{2}\right)}$.

Completing the square allows us to derive a formula for the vertex of the graph of any quadratic function given in its standard form, $f(x) = ax^2 + bx + c$, where $a \neq 0$. Applying the same procedure as in *Example 1*, we calculate

$$f(x) = ax^2 + bx + c \qquad / \div a$$

$$\frac{f(x)}{a} = x^2 + \frac{b}{a}x + \frac{c}{a}$$

$$\frac{f(x)}{a} = \left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}$$

$$\frac{f(x)}{a} = \left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2} \qquad / \cdot a$$

Recall: This is the **discriminant** **Δ!**

$$f(x) = a\left(x - \left(-\frac{b}{2a}\right)\right)^2 + \frac{-(b^2 - 4ac)}{4a}$$

Thus, the coordinates of the vertex $(\boldsymbol{p}, \boldsymbol{q})$ are $\boldsymbol{p} = -\frac{\boldsymbol{b}}{\boldsymbol{2a}}$ and $\boldsymbol{q} = \frac{-(\boldsymbol{b^2 - 4ac})}{\boldsymbol{4a}} = \frac{-\Delta}{\boldsymbol{4a}}$.

Observation: Notice that the expression for q can also be found by evaluating f at $x = -\frac{\boldsymbol{b}}{\boldsymbol{2a}}$.

So, the vertex of the parabola can also be expressed as $\left(-\frac{\boldsymbol{b}}{\boldsymbol{2a}}, \boldsymbol{f}\left(-\frac{\boldsymbol{b}}{\boldsymbol{2a}}\right)\right)$.

VERTEX FORMULA

Summarizing, the **vertex** of a parabola defined by $f(x) = \boldsymbol{a}x^2 + \boldsymbol{b}x + \boldsymbol{c}$, where $\boldsymbol{a} \neq 0$, can be calculated by following one of the formulas:

$$\left(-\frac{\boldsymbol{b}}{\boldsymbol{2a}}, \frac{-(\boldsymbol{b^2 - 4ac})}{\boldsymbol{4a}}\right) = \left(-\frac{\boldsymbol{b}}{\boldsymbol{2a}}, \frac{-\Delta}{\boldsymbol{4a}}\right) = \left(-\frac{\boldsymbol{b}}{\boldsymbol{2a}}, \boldsymbol{f}\left(-\frac{\boldsymbol{b}}{\boldsymbol{2a}}\right)\right)$$

Example 2 ► **Using the Vertex Formula to Find the Vertex of a Parabola**

Use the vertex formula to find the vertex of the graph of $f(x) = -x^2 - x + 1$.

Solution ►

The first coordinate of the vertex is equal to $-\frac{b}{2a} = -\frac{-1}{2\cdot(-1)} = -\frac{\boldsymbol{1}}{\boldsymbol{2}}$.

The second coordinate can be calculated by following the formula

$$\frac{-\Delta}{4a} = \frac{-((-1)^2 - 4\cdot(-1)\cdot 1)}{4\cdot(-1)} = \frac{\boldsymbol{5}}{\boldsymbol{4}},$$

or by evaluating $f\left(-\frac{1}{2}\right) = -\left(-\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right) + 1 = -\frac{1}{4} + \frac{1}{2} + 1 = \frac{\boldsymbol{5}}{\boldsymbol{4}}$.

So, the vertex is $\left(-\frac{\boldsymbol{1}}{\boldsymbol{2}}, \frac{\boldsymbol{5}}{\boldsymbol{4}}\right)$.

Example 3 ► **Graphing a Quadratic Function Given in the Standard Form**

Graph each function.

a. $g(x) = \frac{1}{2}x^2 + x + 3$ **b.** $f(x) = -x^2 - x + 1$

Solution ► **a.** The shape of the graph of function g is the same as this of $y = \frac{1}{2}x^2$. Since the leading coefficient is positive, the arms of the parabola **open up**.

The **vertex**, $\left(-\mathbf{1}, \frac{\mathbf{5}}{\mathbf{2}}\right)$, was found in *Example 1b* as a result of completing the square. Since the vertex is in quadrant II and the graph opens up, we do not expect any x-intercepts. However, without much effort, we can find the y-intercept by evaluating $g(0) = \mathbf{3}$. Furthermore, since $(0, 3)$ belongs to the graph, then by symmetry, $(-2, 3)$ must also belong to the graph. So, we graph function g is as in *Figure 4.2*.

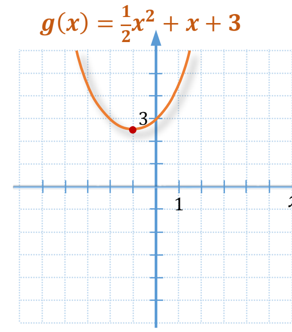


Figure 4.2

b. The graph of function f has the shape of the basic parabola. Since the leading coefficient is negative, the arms of the parabola **open down**.

When plotting points with fractional coordinates, round the values to one place value.

The **vertex**, $\left(-\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{5}}{\mathbf{4}}\right)$, was found in *Example 2* by using the vertex formula. Since the vertex is in quadrant II and the graph opens down, we expect two x-intercepts. Their values can be found via the quadratic formula applied to the equation $-x^2 - x + 1 = 0$. So, the x-intercepts are $x_{1,2} = \frac{1 \pm \sqrt{5}}{-2} \approx \mathbf{-1.6}$ *or* $\mathbf{0.6}$. In addition, the y-intercept of the graph is $f(0) = \mathbf{1}$.

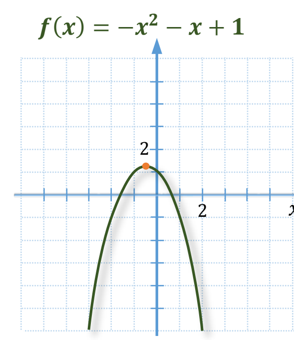


Figure 4.3

Using all this information, we graph function f, as in *Figure 4.3*.

Graphing Quadratic Functions Given in the Factored Form $f(x) = a(x - r_1)(x - r_2)$

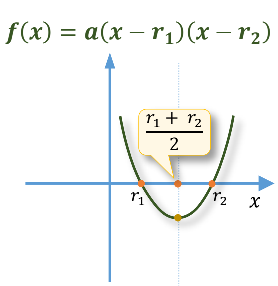


Figure 4.4

What if a quadratic function is given in factored form? Do we have to change it to vertex or standard form in order to find the vertex and graph it?

The factored form, $f(x) = a(x - r_1)(x - r_2)$, allows us to find the roots (or x-intercepts) of such a function. These are r_1 and r_2. A parabola is symmetrical about the axis of symmetry, which is the vertical line passing through its vertex. So, the first coordinate of the vertex is the same as the first coordinate of the midpoint of the line segment connecting the roots, r_1 with r_2, as indicated in *Figure 4.4*. Thus, the first coordinate of the vertex is the average of the two roots, $\frac{r_1 + r_2}{2}$. Then, the second coordinate of the vertex can be found by evaluating $f\left(\frac{r_1 + r_2}{2}\right)$.

Example 4 ► **Graphing a Quadratic Function Given in a Factored Form**

Graph function $g(x) = -(x-2)(x+1)$.

Solution ► First, observe that the graph of function g has the same shape as the graph of the basic parabola, $f(x) = x^2$. Since the leading coefficient is negative, the arms of the parabola **open down**. Also, the graph intersects the x-axis at 2 and -1. So, the first coordinate of the vertex is the average of 2 and -1, which is $\frac{1}{2}$. The second coordinate is

$$g\left(\frac{1}{2}\right) = -\left(\frac{1}{2}-2\right)\left(\frac{1}{2}+1\right) = -\left(-\frac{3}{2}\right)\left(\frac{3}{2}\right) = \frac{9}{4}$$

Therefore, function g can be graphed by connecting the vertex, $\left(\boldsymbol{\frac{1}{2}}, \boldsymbol{\frac{9}{4}}\right)$, and the x-intercepts, $(-1,0)$ and $(-1,0)$, with a parabolic curve, as in *Figure 4.5*. For a more precise graph, we may additionally plot the y-intercept, $g(0) = 2$, and the symmetrical point $g(1) = 2$.

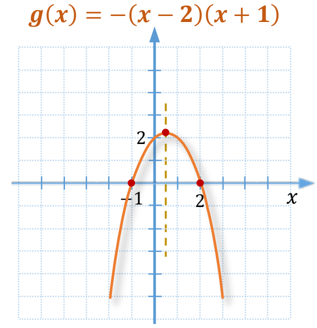


Figure 4.5

Example 5 ► **Using Complete Factorization to Graph a Quadratic Function**

Graph function $f(x) = 4x^2 - 2x - 6$.

Solution ► Since the discriminant $\Delta = (-2)^2 - 4 \cdot 4 \cdot (-6) = 4 + 96 = 100$ is a perfect square number, the defining trinomial is factorable. So, to graph function f, we may want to factor it first. Notice that the GCF of all the terms is 2. So, $f(x) = 2(2x^2 - x - 3)$. Then, using factoring techniques discussed in *Section F2*, we obtain $f(x) = 2(2x-3)(x+1)$. This form allows us to identify the roots (or zeros) of function f, which are $\frac{3}{2}$ and -1. So, the first coordinate of the vertex is the average of $\frac{3}{2} = 1.5$ and -1, which is $\frac{1.5+(-1)}{2} = \frac{0.5}{2} = 0.25$. The second coordinate can be calculated by evaluating

$$f(0.25) = 2(2 \cdot 0.25 - 3)(0.25+1) = 2(0.5-3)(1.25) = 2(-2.5)(1.25) = -6.25$$

So, we can graph function f by connecting its vertex, $(\mathbf{0.25}, \mathbf{-6.25})$, and its x-intercepts, $(-1,0)$ and $(1.5,0)$, with a parabolic curve, as in *Figure 4.6*. For a more precise graph, we may additionally plot the y-intercept, $f(0) = -6$, and by symmetry, $f(0.5) = -6$.

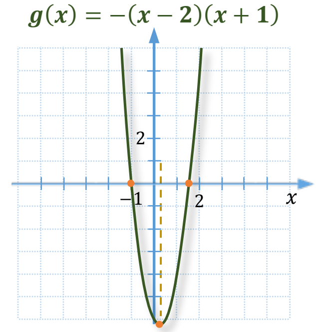


Figure 4.6

Observation: Since x-intercepts of a parabola are the solutions (zeros) of its equation, the equation of a parabola with x-intercepts at $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$ can be written as

$$\boldsymbol{y = a(x - r_1)(x - r_2)},$$

for some real coefficient $\boldsymbol{a} \neq 0$.

Example 6 ► **Finding an Equation of a Quadratic Function Given Its Solutions**

a. Find an equation of a quadratic function whose graph passes the x-axis at -1 and 3.

b. Find an equation of a quadratic function whose graph passes the x-axis at -1 and 3 and the y-axis at -4.

c. Write a quadratic equation with integral coefficients knowing that the solutions of this equation are $\frac{1}{2}$ and $-\frac{2}{3}$.

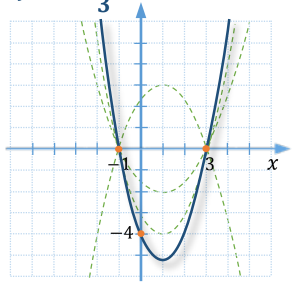
Solution ►

a. x-intercepts of a function are the zeros of this function. So, -1 and 3 are the zeros of the quadratic function. This means that the defining formula for such function should include factors $\big(x-(-1)\big)$ and $(x-3)$. So, it could be

$$f(x) = (x+1)(x-3).$$

Notice that this is indeed a quadratic function with x-intercepts at -1 and 3. Hence, it satisfies the conditions of the problem.

b. Using the solution to *Example 6a*, notice that any function of the form



$$f(x) = a(x+1)(x-3),$$

where a is a nonzero real number, is a quadratic function with x-intercepts at -1 and 3. To guarantee that the graph of our function passes through the point $(0,-4)$, we need to find the particular value of the coefficient a. This can be done by substituting $x = 0$ and $f(x) = -4$ into the function's equation and solving it for a. Thus,

$$-4 = a(0+1)(0-3)$$

$$-4 = -3a$$

$$a = \frac{4}{3},$$

and the desired function is $f(x) = \frac{4}{3}(x+1)(x-3)$.

c. First, observe that $\frac{1}{2}$ is a solution to the linear equation $2x - 1 = 0$. Similarly, $-\frac{2}{3}$ is a solution to the equation $3x + 2 = 0$. Multiplying these two equations side by side, we obtain a quadratic equation

$$(2x-1)(3x+2) = 0$$

that satisfies the conditions of the problem.

Note: Here, we could create the desired equation by writing

$$\left(x-\frac{1}{2}\right)\left(x-\left(-\frac{2}{3}\right)\right) = 0 \qquad / \cdot 6$$

and then multiplying it by the $LCD = \mathbf{6} = \mathbf{2} \cdot \mathbf{3}$

$$\mathbf{2}\left(x-\frac{1}{2}\right)\mathbf{3}\left(x+\frac{2}{3}\right) = 0$$

$$(2x-1)(3x+2) = 0$$

Optimization Problems

In many applied problems we are interested in **maximizing** or **minimizing** some quantity under specific conditions, called **constraints**. For example, we might be interested in finding the greatest area that can be fenced in by a given length of fence, or minimizing the cost of producing a container of a given shape and volume. These types of problems are called **optimization problems**.

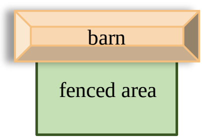
Since the vertex of the graph of a quadratic function is either the highest or the lowest point of the parabola, it can be used in solving optimization problems that can be modeled by a quadratic function.

The vertex of a parabola provides the following information.

- The y-value of the vertex gives the maximum or minimum value of y.
- The x-value tells where the maximum or minimum occurs.

Example 7 ►

Maximizing Area of a Rectangular Region



John has 60 meters of fencing to enclose a rectangular field by his barn. Assuming that the barn forms one side of the rectangle, find the maximum area he can enclose and the dimensions of the enclosed field that yields this area.

Solution ►

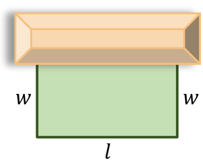


Figure 4.7

Let l and w represent the length and width of the enclosed area correspondingly, as indicated in *Figure 4.7*. The 60 meters of fencing is used to cover the distance of twice along the width and once along the length. So, we can form the constraint equation

$$2W + l = 60 \qquad (1)$$

To analyse the area of the field,

$$A = lw, \qquad (2)$$

we would like to express it as a function of one variable, for example w. To do this, we can solve the constraint equation (1) for l and substitute the obtained expression into the equation of area, (2). Since $l = 60 - 2w$, then

$$A = lw = (60 - 2w)w$$

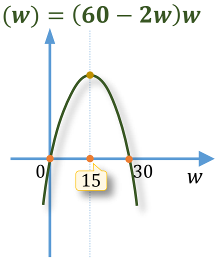


Figure 4.8

Observe that the graph of the function $A(w) = (60 - 2w)w$ is a parabola that opens down and intersects the x-axis at 0 and 30. This is because the leading coefficient of $(60 - 2w)w$ is negative and the roots to the equation $(60 - 2w)w = 0$ are 0 and 30. These roots are symmetrical in the axis of symmetry, which also passes through the vertex of the parabola, as illustrated in *Figure 4.8*. So, the first coordinate of the vertex is the average of the two roots, which is $\frac{0+30}{2} = 15$. Thus, the width that would maximize the enclosed area is $w_{max} = \mathbf{15}$ **meters**. Consequently, the length that would maximize the enclosed area is $l_{max} = 60 - 2w_{max} = 60 - 2 \cdot 15 = \mathbf{30}$ **meters**. The maximum area represented by the second coordinate of the vertex can be obtained by evaluating the function of area at the width of 15 meters.

$$A(15) = (60 - 2 \cdot 15)15 = 30 \cdot 15 = 450 \text{ m}^2$$

So, the maximum area that can be enclosed by 60 meters of fencing is **450 square meters**, and the dimensions of this rectangular area are **15 by 30 meters**.

Example 8 ► **Minimizing Average Unit Cost**

A company producing skateboards has determined that when x hundred skateboards are produced, the average cost of producing one skateboard can be modelled by the function

$$C(x) = 0.15x^2 - 0.75x + 1.5,$$

where $C(x)$ is in hundreds of dollars. How many skateboards should be produced to minimize the average cost of producing one skateboard? What would this cost be?

Solution ► Since $C(x)$ is a quadratic function, to find its minimum, it is enough to find the vertex of the parabola $C(x) = 0.15x^2 - 0.75x + 1.5$. This can be done either by completing the square method or by using the formula for the vertex, $\left(\frac{-b}{2a}, \frac{-\Delta}{4a}\right)$. We will do the latter. So, the vertex is

$$\left(\frac{-b}{2a}, \frac{-\Delta}{4a}\right) = \left(\frac{0.75}{0.3}, \frac{-(0.75^2 - 4 \cdot 0.15 \cdot 1.5)}{0.6}\right) = \left(2.5, \frac{-(0.5625 - 1.35)}{0.6}\right)$$
$$= \left(2.5, \frac{0.3375}{0.6}\right) = (\mathbf{2.5}, \mathbf{0.5625}).$$

This means that the lowest average unit cost can be achieved when 250 skateboards are produced, and then the average cost of a skateboard would be $56.25.

Q.4 Exercises

*Convert each quadratic function to its **vertex form**. Then, state the coordinates of the **vertex**.*

1. $f(x) = x^2 + 6x + 10$
2. $f(x) = x^2 - 4x - 5$
3. $f(x) = x^2 + x - 3$
4. $f(x) = x^2 - x + 7$
5. $f(x) = -x^2 + 7x + 3$
6. $f(x) = 2x^2 - 4x + 1$
7. $f(x) = -3x^2 + 6x + 12$
8. $f(x) = -2x^2 - 8x + 10$
9. $f(x) = \frac{1}{2}x^2 + 3x - 1$

*Use the vertex formula, $\left(-\frac{b}{2a}, \frac{-\Delta}{4a}\right)$, to find the coordinates of the **vertex** of each parabola.*

10. $f(x) = x^2 + 6x + 3$
11. $f(x) = -x^2 + 3x - 5$
12. $f(x) = \frac{1}{2}x^2 - 4x - 7$
13. $f(x) = -3x^2 + 6x + 5$
14. $f(x) = 5x^2 - 7x$
15. $f(x) = 3x^2 + 6x - 20$

*For each parabola, state its **vertex**, **opening** and **shape**. Then **graph** it and state the **domain** and **range**.*

16. $f(x) = x^2 - 5x$

17. $f(x) = x^2 + 3x$

18. $f(x) = x^2 - 2x - 5$

19. $f(x) = -x^2 + 6x - 3$

20. $f(x) = -x^2 - 3x + 2$

21. $f(x) = 2x^2 + 12x + 18$

22. $f(x) = -2x^2 + 3x - 1$

23. $f(x) = -2x^2 + 4x + 1$

24. $f(x) = 3x^2 + 4x + 2$

*For each quadratic function, state its **zeros** (roots), coordinates of the **vertex**, **opening** and **shape**. Then **graph** it and identify its **extreme** (minimum or maximum) **value** as well as where it occurs.*

25. $f(x) = (x - 2)(x + 2)$

26. $f(x) = -(x + 3)(x - 1)$

27. $f(x) = x^2 - 4x$

28. $f(x) = x^2 + 5x$

29. $f(x) = x^2 - 8x + 16$

30. $f(x) = -x^2 - 4x - 4$

31. $f(x) = -3(x^2 - 1)$

32. $f(x) = \frac{1}{2}(x + 3)(x - 4)$

33. $f(x) = -\frac{3}{2}(x - 1)(x - 5)$

Find an equation of a quadratic function satisfying the given conditions.

34. passes the x-axis at -2 and 5

35. has x-intercepts at 0 and $\frac{2}{5}$

36. passes the x-axis at -3 and -1 and y-axis at 2

37. $f(1) = 0$, $f(4) = 0$, $f(0) = 3$

Write a quadratic equation with the indicated solutions using only integral coefficients.

38. -5 and 6

39. 0 and $\frac{1}{3}$

40. $-\frac{2}{5}$ and $\frac{3}{4}$

41. 2

42. Suppose the x-intercepts of the graph of a parabola are $(x_1, 0)$ and $(x_2, 0)$. What is the equation of the axis of symmetry of this graph?

43. How can we determine the number of x-intercepts of the graph of a quadratic function without graphing the function?

True or false? Explain.

44. The domain and range of a quadratic function are both the set of real numbers.

45. The graph of every quadratic function has exactly one y-intercept.

46. The graph of $y = -2(x - 1)^2 - 5$ has no x-intercepts.

47. The maximum value of y in the function $y = -4(x - 1)^2 + 9$ is 9.

48. The value of the function $f(x) = x^2 - 2x + 1$ is at its minimum when $x = 0$.

49. The graph of $f(x) = 9x^2 + 12x + 4$ has one x-intercept and one y-intercept.

50. If a parabola opens down, it has two x-intercepts.

Solve each problem.

51. A ball is projected from the ground straight up with an initial velocity of 24.5 m/sec. The function $h(t) = -4.9t^2 + 24.5t$ allows for calculating the height $h(t)$, in meters, of the ball above the ground after t seconds. What is the maximum height reached by the ball? In how many seconds should we expect the ball to come back to the ground?

52. A firecracker is fired straight up and explodes at its maximum height above the ground. The function $h(t) = -4.9t^2 + 98t$ allows for calculating the height $h(t)$, in meters, of the firecracker above the ground t seconds after it was fired. In how many seconds after firing should we expect the firecracker to explode and at what height?

53. Antonio prepares and sells his favourite desserts at a market stand. Suppose his daily cost, C, in dollars, to sell n desserts can be modelled by the function $C(n) = 0.5n^2 - 30n + 350$. How many of these desserts should he sell to minimize the cost and what is the minimum cost?

54. Chris has a hot-dog stand. His daily cost, C, in dollars, to sell n hot-dogs can be modelled by the function $C(n) = 0.1n^2 - 15n + 700$. How many hotdogs should he sell to minimize the cost and what is the minimum cost?

55. Find two positive numbers with a sum of 32 that would produce the maximum product.

56. Find two numbers with a difference of 32 that would produce the minimum product.

57. Luke uses 16 meters of fencing to enclose a rectangular area for his baby goats. The enclosure shares one side with a large barn, so only 3 sides need to be fenced. If Luke wishes to enclose the greatest area, what should the dimensions of the enclosure be?

58. Ryan uses 60 meters of fencing to enclose a rectangular area for his livestock. He plans to subdivide the area by placing additional fence down the middle of the rectangle to separate different types of livestock. What dimensions of the overall rectangle will maximize the total area of the enclosure?

59. Julia works as a tour guide. She charges \$58 for an individual tour. When more people come for a tour, she charges \$2 less per person for each additional person, up to 25 people.

a. Express the price per person P as a function of the number of people n, for $n \in \{1,2, \dots ,25\}$.

b. Express her revenue, R, as a function of the number of people on tour.

c. How many people on tour would maximize Julia's revenue?

d. What is the highest revenue she can achieve?

60. One-day adult passes for The Mission Folk Festival cost \$50. At this price, the organizers of the festival expect about 1300 people to purchase the pass. Suppose that the organizers observe that every time they increase the cost per pass by 5\$, the number of passes sold decrease by about 100.

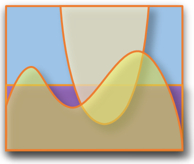
a. Express the number of passes sold, N, as a function of the cost, c, of a one-day pass.

b. Express the revenue, R, as a function of the cost, c, of a one-day pass.

c. How much should a one-day pass costs to maximize the revenue?

d. What is the maximum revenue?

Q5 Polynomial and Rational Inequalities



In *Sections L4* and *L5*, we discussed solving linear inequalities in one variable as well as solving systems of such inequalities. In this section, we examine polynomial and rational inequalities in one variable. Such inequalities can be solved using either graphical or analytic methods. Below, we discuss both types of methods with a particular interest in the analytic one.

Solving Quadratic Inequalities by Graphing

Definition 5.1 ► A **quadratic inequality** is any inequality that can be written in one of the forms

$$\boldsymbol{ax^2+bx+c>(\geq)\ 0},\quad \text{or}\quad \boldsymbol{ax^2+bx+c<(\leq)\ 0},\quad \text{or}\quad \boldsymbol{ax^2+bx+c\neq\ 0}$$

where a, b, and c are real numbers, with $a \neq 0$.

To solve a quadratic inequality, it is useful to solve the related quadratic equation first. For example, to solve $x^2+2x-3>(\geq)\ 0$, or $x^2+2x-3<(\leq)\ 0$, we may consider solving the related equation:

$$x^2+2x-3=0$$

$$(x+3)(x-1)=0$$

$$x=-3 \text{ or } x=1.$$

This helps us to sketch an approximate graph of the related function $f(x)=x^2+x-2$, as in *Figure 1.1*. The graph of function f is a parabola that crosses the x-axis at $x=-3$ and $x=1$, and is directed upwards. Observe that the graph extends below the x-axis for x-values from the interval $(-3,1)$ and above the x-axis for x-values from the set $(-\infty,-3)\cup(1,\infty)$. This allows us to read solution sets of several inequalities, as listed below.

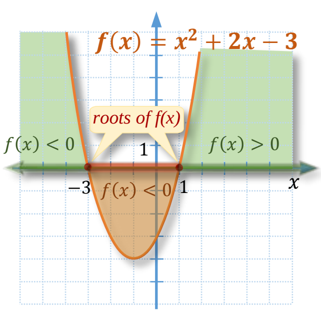


Figure 1.1

Inequality	Solution Set
$\boldsymbol{x^2+2x-3>0}$	$(-\infty,-3)\cup(1,\infty)$
$\boldsymbol{x^2+2x-3\geq 0}$	$(-\infty,-3]\cup[1,\infty)$
$\boldsymbol{x^2+2x-3<0}$	$(-3,1)$
$\boldsymbol{x^2+2x-3\leq 0}$	$[-3,1]$
$\boldsymbol{x^2+2x-3\neq 0}$	$\mathbb{R}\setminus\{-3,1\}$

Note: If the inequalities contain equations $(\geq,\leq)$, the x-values of the intercepts are included in the solution sets. Otherwise, the x-values of the intercepts are excluded from the solution sets.

Solving Polynomial Inequalities

Definition 5.2 ► A **polynomial inequality** is any inequality that can be written in one of the forms

$$\boldsymbol{P(x)>(\geq)\ 0},\quad \text{or}\quad \boldsymbol{P(x)<(\leq)\ 0},\quad \text{or}\quad \boldsymbol{P(x)\neq\ 0}$$

where $P(x)$ is a polynomial with real coefficients.

Note: Linear or quadratic inequalities are special cases of polynomial inequalities.

Polynomial inequalities can be solved graphically or analytically, without the use of a graph. The analytic method involves determining the sign of the polynomial by analysing signs of the polynomial factors for various x-values, as in the following example.

Example 1 ► **Solving Polynomial Inequalities Using Sign Analysis**

Solve each inequality using sign analysis.

a. $(x+1)(x-2)x > 0$ **b.** $2x^4 + 8 \leq 10x^2$

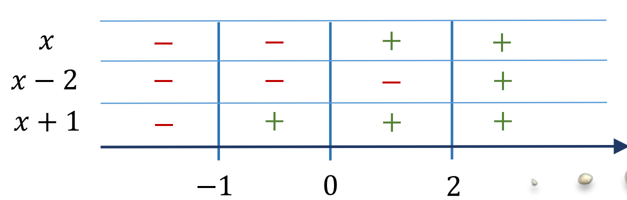
Solution ► **a.** The solution set of $(x+1)(x-2)x > 0$ consists of all x-values that make the product $(x+1)(x-2)x$ positive. To analyse how the sign of this product depends on the x-values, we can visualise the sign behaviour of each factor with respect to the x-value, by recording applicable signs in particular sections of a number line.

For example, the expression $x+1$ changes its sign at $x = -1$.
If $x > -1$, the expression $x+1$ is positive, so we mark "+" in the interval $(-1, \infty)$.
If $x < -1$, the expression $x+1$ is negative, so we mark "−" in the interval $(-\infty, -1)$.

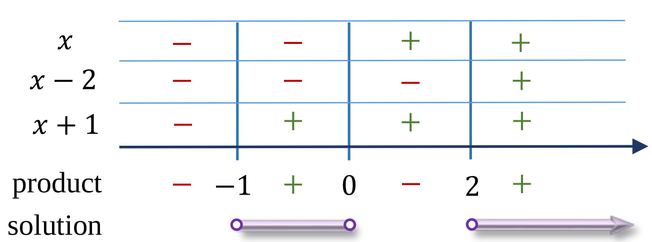
So, the sign behaviour of the expression $x+1$ can be recorded on a number line, as below.

Here, the **zeros** of polynomials are referred to as **critical numbers**. This is because the polynomials may change their signs at these numbers.

A similar analysis can be conducted for the remaining factors, $x-2$ and x. These expressions change their signs at $x = 2$ and $x = 0$, correspondingly. The sign behaviour of all the factors can be visualised by reserving one line of signs per each factor, as shown below.



The sign of the product $(x+1)(x-2)x$ is obtained by multiplying signs in each column. The result is marked beneath the number line, as below.



The signs in the "product" row give us the grounds to state the solution set for the original inequality, $(x+1)(x-2)x > 0$. Before we write the final answer though, it is helpful to visualize the solution set by graphing it in the "solution" row. To satisfy the original inequality, we need the product to be positive, so we look for the intervals within which the product is positive. Since the inequality does not include an equation, the intervals are open. Therefore, the solution set is $(\mathbf{-1,0}) \cup (\mathbf{2,\infty})$.

b. To solve $2x^4 + 8 \le 10x^2$ by sign analysis, we need to keep one side of the inequality equal to zero and factor the other side so that we may identify the critical numbers. To do this, we may change the inequality as below.

$$2x^4 + 8 \le 10x^2 \qquad / -10x^2$$

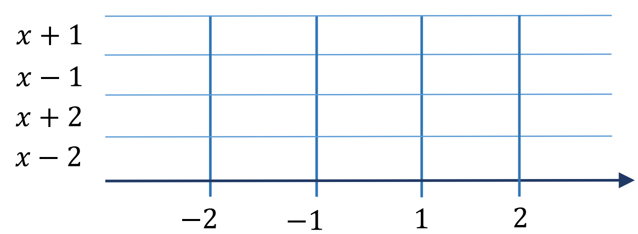
$$2x^4 - 10x^2 + 8 \le 0 \qquad / \div 2$$

$$x^4 - 5x^2 + 4 \le 0$$

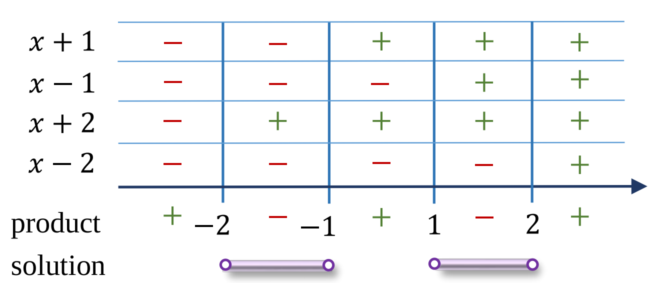
$$(x^2-4)(x^2-1) \le 0$$

$$(x-2)(x+2)(x-1)(x+1) \le 0$$

The critical numbers (the x-values that make the factors equal to zero) are 2, −2, 1, and −1. To create a table of signs, we arrange these numbers on a number line in increasing order and list all the factors in the left column.



Then, we can fill in the table with signs that each factor assumes for the x-values from the corresponding section of the number line.



The signs in each part of the number line can be determined either by:

- **analysing** the behaviour of each factor with respect to its critical number (For example, $x-2>0$ for $x>2$. So, the row of signs assumed by $x-2$ consists of negative signs until the critical number 2 and positive signs after this number.)

or

- **testing** an x-value from each section of the number line (For example, since the expression $x-2$ does not change its sign inside the interval $(-1,1)$, to determine its sign, it is enough to evaluate it for an easy to calculate **test number** from this interval. For instance, when $x = 0 \in (-1,1)$, the value of $x - 2$ is negative. This means that all the values of $x - 2$ are negative between -1 and 1.)

Finally, underneath each column, we record the sign of the product and graph the solution set to the inequality $(x-2)(x+2)(x-1)(x+1) \le 0$

	$(-\infty,-2)$	$(-2,-1)$	$(-1,1)$	$(1,2)$	$(2,\infty)$
$x+1$	−	−	+	+	+
$x-1$	−	−	−	+	+
$x+2$	−	+	+	+	+
$x-2$	−	−	−	−	+
product	+	−	+	−	+

solution

Note: Since the inequality contains an equation, the endpoints of the resulting intervals belong to the solution set as well. Hence, they are marked by **filled-in circles** and notated with **square brackets** in interval notation.

So, the solution set is $[\mathbf{-2}, \mathbf{-1}] \cup [\mathbf{1}, \mathbf{2}]$.

Solving Rational Inequalities

Definition 5.3 ► A **rational inequality** is any inequality that can be written in one of the forms

$$\frac{P(x)}{Q(x)} > (\ge)\ \mathbf{0}, \quad \text{or} \quad \frac{P(x)}{Q(x)} < (\le)\ \mathbf{0}, \quad \text{or} \quad \frac{P(x)}{Q(x)} \ne \mathbf{0}$$

where $P(x)$ and $Q(x)$ are polynomials with real coefficients.

Rational inequalities can be solved similarly as polynomial inequalities. To solve a rational inequality using the sign analysis method, we need to make sure that one side of the inequality is **zero** and the other side is expressed as a **single algebraic fraction** with a completely factored numerator and denominator.

Example 2 ► **Solving Rational Inequalities Using Sign Analysis**

Solve each inequality using sign analysis.

a. $\frac{(x-2)x}{x+1} \ge 0$ **b.** $\frac{4-x}{x+2} \ge x$

Solution ► **a.** The right side of the inequality $\frac{(x-2)x}{x+1} > 0$ is zero, and the left side is a single fraction with both numerator and denominator in factored form. So, to solve this inequality, it is enough to analyse signs of the expression $\frac{(x-2)x}{x+1}$ at different intervals of the domain.

These intervals are determined by the **critical numbers** (the zeros of the numerator and denominator), which are -1, 0, and 2.

x	$-$		$-$		$+$		$+$
$x-2$	$-$		$-$		$-$		$+$
$x+1$	$-$		$+$		$+$		$+$
product	$-$	-1	$+$	0	$-$	2	$+$
solution		○	———	●		●	———→

As indicated in the above table of signs, the solution set to the inequality $\frac{(x-2)x}{x+1} \geq 0$ contains numbers between -1 and 0 and numbers higher than 2. In addition, since the inequality includes an equation, $x = 0$ and $x = 2$ are also solutions. However, $x = -1$ is not a solution because -1 does not belong to the domain of the expression $\frac{(x-2)x}{x+1}$ since it would make the denominator 0. So, the solution set is $(\mathbf{-1,0}] \cup [\mathbf{2},\infty)$.

Attention: **Solutions** to a rational inequality **must belong to the domain** of the inequality. This means that any number that makes the denominator 0 must be excluded from the solution set.

b. To solve $\frac{4-x}{x+2} \geq x$ by the sign analysis method, first, we would like to keep the right side equal to zero. So, we rearrange the inequality as below.

When working with inequalities, **avoid multiplying by the denominator** as it can be positive or negative for different x-values!

$$\frac{4-x}{x+2} \geq x \qquad / -x$$

$$\frac{4-x}{x+2} - x \geq 0$$

$$\frac{4-x-x(x+2)}{x+2} \geq 0$$

$$\frac{4-x-x^2-2x}{x+2} \geq 0$$

$$\frac{-x^2-3x+4}{x+2} \geq 0$$

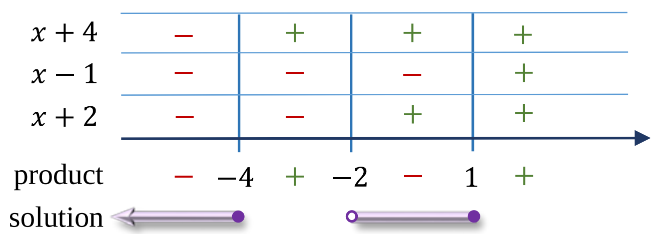
$$\frac{-(x^2+3x-4)}{x+2} \geq 0$$

$$\frac{-(x+4)(x-1)}{x+2} \geq 0 \qquad / \cdot(-1)$$

$$\frac{(x+4)(x-1)}{x+2} \leq 0$$

When multiplying by a **negative** number, remember to **reverse** the inequality sign!

Now, we can analyse the signs of the expression $\frac{(x+4)(x-1)}{x+2}$, using the table of signs with the critical numbers -4, -2, and 1.



So, the solution set for the inequality $\frac{(x+4)(x-1)}{x+2} \le 0$, which is equivalent to $\frac{4-x}{x+2} \ge x$, contains numbers lower than -4 and numbers between -2 and 1. Since the inequality includes an equation, $x = -4$ and $x = 1$ are also solutions. However, $x = -2$ is not in the domain of $\frac{(x+4)(x-1)}{x+2}$, and therefore it is not a solution.

Thus, the solution set to the original inequality is $(-\infty, \mathbf{-4}] \cup (\mathbf{-2}, \mathbf{1}]$.

Summary of Solving Polynomial or Rational Inequalities

1. **Write** the inequality so that **one of its sides is zero** and the other side is expressed as the **product or quotient of prime polynomials**.
2. **Determine the critical numbers**, which are the roots of all the prime polynomials appearing in the inequality.
3. **Divide the number line into intervals** formed by the set of critical numbers.
4. **Create a table of signs** for all prime factors in all intervals formed by the set of critical numbers. This can be done by analysing the sign of each factor, or by testing a number from each interval.
5. **Determine the sign of the overall expression** in each of the intervals.
6. **Graph the intervals of numbers that satisfy the inequality**. Make sure to **exclude endpoints that are not in the domain** of the inequality.
7. **State the solution set** to the original inequality **in interval notation**.

Solving Special Cases of Polynomial or Rational Inequalities

Some inequalities can be solved without the use of a graph or a table of signs.

Example 3 ► **Solving Special Cases of Inequalities**

Solve each inequality.

a. $(3x+2)^2 > -1$

b. $\frac{(x-4)^2}{x^2} \le 0$

Solution ► **a.** First, notice that the left side of the inequality $(3x + 2)^2 > -1$ is a perfect square and as such, it assumes a nonnegative value for any input x. Since a nonnegative quantity is always bigger than -1, the inequality is satisfied by any real number x. So, the solution set is $\mathbb{R}$.

Note: The solution set of an inequality that is **always true** is the set of all real numbers, $\mathbb{R}$. For example, inequalities that take one of the following forms

$$\textbf{\textit{nonnegative}} > \textbf{\textit{negative}}$$

$$\textbf{\textit{positive}} \geq \textbf{\textit{negative}}$$

$$\textbf{\textit{positive}} > \textbf{\textit{nonpositive}}$$

$$\textbf{\textit{positive}} > \mathbf{0}$$

$$\textbf{\textit{negative}} < \mathbf{0}$$

are **always true**. So their solution set is $\mathbb{R}$.

b. Since the left side of the inequality $\frac{(x-4)^2}{x^2} \leq 0$ is a perfect square, it is bigger or equal to zero for all x-values. So, we have

$$0 \leq \frac{(x-4)^2}{x^2} \leq 0,$$

which can be true only if $\frac{(x-4)^2}{x^2} = 0$. Since a fraction equals to zero only when its numerator equals to zero, the solution to the last equation is $x = 4$. Thus, the solution set for the original inequality is $\{\mathbf{4}\}$.

Observation: Notice that the inequality $\frac{(x-4)^2}{x^2} < 0$ has no solution as a perfect square is never negative.

Note: The solution set of an inequality that is **never true** is the empty set, $\emptyset$. For example, inequalities that take one of the following forms

$$\textbf{\textit{positive}}\ (\textbf{\textit{or nonnegative}}) \leq \textbf{\textit{negative}}$$

$$\textbf{\textit{nonnegative}} < \textbf{\textit{nonpositive}}$$

$$\textbf{\textit{positive}} \leq \textbf{\textit{nonpositive}}$$

$$\textbf{\textit{positive}} \leq \mathbf{0} \text{ or } \textbf{\textit{negative}} \geq \mathbf{0}$$

$$\textbf{\textit{nonnegative}} < \mathbf{0} \text{ or } \textbf{\textit{nonpositive}} > \mathbf{0}$$

are **never true**. So, their solution sets are $\emptyset$.

Polynomial and Rational Inequalities in Application Problems

Some application problems involve solving polynomial or rational inequalities.

Example 4 ► **Finding the Range of Values Satisfying the Given Condition**

The manager of a shoe store observed that the weekly revenue, R, for selling rain boots at p dollars per pair can be modelled by the function $R(p) = 170p - 2p^2$. For what range of prices p will the weekly revenue be at least \$3000?

Solution ► Since the revenue must be at least \$3000, we can set up the inequality

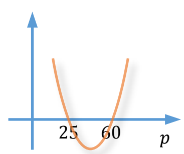
$$R(p) = 170p - 2p^2 \geq 3000,$$

and solve it for p. So, we have

$$-2p^2 + 170p - 3000 \geq 0 \qquad / \cdot (-2)$$

$$p^2 - 85p + 1500 \leq 0$$

$$(p - 25)(p - 60) \leq 0$$



Since the left-hand side expression represents a directed upwards parabola with roots at $p = 25$ and $p = 60$, its graph looks like in the accompanying figure. The graph extends below the p-axis for p-values between 25 and 60. So, to generate weekly revenue of at least \$3000, the price p of a pair of rain boots must take a value within the interval $[\mathbf{25\$}, \mathbf{60\$}]$.

Q.5 Exercises

True or False.

1. To determine if the value of an expression is greater than or less than 0 in a given interval, a test number can be used.

2. If the solution to the inequality $P(x) \geq 0$, where $P(x)$ is a polynomial with real coefficients, is $[2, 5)$, then the solution to the inequality $P(x) < 0$ is $(-\infty, 2) \cup [5, \infty)$.

3. The inequalities $(x - 1)(x + 3) \leq 0$ and $\frac{(x-1)}{(x+3)} \leq 0$ have the same solutions.

4. The solution set of the inequality $(x - 1)^2 > 0$ is the set of all real numbers.

5. The inequality $x^2 + 1 \leq 0$ has no solution.

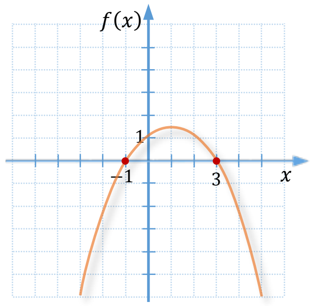
6. The solution set of the inequality $\frac{(x-1)^2}{(x+1)^2} \geq 0$ is the set of all real numbers.

Given the graph of function f, state the solution set for each inequality

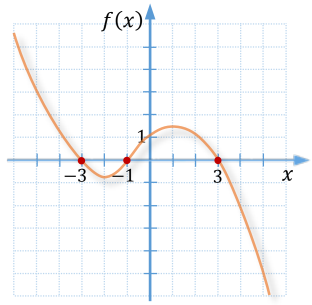
a. $f(x) \geq 0$

b. $f(x) < 0$

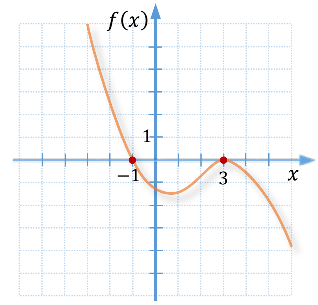
7.



8.



9.



Solve *each inequality by sketching an approximate graph for the related equation.*

10. $(x+4)(x-2) > 0$

11. $(x+1)(x-2) < 0$

12. $x^2 - 4x + 3 \geq 0$

13. $\frac{1}{2}(x^2 - 3x - 10) \leq 0$

14. $4 - 9x^2 > 0$

15. $-x^2 - 2x < 0$

Solve *each inequality using* ***sign analysis****.*

16. $(x-3)(x+2) > 0$

17. $(x+4)(x-5) < 0$

18. $x^2 + 2x - 7 \leq 8$

19. $x^2 - x - 2 \geq 10$

20. $3x^2 + 10x > 8$

21. $2x^2 + 5x < -2$

22. $x^2 + 9 > -6x$

23. $x^2 + 4 \leq 4x$

24. $6 + x - x^2 \leq 0$

25. $20 - x - x^2 < 0$

26. $(x-1)(x+2)(x-3) \geq 0$

27. $(x+3)(x-2)(x-5) \leq 0$

28. $x(x+3)(2x-1) > 0$

29. $x^2(x-2)(2x-1) \geq 0$

30. $x^4 - 13x^2 + 36 \leq 0$

Solve each inequality using sign analysis.

31. $\frac{x}{x+1} > 0$

32. $\frac{x+1}{x-2} < 0$

33. $\frac{2x-1}{x+3} \leq 0$

34. $\frac{2x-3}{x+1} \geq 0$

35. $\frac{3}{y+5} > 1$

36. $\frac{5}{t-1} \leq 2$

37. $\frac{x-1}{x+2} \leq 3$

38. $\frac{a+4}{a+3} \geq 2$

39. $\frac{2t-3}{t+3} < 4$

40. $\frac{3y+9}{2y-3} < 3$

41. $\frac{1-2x}{2x+5} \leq 2$

42. $\frac{2x+3}{1-x} \leq 1$

43. $\frac{4x}{2x-1} \leq x$

44. $\frac{-x}{x+2} > 2x$

45. $\frac{2x-3}{(x+1)^2} \leq 0$

46. $\frac{2x-3}{(x-2)^2} \geq 0$

47. $\frac{x^2+1}{5-x^2} > 0$

48. $x < \frac{3x-8}{5-x}$

49. $\frac{1}{x+2} \geq \frac{1}{x-3}$

50. $\frac{2}{x+3} \leq \frac{1}{x-1}$

51. $\frac{(x-3)(x+1)}{4-x} \geq 0$

52. $\frac{(x+2)(x-1)}{(x+4)^2} \geq 0$

53. $\frac{x^2-2x-8}{x^2+10x+25} > 0$

54. $\frac{x^2-4x}{x^2-x-6} \leq 0$

Solve each inequality.

55. $(4-3x)^2 \geq -2$

56. $(5+2x)^2 < -1$

57. $\frac{(1-2x)^2}{2x^4} \leq 0$

58. $\frac{(1-2x)^2}{(x+2)^2} > -3$

59. $\frac{-2x^2}{(x+2)^2} \geq 0$

60. $\frac{-x^2}{(x-3)^2} < 0$

Solve each problem.

61. Sonia tossed a dice upwards with an initial velocity of 4 m/sec. Suppose the height, h, in meters, of the dice above the ground t seconds after it was tossed is modelled by the function $h(t) = -4.9t^2 + 12t + 1$. If the dice landed on a 0.7 meters high table, estimate the interval of time during which the dice was above the table? *Round the answer to two decimal places.*

62. A company producing furniture observes that the weekly cost, C, for producing n accent glass tables can be modelled by the function $C(x) = 2n^2 - 60n + 800$. How many of these tables should the company produce to decrease the weekly cost below $400?

63. Anna wishes to create a rectangular flower bed and install a rubber edge around its perimeter. She bought 36 meters of edging. If she intends to use all the edging, how long could be the flower bed enclosure so that its area is at least 72 m^2?

64. Suppose that when a company producing furniture sells n chairs, the average cost per chair, C, in dollars, is modelled by the function $C(n) = \frac{450+2x}{x}$. For what number of chairs n will the average cost per chair be less than $12?

65. A software developing company has a revenue of $22 million this year. Suppose R, in millions of dollars, is the company's last year revenue. The company's percent revenue growth, P, in percent, is given by the function $P(R) = \frac{2200-100R}{R}$. For what revenues, R, would the company's revenue grow by more than 10%?

Attributions