

# Quadratic Equations and Functions



In this chapter, we discuss various ways of solving quadratic equations,  $ax^2 + bx + c = 0$ , including equations quadratic in form, such as  $x^{-2} + x^{-1} - 20 = 0$ , and solving formulas for a variable that appears in the first and second power, such as  $k$  in  $k^2 - 3k = 2N$ . Frequently used strategies of solving quadratic equations include the **completing the square** procedure and its generalization in the form of the **quadratic formula**. Completing the square allows for rewriting quadratic functions in vertex form,  $f(x) = a(x - h)^2 + k$ , which is very useful for graphing as it provides information about the location, shape, and direction of the parabola.

In the second part of this chapter, we examine properties and graphs of quadratic functions, including basic transformations of these graphs.

Finally, these properties are used in solving application problems, particularly problems involving **optimization**. In the last section of this chapter, we study how to solve polynomial and rational inequalities using **sign analysis**.

## Q.1

## Methods of Solving Quadratic Equations

As defined in *section F4*, a quadratic equation is a second-degree polynomial equation in one variable that can be written in standard form as

$$ax^2 + bx + c = 0,$$

where  $a$ ,  $b$ , and  $c$  are real numbers and  $a \neq 0$ . Such equations can be solved in many different ways, as presented below.

### Solving by Graphing

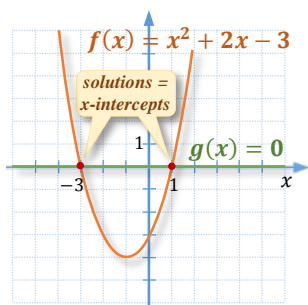


Figure 1.1

To solve a quadratic equation, for example  $x^2 + 2x - 3 = 0$ , we can consider its left side as a function  $f(x) = x^2 + 2x - 3$  and the right side as a function  $g(x) = 0$ . To satisfy the original equation, both function values must be equal. After graphing both functions on the same grid, one can observe that this happens at points of intersection of the two graphs.

So the **solutions** to the original equation are the  $x$ -coordinates of the intersection points of the two graphs. In our example, these are the  **$x$ -intercepts** or the **roots** of the function  $f(x) = x^2 + 2x - 3$ , as indicated in *Figure 1.1*.

Thus, the solutions to  $x^2 + 2x - 3 = 0$  are  $x = -3$  and  $x = 1$ .

**Note:** Notice that the graphing method, although visually appealing, is not always reliable. For example, the solutions to the equation  $49x^2 - 4 = 0$  are  $x = \frac{2}{7}$  and  $x = -\frac{2}{7}$ . Such numbers would be very hard to read from the graph.

Thus, the graphing method is advisable to use when searching for integral solutions or estimations of solutions.

To find exact solutions, we can use one of the algebraic methods presented below.

## Solving by Factoring

Many quadratic equations can be solved by factoring and employing the zero-product property, as in *section F4*.

For example, the equation  $x^2 + 2x - 3 = 0$  can be solved as follows:

$$(x + 3)(x - 1) = 0$$

so, by zero-product property,

$$x + 3 = 0 \text{ or } x - 1 = 0,$$

which gives us the solutions

$$x = -3 \text{ or } x = 1.$$

## Solving by Using the Square Root Property

Quadratic equations of the form  $ax^2 + c = 0$  can be solved by applying the **square root property**.

### Square Root Property:

For any positive real number  $a$ , if  $x^2 = a$ , then  $x = \pm\sqrt{a}$ .

This is because  $\sqrt{x^2} = |x|$ . So, after applying the square root operator to both sides of the equation  $x^2 = a$ , we have

$$\begin{aligned}\sqrt{x^2} &= \sqrt{a} \\ |x| &= \sqrt{a} \\ x &= \pm\sqrt{a}\end{aligned}$$

The  $\pm\sqrt{a}$  is a shorter recording of two solutions:  $\sqrt{a}$  and  $-\sqrt{a}$ .

For example, the equation  $49x^2 - 4 = 0$  can be solved as follows:

$$49x^2 - 4 = 0 \quad / +4$$

$$49x^2 = 4 \quad / \div 49$$

$$x^2 = \frac{4}{49}$$

$$\sqrt{x^2} = \sqrt{\frac{4}{49}}$$

$$x = \pm\sqrt{\frac{4}{49}}$$

$$x = \pm\frac{2}{7}$$

Here we use the square root property. Remember the  $\pm$  sign!

apply square root to both sides of the equation

**Note:** Using the square root property is a common solving strategy for quadratic equations where **one side is a perfect square** of an unknown quantity and the **other side is a constant** number.

**Example 1** ▶ **Solve by the Square Root Property**

Solve each equation using the square root property.

a.  $(x - 3)^2 = 49$

b.  $2(3x - 6)^2 - 54 = 0$

**Solution** ▶ a. Applying the square root property, we have

$$\sqrt{(x - 3)^2} = \sqrt{49}$$

$$x - 3 = \pm 7 \quad / +3$$

$$x = 3 \pm 7$$

so

$$x = 10 \text{ or } x = -4$$

b. To solve  $2(3x - 6)^2 - 54 = 0$ , we isolate the perfect square first and then apply the square root property. So,

$$2(3x - 6)^2 - 54 = 0 \quad / +54, \div 2$$

$$(3x - 6)^2 = \frac{54}{2}$$

$$\sqrt{(3x - 6)^2} = \sqrt{27}$$

$$3x - 6 = \pm 3\sqrt{3} \quad / +6$$

$$3x = 6 \pm 3\sqrt{3} \quad / \div 3$$

$$x = \frac{6 \pm 3\sqrt{3}}{3}$$

$$x = \frac{3(2 \pm \sqrt{3})}{3}$$

$$x = 2 \pm \sqrt{3}$$

Thus, the solution set is  $\{2 - \sqrt{3}, 2 + \sqrt{3}\}$ .**Caution:** To simplify expressions such as  $\frac{6+3\sqrt{3}}{3}$ , we **factor the numerator** first. The common errors to avoid are

*incorrect order of operations* ←  $\frac{6+3\sqrt{3}}{3} = \frac{9\sqrt{3}}{3} = 3\sqrt{3}$

or

*incorrect canceling* ←  $\frac{6+3\sqrt{3}}{3} = 6 + \sqrt{3}$

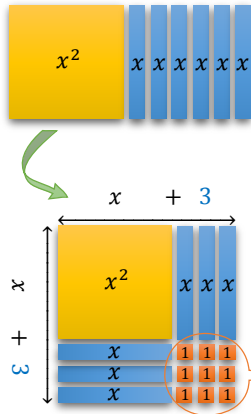
or

*incorrect canceling* ←  $\frac{6+3\sqrt{3}}{3} = 2 + 3\sqrt{3}$

## Solving by Completing the Square

So far, we have seen how to solve quadratic equations,  $ax^2 + bx + c = 0$ , if the expression  $ax^2 + bx + c$  is factorable or if the coefficient  $b$  is equal to zero. To solve other quadratic equations, we may try to rewrite the variable terms in the form of a perfect square, so that the resulting equation can already be solved by the square root property.

For example, to solve  $x^2 + 6x - 3 = 0$ , we observe that the variable terms  $x^2 + 6x$  could be written in **perfect square** form if we add 9, as illustrated in *Figure 1.2*. This is because



$$x^2 + 6x + 9 = (x + 3)^2$$

observe that 3 comes from taking half of 6

Since the original equation can only be changed to an equivalent form, if we add 9, we must subtract 9 as well. (Alternatively, we could add 9 to both sides of the equation.) So, the equation can be transformed as follows:

**Completing the Square Procedure**

$$x^2 + 6x - 3 = 0 \quad / +12$$

$$\underbrace{x^2 + 6x + 9}_{\text{perfect square}} - 9 - 3 = 0$$

$$(x + 3)^2 = 12$$

square root property

$$\sqrt{(x + 3)^2} = \sqrt{12}$$

$$x + 3 = \pm 2\sqrt{3}$$

$$x = -3 \pm 2\sqrt{3}$$

Figure 1.2

Generally, to **complete the square** for the first two terms of the equation

$$x^2 + bx + c = 0,$$

we take **half of the  $x$ -coefficient**, which is  $\frac{b}{2}$ , and **square it**. Then, we **add** and **subtract** that number,  $\left(\frac{b}{2}\right)^2$ . (Alternatively, we could add  $\left(\frac{b}{2}\right)^2$  to both sides of the equation.) This way, we produce an equivalent equation

$$x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c = 0,$$

and consequently,

$$\left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4} + c = 0.$$

We can write this equation directly, by following the rule:

**Write the sum of  $x$  and half of the middle coefficient, square the binomial, and subtract the perfect square of the constant appearing in the bracket.**

To **complete the square** for the first two terms of a quadratic equation with a leading coefficient of  $a \neq 1$ ,

$$ax^2 + bx + c = 0,$$

we

- divide the equation by  $a$  (alternatively, we could factor  $a$  out of the first two terms) so that the leading coefficient is 1, and then
- complete the square as in the previous case, where  $a = 1$ .

So, after division by  $a$ , we obtain

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

Since half of  $\frac{b}{a}$  is  $\frac{b}{2a}$ , then we complete the square as follows:

$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0.$$

Remember to **subtract the perfect square of the constant** appearing in the bracket!

### Example 2 ▶ Solve by Completing the Square

Solve each equation using the completing the square method.

a.  $x^2 + 5x - 1 = 0$

b.  $3x^2 - 12x - 5 = 0$

- Solution** ▶ a. First, we complete the square for  $x^2 + 5x$  by adding and subtracting  $\left(\frac{5}{2}\right)^2$  and then we apply the square root property. So, we have

$$\underbrace{x^2 + 5x + \left(\frac{5}{2}\right)^2}_{\text{perfect square}} - \left(\frac{5}{2}\right)^2 - 1 = 0$$

$$\left(x + \frac{5}{2}\right)^2 - \frac{25}{4} - 1 \cdot \frac{4}{4} = 0$$

$$\left(x + \frac{5}{2}\right)^2 - \frac{29}{4} = 0 \quad / + \frac{29}{4}$$

apply square root to both sides of the equation

$$\left(x + \frac{5}{2}\right)^2 = \frac{29}{4}$$

$$x + \frac{5}{2} = \pm \sqrt{\frac{29}{4}}$$

remember to use the  $\pm$  sign!

$$x + \frac{5}{2} = \pm \frac{\sqrt{29}}{2} \quad / -\frac{5}{2}$$

$$x = \frac{-5 \pm \sqrt{29}}{2}$$

Thus, the solution set is  $\left\{\frac{-5-\sqrt{29}}{2}, \frac{-5+\sqrt{29}}{2}\right\}$ .

**Note:** Unless specified otherwise, we are expected to state the **exact solutions** rather than their calculator approximations. Sometimes, however, especially when solving application problems, we may need to use a calculator to approximate the solutions. The reader is encouraged to check that the two decimal **approximations** of the above solutions are

$$\frac{-5-\sqrt{29}}{2} \approx -5.19 \quad \text{and} \quad \frac{-5+\sqrt{29}}{2} \approx 0.19$$

- b. In order to apply the strategy as in the previous example, we divide the equation by the leading coefficient, 3. So, we obtain

$$3x^2 - 12x - 5 = 0 \quad / \div 3$$

$$x^2 - 4x - \frac{5}{3} = 0$$

Then, to complete the square for  $x^2 - 4x$ , we may add and subtract 4. This allows us to rewrite the equation equivalently, with the variable part in perfect square form.

$$(x - 2)^2 - 4 - \frac{5}{3} = 0$$

$$(x - 2)^2 = 4 \cdot \frac{3}{3} + \frac{5}{3}$$

$$(x - 2)^2 = \frac{17}{3}$$

$$x - 2 = \pm \sqrt{\frac{17}{3}}$$

$$x = 2 \pm \frac{\sqrt{17}}{\sqrt{3}}$$

**Note:** The final answer could be written as a single fraction as shown below:

$$x = \frac{2\sqrt{3} \pm \sqrt{17}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{6 \pm \sqrt{51}}{3}$$

## Solving with Quadratic Formula

Applying the completing the square procedure to the quadratic equation

$$ax^2 + bx + c = 0,$$

with real coefficients  $a \neq 0$ ,  $b$ , and  $c$ , allows us to develop a general formula for finding the solution(s) to any such equation.

**Quadratic Formula**

▶ The solution(s) to the equation  $ax^2 + bx + c = 0$ , where  $a \neq 0$ ,  $b$ ,  $c$  are real coefficients, are given by the formula

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Here  $x_{1,2}$  denotes the two solutions,  $x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ , and  $x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$ .

**Proof:**

▶ First, since  $a \neq 0$ , we can divide the equation  $ax^2 + bx + c = 0$  by  $a$ . So, the equation to solve is

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Then, we complete the square for  $x^2 + \frac{b}{a}x$  by adding and subtracting the perfect square of half of the middle coefficient,  $\left(\frac{b}{2a}\right)^2$ . So, we obtain

$$x^2 + \frac{b}{a}x + \underbrace{\left(\frac{b}{2a}\right)^2}_{\text{perfect square}} - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0 \quad / + \left(\frac{b}{2a}\right)^2, -\frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} \cdot \frac{4a}{4a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a} \quad / -\frac{b}{2a}$$

and finally,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

which concludes the proof.

**QUADRATIC FORMULA****Example 3****Solving Quadratic Equations with the Use of the Quadratic Formula**

Using the Quadratic Formula, solve each equation, if possible. Then visualize the solutions graphically.

a.  $2x^2 + 3x - 20 = 0$

b.  $3x^2 - 4 = 2x$

c.  $x^2 - \sqrt{2}x + 3 = 0$

**Solution**

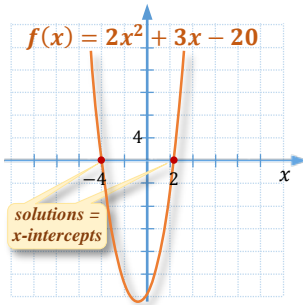
- a. To apply the quadratic formula, first, we identify the values of  $a$ ,  $b$ , and  $c$ . Since the equation is in standard form,  $a = 2$ ,  $b = 3$ , and  $c = -20$ . The solutions are equal to

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{3^2 - 4 \cdot 2(-20)}}{2 \cdot 2} = \frac{-3 \pm \sqrt{9 + 160}}{4}$$

$$= \frac{-3 \pm 13}{4} = \begin{cases} \frac{-3 + 13}{4} = \frac{10}{4} = \frac{5}{2} \\ \frac{-3 - 13}{4} = \frac{-16}{4} = -4 \end{cases}$$

Thus, the solution set is  $\{-4, \frac{5}{2}\}$ .

These solutions can be seen as  $x$ -intercepts of the function  $f(x) = 2x^2 + 3x - 20$ , as shown in *Figure 1.3*.



**Figure 1.3**

- b. Before we identify the values of  $a$ ,  $b$ , and  $c$ , we need to write the given equation  $3x^2 - 4 = 2x$  in standard form. After subtracting  $4x$  from both sides of the given equation, we obtain

$$3x^2 - 2x - 4 = 0$$

Since  $a = 3$ ,  $b = -2$ , and  $c = -4$ , we evaluate the quadratic formula,

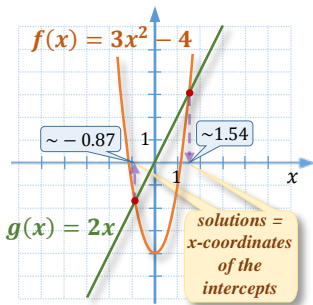
$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{(-2)^2 - 4 \cdot 3(-4)}}{2 \cdot 3} = \frac{2 \pm \sqrt{4 + 48}}{6} = \frac{2 \pm \sqrt{52}}{6}$$

$$= \frac{2 \pm \sqrt{4 \cdot 13}}{6} = \frac{2 \pm 2\sqrt{13}}{6} = \frac{2(1 \pm \sqrt{13})}{6} = \frac{1 \pm \sqrt{13}}{3}$$

So, the solution set is  $\{\frac{1-\sqrt{13}}{3}, \frac{1+\sqrt{13}}{3}\}$ .

*simplify by factoring*

We may visualize solutions to the original equation,  $3x^2 - 4 = 2x$ , by graphing functions  $f(x) = 3x^2 - 4$  and  $g(x) = 2x$ . The  $x$ -coordinates of the intersection points are the solutions to the equation  $f(x) = g(x)$ , and consequently to the original equation. As indicated in *Figure 1.4*, the approximations of these solutions are  $\frac{1-\sqrt{13}}{3} \approx -0.87$  and  $\frac{1+\sqrt{13}}{3} \approx 1.54$ .



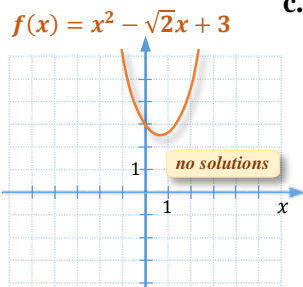
**Figure 1.4**

- c. Substituting  $a = 1$ ,  $b = -\sqrt{2}$ , and  $c = 3$  into the Quadratic Formula, we obtain

$$x_{1,2} = \frac{\sqrt{2} \pm \sqrt{(-\sqrt{2})^2 - 4 \cdot 1 \cdot 3}}{2 \cdot 1} = \frac{\sqrt{2} \pm \sqrt{4 - 12}}{2} = \frac{\sqrt{2} \pm \sqrt{-8}}{2}$$

*not a real number!*

Since a square root of a negative number is not a real value, we have **no real solutions**. Thus the solution set to this equation is  $\emptyset$ . In a graphical representation, this means that the graph of the function  $f(x) = x^2 - \sqrt{2}x + 3$  does not cross the  $x$ -axis. See *Figure 1.5*.



**Figure 1.5**



**Observation:** Notice that we could find the solution set in *Example 3c* just by evaluating the radicand  $b^2 - 4ac$ . Since this radicand was negative, we concluded that there is no solution to the given equation as a root of a negative number is not a real number. There was no need to evaluate the whole expression of Quadratic Formula.

So, the radicand in the Quadratic Formula carries important information about the number and nature of roots. Because of it, this radicand earned a special name, the discriminant.

**Definition 1.1** ▶ The radicand  $b^2 - 4ac$  in the Quadratic Formula is called the **discriminant** and it is denoted by  $\Delta$ .

Notice that in terms of  $\Delta$ , the Quadratic Formula takes the form

$$x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a}$$

Observing the behaviour of the expression  $\sqrt{\Delta}$  allows us to classify the number and type of solutions (roots) of a quadratic equation with rational coefficients.

### Characteristics of Roots (Solutions) Depending on the Discriminant

Suppose  $ax^2 + bx + c = 0$  has **rational** coefficients  $a \neq 0$ ,  $b$ ,  $c$ , and  $\Delta = b^2 - 4ac$ .

- ▶ If  $\Delta < 0$ , then the equation has **no real solutions**, as  $\sqrt{\text{negative}}$  is not a real number.
- ▶ If  $\Delta = 0$ , then the equation has **one rational solution**,  $\frac{-b}{2a}$ .
- ▶ If  $\Delta > 0$ , then the equation has **two solutions**,  $\frac{-b - \sqrt{\Delta}}{2a}$  and  $\frac{-b + \sqrt{\Delta}}{2a}$ .

These solutions are

- **irrational**, if  $\Delta$  is **not a perfect square number**
- **rational**, if  $\Delta$  is a **perfect square number** (as  $\sqrt{\text{perfect square}} = \text{integer}$ )

In addition, if  $\Delta \geq 0$  is a **perfect square number**, then the equation could be solved by **factoring**.

### Example 4 ▶ Determining the Number and Type of Solutions of a Quadratic Equation

Using the discriminant, determine the number and type of solutions of each equation without solving the equation. If the equation can be solved by factoring, show the factored form of the trinomial.

a.  $2x^2 + 7x - 15 = 0$

b.  $4x^2 - 12x + 9 = 0$

c.  $3x^2 - x + 1 = 0$

d.  $2x^2 - 7x + 2 = 0$

**Solution**

▶ a.  $\Delta = 7^2 - 4 \cdot 2 \cdot (-15) = 49 + 120 = 169$

Since 169 is a perfect square number, the equation has **two rational solutions** and it can be solved by factoring. Indeed,  $2x^2 + 7x - 15 = (2x - 3)(x + 5)$ .

b.  $\Delta = (-12)^2 - 4 \cdot 4 \cdot 9 = 144 - 144 = 0$

$\Delta = 0$  indicates that the equation has **one rational solution** and it can be solved by factoring. Indeed, the expression  $4x^2 - 12x + 9$  is a perfect square,  $(2x - 3)^2$ .

c.  $\Delta = (-1)^2 - 4 \cdot 3 \cdot 1 = 1 - 12 = -11$

Since  $\Delta < 0$ , the equation has **no real solutions** and therefore it can not be solved by factoring.

d.  $\Delta = (-7)^2 - 4 \cdot 2 \cdot 2 = 49 - 16 = 33$

Since  $\Delta > 0$  but it is not a perfect square number, the equation has **two real solutions** but it cannot be solved by factoring.

**Example 5**

▶ **Solving Equations Equivalent to Quadratic**

Solve each equation.

a.  $2 + \frac{7}{x} = \frac{5}{x^2}$

b.  $2x^2 = (x + 2)(x - 1) + 1$

**Solution**

▶ a. This is a rational equation, with the set of  $\mathbb{R} \setminus \{0\}$  as its domain. To solve it, we multiply the equation by the  $LCD = x^2$ . This brings us to a quadratic equation

$$2x^2 + 7x = 5$$

or equivalently

$$2x^2 + 7x - 5 = 0,$$

which can be solved by following the Quadratic Formula for  $a = 2$ ,  $b = 7$ , and  $c = -5$ . So, we have

$$x_{1,2} = \frac{-7 \pm \sqrt{(-7)^2 - 4 \cdot 2(-5)}}{2 \cdot 2} = \frac{-7 \pm \sqrt{49 + 40}}{4} = \frac{-7 \pm \sqrt{89}}{4}$$

Since both solutions are in the domain, the solution set is  $\left\{ \frac{-7 - \sqrt{89}}{4}, \frac{-7 + \sqrt{89}}{4} \right\}$

b. To solve  $1 - 2x^2 = (x + 2)(x - 1)$ , we simplify the equation first and rewrite it in standard form. So, we have

$$1 - 2x^2 = x^2 + x - 2 \quad / -x, +2$$

$$-3x^2 - x + 3 = 0 \quad / \cdot (-1)$$

$$3x^2 + x - 3 = 0$$

Since the left side of this equation is not factorable, we may use the Quadratic Formula. So, the solutions are

$$x_{1,2} = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 3(-3)}}{2 \cdot 3} = \frac{1 \pm \sqrt{1 + 36}}{6} = \frac{1 \pm \sqrt{37}}{6}.$$

## Q.1 Exercises

**Concept Check** True or False.

1. A quadratic equation is an equation that can be written in the form  $ax^2 + bx + c = 0$ , where  $a$ ,  $b$ , and  $c$  are any real numbers.
2. If the graph of  $f(x) = ax^2 + bx + c$  intersects the  $x$ -axis twice, the equation  $ax^2 + bx + c = 0$  has two solutions.
3. If the equation  $ax^2 + bx + c = 0$  has no solution, the graph of  $f(x) = ax^2 + bx + c$  does not intersect the  $x$ -axis.
4. The Quadratic Formula cannot be used to solve the equation  $x^2 - 5 = 0$  because the equation does not contain a linear term.
5. The solution set for the equation  $x^2 = 16$  is  $\{4\}$ .
6. To complete the square for  $x^2 + bx$ , we add  $\left(\frac{b}{2}\right)^2$ .
7. If the discriminant is positive, the equation can be solved by factoring.

**Concept Check**

For each function  $f$ ,

- a) graph  $f(x)$  using a table of values;
- b) find the  $x$ -intercepts of the graph;
- c) solve the equation  $f(x) = 0$  by factoring and compare these solutions to the  $x$ -intercepts of the graph.

- |                            |                            |   |
|----------------------------|----------------------------|---|
| 8. $f(x) = -x^2 - 3x + 2$  | 9. $f(x) = x^2 + 2x - 3$   | 10. $f(x) = 3x + x(x - 2)$                |
| 11. $f(x) = 2x - x(x - 3)$ | 12. $f(x) = 4x^2 - 4x - 3$ | 13. $f(x) = -\frac{1}{2}(2x^2 + 5x - 12)$ |

Solve each equation using the **square root property**.

- |                    |                     |                      |
|--------------------|---------------------|----------------------|
| 14. $x^2 = 49$     | 15. $x^2 = 32$      | 16. $a^2 - 50 = 0$   |
| 17. $n^2 - 24 = 0$ | 18. $3x^2 - 72 = 0$ | 19. $5y^2 - 200 = 0$ |

20. $(x - 4)^2 = 64$	21. $(x + 3)^2 = 16$	22. $(3n - 1)^2 = 7$
23. $(5t + 2)^2 = 12$	24. $x^2 - 10x + 25 = 45$	25. $y^2 + 8y + 16 = 44$
26. $4a^2 + 12a + 9 = 32$	27. $25(y - 10)^2 = 36$	28. $16(x + 4)^2 = 81$
29. $(4x + 3)^2 = -25$	30. $(3n - 2)(3n + 2) = -5$	31. $2x - 1 = \frac{18}{2x-1}$

Solve each equation using the **completing the square** procedure.

32. $x^2 + 12x = 0$	33. $y^2 - 3y = 0$	34. $x^2 - 8x + 2 = 0$
35. $n^2 + 7n = 3n - 4$	36. $p^2 - 4p = 4p - 16$	37. $y^2 + 7y - 1 = 0$
38. $2x^2 - 8x = -4$	39. $3a^2 + 6a = -9$	40. $3y^2 - 9y + 15 = 0$
41. $5x^2 - 60x + 80 = 0$	42. $2t^2 + 6t - 10 = 0$	43. $3x^2 + 2x - 2 = 0$
44. $2x^2 - 16x + 25 = 0$	45. $9x^2 - 24x = -13$	46. $25n^2 - 20n = 1$
47. $x^2 - \frac{4}{3}x = -\frac{1}{9}$	48. $x^2 + \frac{5}{2}x = -1$	49. $x^2 - \frac{2}{5}x - 3 = 0$

Given  $f(x)$  and  $g(x)$ , find all values of  $x$  for which  $f(x) = g(x)$ .

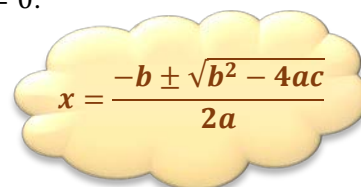
50. $f(x) = x^2 - 9$ and $g(x) = 4x - 6$	51. $f(x) = 2x^2 - 5x$ and $g(x) = -x + 14$
--	---

### Discussion Point

52. Explain the errors in the following solutions of the equation  $5x^2 - 8x + 2 = 0$ :

a.  $x = \frac{8 \pm \sqrt{-8^2 - 4 \cdot 5 \cdot 2}}{2 \cdot 8} = \frac{8 \pm \sqrt{64 - 40}}{16} = \frac{8 \pm \sqrt{24}}{16} = \frac{8 \pm 2\sqrt{6}}{16} = \frac{1}{2} \pm 2\sqrt{6}$

b.  $x = \frac{8 \pm \sqrt{(-8)^2 - 4 \cdot 5 \cdot 2}}{2 \cdot 8} = \frac{8 \pm \sqrt{64 - 40}}{16} = \frac{8 \pm \sqrt{24}}{16} = \frac{8 \pm 2\sqrt{6}}{16} = \begin{cases} \frac{10\sqrt{6}}{16} = \frac{5\sqrt{6}}{8} \\ \frac{6\sqrt{6}}{16} = \frac{3\sqrt{6}}{8} \end{cases}$



$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Solve each equation with the aid of the **Quadratic Formula**, if possible. Illustrate your solutions graphically, using a table of values or a graphing utility.

53. $x^2 + 3x + 2 = 0$	54. $y^2 - 2 = y$	55. $x^2 + x = -3$
56. $2y^2 + 3y = -2$	57. $x^2 - 8x + 16 = 0$	58. $4n^2 + 1 = 4n$

Solve each equation with the aid of the **Quadratic Formula**. Give the **exact** and **approximate** solutions up to two decimal places.

59. $a^2 - 4 = 2a$	60. $2 - 2x = 3x^2$	61. $0.2x^2 + x + 0.7 = 0$
62. $2t^2 - 4t + 2 = 3$	63. $y^2 + \frac{y}{3} = \frac{1}{6}$	64. $\frac{x^2}{4} - \frac{x}{2} = 1$
65. $5x^2 = 17x - 2$	66. $15y = 2y^2 + 16$	67. $6x^2 - 8x = 2x - 3$

**Concept Check** Use the discriminant to determine the **number and type of solutions** for each equation. Also, decide whether the equation can be solved by **factoring** or whether the quadratic formula should be used. Do not actually solve.

68.  $3x^2 - 5x - 2 = 0$

69.  $4x^2 = 4x + 3$

70.  $x^2 + 3 = -2\sqrt{3}x$

71.  $4y^2 - 28y + 49 = 0$

72.  $3y^2 - 10y + 15 = 0$

73.  $9x^2 + 6x = -1$

Find the value(s) of the constant  $k$ , so that each equation will have exactly one rational solution.

74.  $x^2 + ky + 49 = 0$

75.  $9y^2 - 30y + k = 0$

76.  $kx^2 + 8x + 1 = 0$

### Discussion Point

77. Is it possible for the solution of a quadratic equation with integral coefficients to include only one irrational number? Why or why not?

Solve each equation using any algebraic method. State the solutions in their exact form.

78.  $-2x(x + 2) = -3$

79.  $(x + 2)(x - 4) = 1$

80.  $(x + 2)(x + 6) = 8$

81.  $(2x - 3)^2 = 8(x + 1)$

82.  $(3x + 1)^2 = 2(1 - 3x)$

83.  $2x^2 - (x + 2)(x - 3) = 12$

84.  $(x - 2)^2 + (x + 1)^2 = 0$

85.  $1 + \frac{2}{x} + \frac{5}{x^2} = 0$

86.  $x = \frac{2(x+3)}{x+5}$

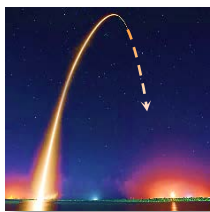
87.  $2 + \frac{1}{x} = \frac{3}{x^2}$

88.  $\frac{3}{x} + \frac{x}{3} = \frac{5}{2}$

89.  $\frac{1}{x} + \frac{1}{x+4} = \frac{1}{7}$

## Q.2

## Applications of Quadratic Equations



Some polynomial, rational or even radical equations are **quadratic in form**. As such, they can be solved using techniques described in the previous section. For instance, the rational equation  $\frac{1}{x^2} + \frac{1}{x} - 6 = 0$  is quadratic in form because if we replace  $\frac{1}{x}$  with a single variable, say  $a$ , then the equation becomes quadratic,  $a^2 + a - 6 = 0$ . In this section, we explore applications of quadratic equations in solving equations quadratic in form as well as solving formulas containing variables in the second power.

We also revisit application problems that involve solving quadratic equations. Some of the application problems that are typically solved with the use of quadratic or polynomial equations were discussed in *sections F4* and *RT6*. However, in the previous sections, the equations used to solve such problems were all possible to solve by factoring. In this section, we include problems that require the use of methods other than factoring.

## Equations Quadratic in Form

**Definition 2.1** ▶ A nonquadratic equation is referred to as **quadratic in form** or **reducible to quadratic** if it can be written in the form

$$au^2 + bu + c = 0,$$

where  $a \neq 0$  and  $u$  represents any *algebraic expression*.

Equations quadratic in form are usually easier to solve by using strategies for solving the related quadratic equation  $au^2 + bu + c = 0$  for the expression  $u$ , and then solve for the original variable, as shown in the example below.

**Example 1** ▶ Solving Equations Quadratic in Form

Solve each equation.

a.  $(x^2 - 1)^2 - (x^2 - 1) = 2$                       b.  $x - 3\sqrt{x} = 10$

c.  $\frac{1}{(a+2)^2} + \frac{1}{a+2} - 6 = 0$

**Solution** ▶ a. First, observe that the expression  $x^2 - 1$  appears in the given equation in the first and second power. So, it may be useful to replace  $x^2 - 1$  with a new variable, for example  $u$ . After this substitution, the equation becomes quadratic,

$$u^2 - u = 2, \quad / -2$$

and can be solved via factoring

$$u^2 - u - 2 = 0$$

$$(u - 2)(u + 1) = 0$$

$$u = 2 \text{ or } u = -1$$

Since we need to solve the original equation for  $x$ , not for  $u$ , we replace  $u$  back with  $x^2 - 1$ . This gives us

This can be any letter, as long as it is different than the original variable.

$$x^2 - 1 = 2 \quad \text{or} \quad x^2 - 1 = -1$$

$$x^2 = 3 \quad \text{or} \quad x^2 = 0$$

$$x = \pm\sqrt{3} \quad \text{or} \quad x = 0$$

Thus, the solution set is  $\{-\sqrt{3}, 0, \sqrt{3}\}$ .

- b. If we replace  $\sqrt{x}$  with, for example,  $a$ , then  $x = a^2$ , and the equation becomes

$$a^2 - 3a = 10, \quad / -10$$

which can be solved by factoring

$$a^2 - 3a - 10 = 0$$

$$(a + 2)(a - 5) = 0$$

$$a = -2 \quad \text{or} \quad a = 5$$

After replacing  $a$  back with  $\sqrt{x}$ , we have

$$\sqrt{x} = -2 \quad \text{or} \quad \sqrt{x} = 5.$$

The first equation,  $\sqrt{x} = -2$ , does not give us any solution as the square root cannot be negative. After squaring both sides of the second equation, we obtain  $x = 25$ . So, the solution set is **{25}**.

- c. The equation  $\frac{1}{(a+2)^2} + \frac{1}{a+2} - 6 = 0$  can be solved as any other rational equation, by clearing the denominators via multiplying by the  $LCD = (a + 2)^2$ . However, it can also be seen as a quadratic equation as soon as we replace  $\frac{1}{a+2}$  with, for example,  $x$ . By doing so, we obtain

$$x^2 + x - 6 = 0,$$

which after factoring

$$(x + 3)(x - 2) = 0,$$

gives us

$$x = -3 \quad \text{or} \quad x = 2$$

Remember to use a different letter than the variable in the original equation.

Again, since we need to solve the original equation for  $a$ , we replace  $x$  back with  $\frac{1}{a+2}$ . This gives us

$$\frac{1}{a+2} = -3 \quad \text{or} \quad \frac{1}{a+2} = 2 \quad / \text{ take the reciprocal of both sides}$$

$$a + 2 = \frac{1}{-3} \quad \text{or} \quad a + 2 = \frac{1}{2} \quad / -2$$

$$a = -\frac{7}{3} \quad \text{or} \quad a = -\frac{3}{2}$$

Since both values are in the domain of the original equation, which is  $\mathbb{R} \setminus \{0\}$ , then the solution set is  $\{-\frac{7}{3}, -\frac{3}{2}\}$ .

## Solving Formulas

When solving formulas for a variable that appears in the second power, we use the same strategies as in solving quadratic equations. For example, we may use the square root property or the quadratic formula.

**Example 2** ▶ **Solving Formulas for a Variable that Appears in the Second Power**

Solve each formula for the given variable.

a.  $E = mc^2$ , for  $c$

b.  $N = \frac{k^2 - 3k}{2}$ , for  $k$

**Solution** ▶

- a. To solve for  $c$ , first, we reverse the multiplication by  $m$  via the division by  $m$ . Then, we reverse the operation of squaring by taking the square root of both sides of the equation.

$$E = mc^2 \quad / \div m$$

$$\frac{E}{m} = c^2$$

Then, we reverse the operation of squaring by taking the square root of both sides of the equation. So, we have

$$\sqrt{\frac{E}{m}} = \sqrt{c^2},$$

Remember that  
 $\sqrt{c^2} = |c|$ , so we  
 use the  $\pm$  sign in  
 place of  $| |$ .

and therefore

$$c = \pm \sqrt{\frac{E}{m}}$$

- b. Observe that the variable  $k$  appears in the formula  $N = \frac{k^2 - 3k}{2}$  in two places. Once in the first and once in the second power of  $k$ . This means that we can treat this formula as a quadratic equation with respect to  $k$  and solve it with the aid of the quadratic formula. So, we have

$$N = \frac{k^2 - 3k}{2} \quad / \cdot 2$$

$$2N = k^2 - 3k \quad / -2N$$

$$k^2 - 3k - 2N = 0$$

Substituting  $a = 1$ ,  $b = -3$ , and  $c = -2N$  to the quadratic formula, we obtain

$$k_{1,2} = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \cdot 1 \cdot (-2N)}}{2} = \frac{3 \pm \sqrt{9 + 8N}}{2}$$



## Application Problems

Many application problems require solving quadratic equations. Sometimes this can be achieved via factoring, but often it is helpful to use the quadratic formula.

### Example 3 ▶ Solving a Distance Problem with the Aid of the Quadratic Formula



Three towns  $A$ ,  $B$ , and  $C$  are situated as shown in the accompanying figure. The roads at  $A$  form a right angle. The towns  $A$  and  $C$  are connected by a straight road as well. The distance from  $A$  to  $B$  is 7 kilometers less than the distance from  $B$  to  $C$ . The distance from  $A$  to  $C$  is 20 km. Approximate the remaining distances between the towns up to the tenth of a kilometer.

**Solution** ▶ Since the roads between towns form a right triangle, we can employ the Pythagorean equation

$$AC^2 = AB^2 + BC^2$$

Suppose that  $BC = x$ . Then  $AB = x - 7$ , and we have

$$20^2 = (x - 7)^2 + x^2$$

$$400 = x^2 - 14x + 49 + x^2$$

$$2x^2 - 14x - 351 = 0$$

Applying the quadratic formula, we obtain

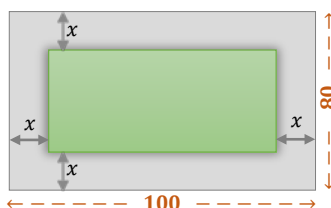
$$x_{1,2} = \frac{14 \pm \sqrt{14^2 + 4 \cdot 2 \cdot 351}}{4} = \frac{14 \pm \sqrt{196 + 2808}}{4} = \frac{14 \pm \sqrt{3004}}{4} \approx 17.2 \text{ or } -10.2$$

Since  $x$  represents a distance, it must be positive. So, the only solution is  $x \approx 17.2$  km. Thus, the distance  $BC \approx 17.2$  km and hence  $AB \approx 17.2 - 7 = 10.2$  km.

### Example 4 ▶ Solving a Geometry Problem with the Aid of the Quadratic Formula

A rectangular garden is 80 ft by 100 ft. Part of the garden is torn up to install a sidewalk of uniform width around it. The area of the new garden is  $\frac{2}{3}$  of the old area. Approximately how wide is the sidewalk?

**Solution** ▶ To visualize the situation, we may draw a diagram as below.



Suppose  $x$  represents the width of the sidewalk. Then, the area of the new garden (the green area) can be expressed as  $(100 - 2x)(80 - 2x)$ . Since the green area is  $\frac{2}{3}$  of the gray area (the original garden), we can form the equation

$$(100 - 2x)(80 - 2x) = \frac{2}{3}(100 \cdot 80)$$

To solve it, we may want to lower the coefficients by dividing both sides of the equation by 4 first. This gives us

$$\frac{\cancel{2}(50 - x) \cdot \cancel{2}(40 - x)}{\cancel{4}} = \frac{2}{3} \cdot \frac{(100 \cdot \cancel{80})}{\cancel{4}} \quad / \cdot 3$$

$$3(50 - x)(40 - x) = 4000$$

$$3(2000 - 90x + x^2) = 4000$$

$$3x^2 - 270x + 6000 = 4000 \quad / -2000$$

$$3x^2 - 270x + 2000 = 0,$$

which can be solved using the Quadratic Formula:

$$x_{1,2} = \frac{270 \pm \sqrt{270^2 - 4 \cdot 3 \cdot 2000}}{2 \cdot 3} = \frac{270 \pm \sqrt{48900}}{6} \approx \frac{270 \pm 221.13}{6}$$

$$= \begin{cases} \frac{491.13}{6} = 81.9 \\ \frac{48.87}{6} = 8.1 \end{cases}$$

The width  $x$  must be smaller than 40, so this value is too large to be considered.

Thus, the sidewalk is approximately **8.1** feet wide.

### Example 5

### Solving a Motion Problem That Requires the Use of the Quadratic Formula



The Columbia River flows at a rate of 2 mph for the length of a popular boating route. In order for a boat to travel 3 miles upriver and return in a total of 4 hours, how fast must the boat be able to travel in still water?

### Solution

Suppose the rate of the boat moving in still water is  $r$ . Then,  $r - 2$  represents the rate of the boat moving upriver and  $r + 2$  represents the rate of the boat moving downriver. We can arrange these data in the table below.

	$R$	$T$	$= D$
upriver	$r - 2$	$\frac{3}{r - 2}$	3
downriver	$r + 2$	$\frac{3}{r + 2}$	3
total		4	

We fill the time-column by following the formula  $T = \frac{D}{R}$ .

By adding the times needed for traveling upriver and downriver, we form the rational equation

$$\frac{3}{r-2} + \frac{3}{r+2} = 4, \quad / \cdot (r^2 - 4)$$

which after multiplying by the  $LCD = r^2 - 4$  becomes a quadratic equation.

$$\begin{aligned}
 3(r + 2) + 3(r - 2) &= 4(r^2 - 4) \\
 3r + 6 + 3r - 6 &= 4r^2 - 16 && / \cdot -6r \\
 0 &= 4r^2 - 6r - 16 && / \div 2 \\
 0 &= 2r^2 - 3r - 8
 \end{aligned}$$

Using the Quadratic Formula, we have

$$r_{1,2} = \frac{3 \pm \sqrt{(-3)^2 + 4 \cdot 2 \cdot 8}}{2 \cdot 2} = \frac{3 \pm \sqrt{9 + 64}}{4} = \frac{3 \pm \sqrt{73}}{4} \approx \begin{cases} 2.9 \\ -1.4 \end{cases}$$

Since the rate cannot be negative, the boat moves in still water at approximately **2.9 mph**.

### Example 6 ▶ Solving a Work Problem That Requires the Use of the Quadratic Formula

John can work through a stack of invoices in 1 hr less time than Bob can. Working together, they take 1.5 hr. To the nearest tenth of an hour, how long would it take each person working alone?

**Solution** ▶ Suppose the time needed for Bob to complete the job is  $t$ . Then,  $t - 1$  represents the time needed for John to complete the same job. Now, we can arrange the given data in a table, as below.

	$R$	$T$	$= Job$
Bob	$\frac{1}{t}$	$t$	1
John	$\frac{1}{t-1}$	$t-1$	1
together	$\frac{2}{3}$	$\frac{3}{2}$	1

We fill the rate-column by following the formula  $R = \frac{Job}{T}$ .

By adding the rates of work for each person, we form the rational equation

$$\frac{1}{t} + \frac{1}{t-1} = \frac{2}{3}, \quad / \cdot 3t(t-1)$$

which after multiplying by the  $LCD = 3t(t-1)$  becomes a quadratic equation.

$$\begin{aligned}
 3(t-1) + 3t &= 2(t^2 - t) \\
 3t - 3 + 3t &= 2t^2 - 2t && / -6t, +3 \\
 0 &= 2t^2 - 8t + 3
 \end{aligned}$$

Using the Quadratic Formula, we have

$$t_{1,2} = \frac{8 \pm \sqrt{(-8)^2 - 4 \cdot 2 \cdot 3}}{2 \cdot 2} = \frac{8 \pm \sqrt{64 - 24}}{4} = \frac{8 \pm \sqrt{40}}{4} \approx \begin{cases} 3.6 \\ 0.4 \end{cases}$$

Since the time needed for Bob cannot be shorter than 1 hr, we reject the 0.4 possibility. So, working alone, Bob requires approximately 3.6 hours, while John can do the same job in approximately 2.6 hours.

**Example 7** ▶ **Solving a Projectile Problem Using a Quadratic Function**

If an object is projected upward from the top of a 120-ft building at 60 ft/sec, its position above the ground,  $h$  in feet, is given by the function  $h(t) = -16t^2 + 60t + 120$ , where  $t$  is the time in seconds after it was projected. To the nearest tenth of a second, when does the object hit the ground?

**Solution**

▶ The object hits the ground when the height  $h$  above the ground is equal to zero. So, we look for the solutions to the equation

$$h(t) = 0$$

which is equivalent to

$$-16t^2 + 60t + 120 = 0$$

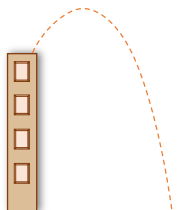
Before applying the Quadratic Formula, we may want to lower the coefficients by dividing both sides of the equation by  $-4$ . So, we have

$$4t^2 - 15t - 30 = 0$$

and

$$t_{1,2} = \frac{15 \pm \sqrt{(-15)^2 + 4 \cdot 4 \cdot 30}}{2 \cdot 4} = \frac{15 \pm \sqrt{225 + 480}}{8} = \frac{15 \pm \sqrt{705}}{8} \approx \begin{cases} 41.6 \\ -1.4 \end{cases}$$

Thus, the object hits the ground in about 41.6 seconds.

**Q.2 Exercises**

**Vocabulary Check** Complete each blank with the most appropriate term or phrase from the given list: *approximate, exact, new, original, Quadratic Formula, quadratic in form, solutions.*

1. An equation is \_\_\_\_\_ if it can be rewritten in the form  $au^2 + bu + c = 0$ , where  $a \neq 0$  and  $u$  represents an algebraic expression.
2. To solve an equation that is quadratic in form, rewrite it in terms of a \_\_\_\_\_ variable. Then, after solving for the new variable, go back to the \_\_\_\_\_ variable.
3. To solve a formula for a variable that appears in the first and second power, we use the \_\_\_\_\_.
4. When solving an application problem with a quadratic equation, check the proposed \_\_\_\_\_ against the information in the original problem.
5. When solving a quadratic equation, we usually leave the answer in the \_\_\_\_\_ form. When solving an application problem, we tend to \_\_\_\_\_ the answer, if needed.

**Discussion Point**

6. Discuss the following solution of the equation  $\left(\frac{1}{x-2}\right)^2 - \frac{1}{x-2} - 2 = 0$ :

Since this equation is quadratic in form, a student solved the related equation  $a^2 - a - 2 = 0$  by factoring

$$(a - 2)(a + 1) = 0.$$

The student found the possible solutions,  $a = 2$  and  $a = -1$ . Since 2 is not in the domain of the original equation, the student gave the final solution set as  $\{-1\}$ .

Solve each equation by treating it as a quadratic in form.

- |   |   |                                |
|---|---|--------------------------------|
| 7. $x^4 - 6x^2 + 9 = 0$   | 8. $x^8 - 29x^4 + 100 = 0$  | 9. $x - 10\sqrt{x} + 9 = 0$    |
| 10. $2x - 9\sqrt{x} + 4 = 0$  | 11. $y^{-2} - 5y^{-1} - 36 = 0$   | 12. $2a^{-2} + a^{-1} - 1 = 0$ |
| 13. $(1 + \sqrt{t})^2 + (1 + \sqrt{t}) - 6 = 0$                         | 14. $(2 + \sqrt{x})^2 - 3(2 + \sqrt{x}) - 10 = 0$                             |                                |
| 15. $(x^2 + 5x)^2 + 2(x^2 + 5x) - 24 = 0$                               | 16. $(t^2 - 2t)^2 - 4(t^2 - 2t) + 3 = 0$                                      |                                |
| 17. $x^{\frac{2}{3}} - 4x^{\frac{1}{3}} - 5 = 0$                        | 18. $x^{\frac{2}{3}} + 2x^{\frac{1}{3}} - 8 = 0$                              |                                |
| 19. $1 - \frac{1}{2p+1} - \frac{1}{(2p+1)^2} = 0$                       | 20. $\frac{2}{(u+2)^2} + \frac{1}{u+2} = 3$                                   |                                |
| 21. $\left(\frac{x+3}{x-3}\right)^2 - \left(\frac{x+3}{x-3}\right) = 6$ | 22. $\left(\frac{y^2-1}{y}\right)^2 - 4\left(\frac{y^2-1}{y}\right) - 12 = 0$ |                                |

Solve each formula for the indicated variable.

- |  |  |  |
|--|--|--|
| 23. $F = \frac{mv^2}{r}$ , for $v$   | 24. $V = \pi r^2 h$ , for $r$              | 25. $A = 4\pi r^2$ , for $r$                   |
| 26. $V = \frac{1}{3}S^2 h$ , for $s$   | 27. $F = \frac{Gm_1 m_2}{r^2}$ , for $r$   | 28. $N = \frac{kq_1 q_2}{s^2}$ , for $s$       |
| 29. $a^2 + b^2 = c^2$ , for $b$  | 30. $I = \frac{703W}{H^2}$ , for $H$       | 31. $A = \pi r^2 + \pi r s$ , for $r$          |
| 32. $A = 2\pi r^2 + 2\pi r h$ , for $r$  | 33. $s = v_0 t + \frac{gt^2}{2}$ , for $t$ | 34. $t = \frac{a}{\sqrt{a^2 + b^2}}$ , for $a$ |
| 35. $P = \frac{A}{(1+r)^2}$ , for $r$  | 36. $P = EI - RI^2$ , for $I$              | 37. $s(6s - t) = t^2$ , for $s$                |
| 38. $m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$ , for $v$ , assuming that $c > 0$ and $m > 0$ |  |  |
| 39. $m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$ , for $c$ , assuming that $v > 0$ and $m > 0$ |  |  |
| 40. $p = \frac{E^2 R}{(r+R)^2}$ , for $E$ , assuming that $(r+R) > 0$                          |  |  |

41. For over 2000 yr, the proportions of a “golden” rectangle have been considered visually appealing. A rectangle of width  $w$  and length  $l$  is considered “golden” if

$$\frac{w}{l} = \frac{l}{w + l}$$

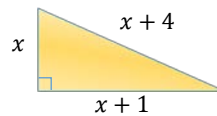
Solve for  $l$ .

**Concept Check** Answer each question.

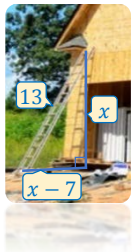
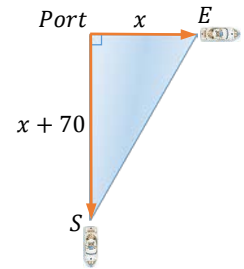
42. A boat goes  $r$  mph in still water, and the rate of the current is  $c$  mph.
- What is the rate of the boat when it travels upstream?
  - What is the rate of the boat when it travels downstream?
43. a. If it takes  $n$  hours to grade a set of papers, what is the grader’s rate in the amount of job completed per hour?  
 b. How much of the job will the grader do in  $h$  hr?

Solve each problem.

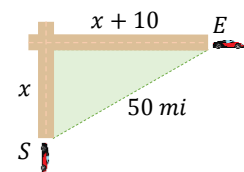
44. Find the lengths of the sides of the triangle.



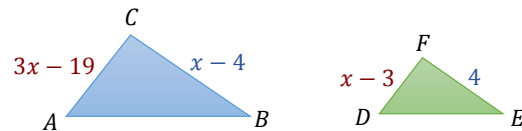
45. Two ships leave port at the same time, one heading due south and the other heading due east. Several hours later, they are 170 mi apart. If the ship traveling south traveled 70 mi farther than the other ship, how many miles did they each travel?
46. The diagonal of a rectangular rug measures 26 ft, and the length is 4 ft more than twice the width. Find the length and width of the rug.
47. The hypotenuse of a right triangle is 25 m long. The length of one leg is 17 m less than the other. Find the lengths of the legs.



48. A 13-ft ladder is leaning against a house. The distance from the bottom of the ladder to the house is 7 ft less than the distance from the top of the ladder to the ground. How far is the bottom of the ladder from the house?
49. Two cars leave an intersection, one traveling south and one traveling east, as shown in the figure to the right. After 1 hour the two cars are 50 miles apart and the car traveling east has traveled 10 miles farther than the car traveling south. How far did each car travel?



50. The width of a rectangular computer screen is 3 inches more than its height. If the area of the screen is 154 square inches, find its dimensions.
51. The length of an American flag that is displayed at a government office is 3 inches less than twice its width. The area is 1710 in<sup>2</sup>. Find the length and the width of the flag.
52. The length of a rectangle is twice the width. The area is 328 cm<sup>2</sup>. Find the exact length and width.
53. Recall that corresponding sides of similar triangles are proportional. The diagram below shows two similar triangles,  $\triangle ABC$  and  $\triangle DEF$ . Given the information in the diagram, find the length  $AC$ . Check all possible solutions. *Caution: The triangles are not drawn to scale here.*



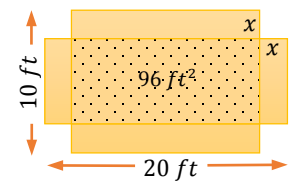
**Analytic Skills** Solve each problem.

54. Ludmila wants to buy a rug for a room that is 20 ft long and 15 ft wide. She wants to leave a strip of uncovered flooring of a uniform width around the edges of the room. How wide would a strip be if she bought a rug with an area of  $234 \text{ ft}^2$ ?



55. A rectangular flower garden in a park is 30 feet wide and 40 feet long. A sidewalk of uniform width is being planned around the perimeter of the garden. The gardener has enough money to install  $624$  square feet of cement sidewalk. Find the width of the sidewalk.

56. An open box is to be made from a 10-ft by 20-ft rectangular piece of cardboard by cutting a square from each corner. The area of the bottom of the box is to be  $96 \text{ ft}^2$ . What is the length of the sides of the squares that are cut from the corners?



57. The outside of a picture frame measures 13 cm by 20 cm, and  $80 \text{ cm}^2$  of the picture shows. Find the width of the frame.
58. A rectangle has a length 2 m less than twice its width. When 5 m are added to the width, the resulting figure is a square with an area of  $144 \text{ m}^2$ . Find the dimensions of the original rectangle.
59. A rectangular piece of sheet metal is 2 inches longer than it is wide. A square piece 3 inches on a side is cut from each corner. The sides are then turned up to form an uncovered box of volume  $765 \text{ in}^3$ . Find the dimensions of the original piece of metal.
60. Karin's motorcycle traveled 300 mi at a certain speed. Had she gone 10 mph faster, she could have made the trip in 1 hr less time. Find her speed.
61. A turbo-jet flies 50 mph faster than a super-prop plane. If a turbo-jet goes 2000 mi in 3 hr less time than it takes the super-prop to go 2800 mi, find the speed of each plane.
62. A Cessna flies 600 mi. A Beechcraft flies 1000 mi at a speed that is 50 mph faster but takes 1 hr longer. If the planes do not fly slower than 200 mph, find the speed of each plane.
63. On a sales trip, Gail drives the 600 mi to Richmond. The return trip is made at a speed that is 10 mph slower. Total travel time for the round trip is 22 hr. How fast did Gail travel on each part of the trip?
64. The current in a typical Mississippi River shipping route flows at a rate of 4 mph. In order for a barge to travel 24 mi upriver and then return in a total of 5 hr, approximately how fast must the barge be able to travel in still water?



65. A camper paddles a canoe 2 miles downstream in a river that has a 2-mile-per-hour current. To return to camp, the canoeist travels upstream on a different branch of the river. It is 4 miles long and has a 1-mile-per-hour current. The total trip (both ways) takes 3 hours. Find the approximate speed of the canoe in still water.
66. One airplane flies east from an airport at an average speed of  $m$  mph. At the same time, a second airplane flies south from the same airport at an average speed that is 42 mph faster than the airplane traveling east.

Thirty minutes after the two airplanes leave, they are 315 mi apart. Find the rate of each airplane to the nearest mile per hour.

67. A pilot flies 500 miles against a 20-mile-per-hour wind. On the next day, the pilot flies back home with a 10-mile-per-hour tailwind. The total trip (both ways) takes 4 hours. Approximate the speed of the airplane without a wind.
68. Working together, two people can clean an office building in 5 hr. One person is new to the job and would take 2 hr longer than the other person to clean the building alone. To the nearest tenth, how long would it take the new worker to clean the building alone?
69. Working together, two people can cut a large lawn in 2 hr. One person can do the job alone in 1 hr less time than the other. To the nearest tenth, how long would it take the faster worker to do the job?
70. Helen and Monica are planting spring flowers. Working alone, Helen would take 2 hr longer than Monica to plant the flowers. Working together, they do the job in 12 hr. Approximately how long would it have taken each person working alone?



71. At Pizza Perfect, Ron can make 100 large pizza crusts in 1.2 hr less than Chad. Together they can do the job in 1.8 hr. To the nearest tenth, how long does it take each to do the job alone?

72. Two pipes together can fill a tank in 2 hr. One of the pipes, used alone, takes 3 hr longer than the other to fill the tank. How long would each pipe take to fill the tank alone?

73. A washing machine can be filled in 6 min if both the hot and cold water taps are fully opened. Filling the washer with hot water alone takes 9 min longer than filling it with cold water alone. How long does it take to fill the washer with cold water?
74. The height  $h$  of a stone  $t$  sec after it is thrown straight downward with an initial velocity of 20 feet per second from a bridge 800 ft above the ground is given by the equation  $h = -16t^2 - 20t + 800$ . When is the stone 300 ft above the ground?
75. A software company's weekly profit,  $P$  (in dollars), for selling  $x$  units of a new video game can be determined by the equation  $P = -0.05x^2 + 48x - 100$ . What is the smaller of the two numbers of units that must be sold in order to make a profit of \$9000?
76. The formula  $A = P(1 + r)^2$  gives the amount  $A$  in dollars that  $P$  dollars will grow to in 2 yr at interest rate  $r$  (where  $r$  is given as a decimal), using compound interest. What interest rate will cause \$2000 to grow to \$2142.45 in 2 yr?



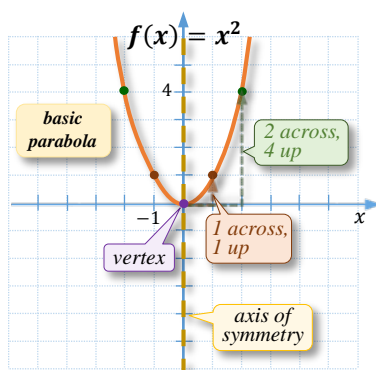
## Q.3

## Properties and Graphs of Quadratic Functions

$$f(x) = a(x-p)^2 + q$$

In this section, we explore an alternative way of graphing quadratic functions. It turns out that if a quadratic function is given in vertex form,  $f(x) = a(x-p)^2 + q$ , its graph can be obtained by transforming the shape of the basic parabola,  $f(x) = x^2$ , by applying a **vertical dilation** by the factor of  $a$ , as well as a **horizontal translation** by  $p$  units and **vertical translation** by  $q$  units. This approach makes the graphing process easier than when using a table of values.

In addition, the vertex form allows us to identify the main characteristics of the corresponding graph such as **shape**, **opening**, **vertex**, and **axis of symmetry**. Then, the additional properties of a quadratic function, such as **domain** and **range**, or where the function increases or decreases can be determined by observing the obtained graph.

 Properties and Graph of the Basic Parabola  $f(x) = x^2$ 


Recall the shape of the **basic parabola**,  $f(x) = x^2$ , as discussed in *section P4*.

$x$	$x^2$
-2	4
-1	1
0	0
1	1
2	4

$\left. \begin{array}{l} \text{symmetry} \\ \text{about the} \\ \text{y-axis} \end{array} \right\}$

Figure 3.1

Observe the relations between the points listed in the table above. If we start with plotting the **vertex**  $(0, 0)$ , then the next pair of points,  $(1, 1)$  and  $(-1, 1)$ , is plotted **1 unit across** from the vertex (both ways) and **1 unit up**. The following pair,  $(2, 4)$  and  $(-2, 4)$ , is plotted **2 units across** from the vertex and **4 units up**. The graph of the parabola is obtained by connecting these 5 main points by a curve, as illustrated in *Figure 3.1*.

The graph of this parabola is symmetric in the  $y$ -axis, so the equation of the **axis of symmetry** is  $x = 0$ .

The **domain** of the basic parabola is the set of all real numbers,  $\mathbb{R}$ , as  $f(x) = x^2$  is a polynomial, and polynomials can be evaluated for any real  $x$ -value.

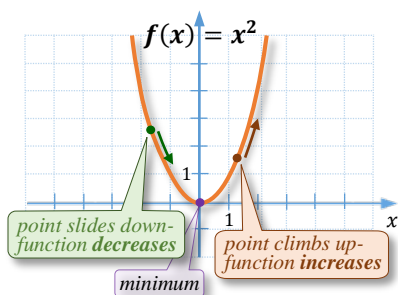


Figure 3.2

The **arms** of the parabola are directed **upwards**, which means that the vertex is the lowest point of the graph. Hence, the **range** of the basic parabola function,  $f(x) = x^2$ , is the interval  $[0, \infty)$ , and the **minimum value** of the function is **0**.

Suppose a point ‘lives’ on the graph and travels from left to right. Observe that in the case of the basic parabola, if  $x$ -coordinates of the ‘travelling’ point are smaller than 0, the point slides down along the graph. Similarly, if  $x$ -coordinates are larger than 0, the point climbs up the graph. (See *Figure 3.2*) To describe this property in mathematical language, we say that the function  $f(x) = x^2$  **decreases** in the interval  $(-\infty, 0]$  and **increases** in the interval  $[0, \infty)$ .

**Properties and Graphs of a Dilated Parabola  $f(x) = ax^2$**

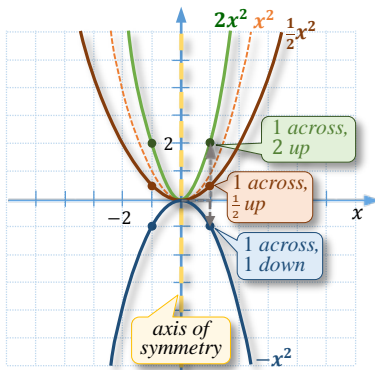


Figure 3.3

Figure 3.3 shows graphs of several functions of the form  $f(x) = ax^2$ . Observe how the shapes of these parabolas change for various values of  $a$  in comparison to the shape of the basic parabola  $y = x^2$ .

The common point for all of these parabolas is the vertex  $(0,0)$ . Additional points, essential for graphing such parabolas, are shown in the table below.

$x$	$ax^2$
-2	$4a$
-1	$a$
0	0 → vertex
1	$a$
2	$4a$

Annotations: A bracket groups the rows for  $x = -1$  and  $x = 1$  with the text "1 unit apart from zero, a units up". A larger bracket groups the rows for  $x = -2$  and  $x = 2$  with the text "2 units apart from zero, 4a units up".

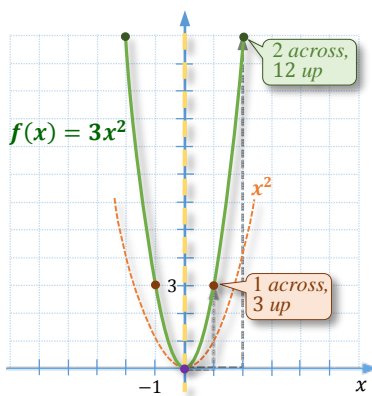


Figure 3.4

For example, to graph  $f(x) = 3x^2$ , it is convenient to plot the **vertex** first, which is at the point  $(0, 0)$ . Then, we may move the pen **1 unit across** from the vertex (either way) and **3 units up** to plot the points  $(-1, 3)$  and  $(1, 3)$ . If the grid allows, we might want to plot the next two points,  $(-2, 12)$  and  $(2, 12)$ , by moving the pen **2 units across** from the vertex and  $4 \cdot 3 = 12$  units **up**, as in Figure 3.4.

Notice that the obtained shape (in green) is **narrower** than the shape of the basic parabola (in orange). However, similarly as in the case of the basic parabola, the shape of the dilated function is still **symmetrical about the y-axis,  $x = 0$** .

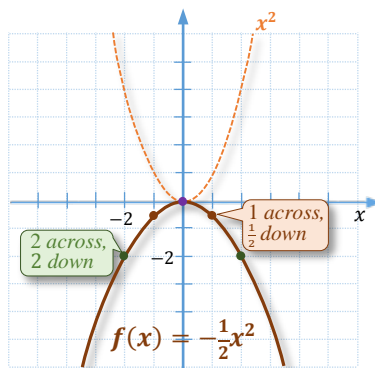


Figure 3.5

Now, suppose we want to graph the function  $f(x) = -\frac{1}{2}x^2$ . As before, we may start by plotting the vertex at  $(0,0)$ . Then, we move the pen **1 unit across** from the vertex (either way) and **half a unit down** to plot the points  $(-1, -\frac{1}{2})$  and  $(1, -\frac{1}{2})$ , as in Figure 3.5. The next pair of points can be plotted by moving the pen **2 units across** from the vertex and **2 units down**, as the ordered pairs  $(-2, -2)$  and  $(2, -2)$  satisfy the equation  $f(x) = -\frac{1}{2}x^2$ .

Notice that this time the obtained shape (in green) is **wider** than the shape of the basic parabola (in orange). Also, as a result of the **negative a-value**, the parabola opens **down**, and the **range** of this function is  $(-\infty, 0]$ .

Generally, the **shape** of a quadratic function of the form  $f(x) = ax^2$  is

- **narrower** than the shape of the basic parabola, if  $|a| > 1$ ;
- **wider** than the shape of the basic parabola, if  $0 < |a| < 1$ ; and
- **the same** as the shape of the **basic parabola**,  $y = x^2$ , if  $|a| = 1$ .

The parabola opens **up**, for  $a > 0$ , and **down**, for  $a < 0$ .

Thus the **vertex** becomes the **lowest point** of the graph, if  $a > 0$ , and the **highest point** of the graph, if  $a < 0$ .

The **range** of  $f(x) = ax^2$  is  $[0, \infty)$ , if  $a > 0$ , and  $(-\infty, 0]$ , if  $a < 0$ .

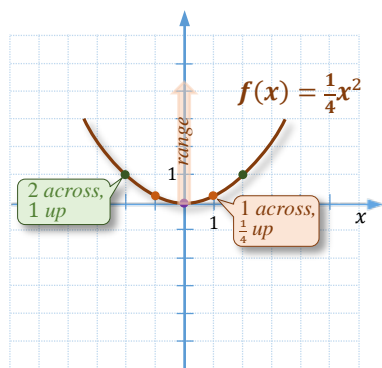
The **axis of symmetry** of the dilated parabola  $f(x) = ax^2$  remains the same as that of the basic parabola, which is  $x = 0$ .

**Example 1** ▶ **Graphing a Dilated Parabola and Describing Its Shape, Opening, and Range**

For each quadratic function, describe its shape and opening. Then graph it and determine its range.

a.  $f(x) = \frac{1}{4}x^2$                       b.  $g(x) = -2x^2$

**Solution** ▶ a. Since the leading coefficient of the function  $f(x) = \frac{1}{4}x^2$  is positive, the parabola



**opens up**. Also, since  $0 < \frac{1}{4} < 1$ , we expect the shape of the parabola to be **wider** than that of the basic parabola.

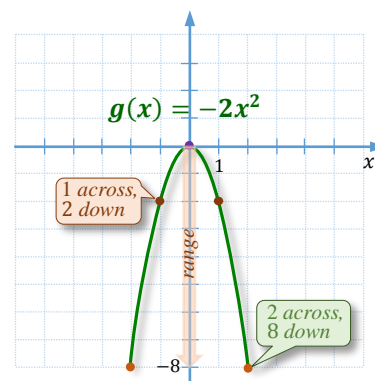
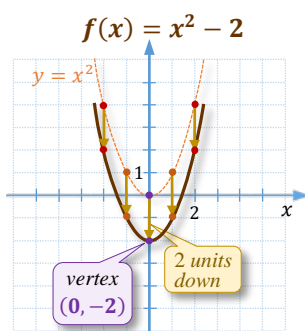
To graph  $f(x) = \frac{1}{4}x^2$ , first we plot the vertex at  $(0,0)$  and then points  $(\pm 1, \frac{1}{4})$  and  $(\pm 2, \frac{1}{4} \cdot 4) = (\pm 2, 1)$ . After connecting these points with a curve, we obtain the graph of the parabola.

By projecting the graph onto the  $y$ -axis, we observe that the range of the function is  $[0, \infty)$ .

b. Since the leading coefficient of the function  $g(x) = -2x^2$  is negative, the parabola **opens down**. Also, since  $|-2| > 1$ , we expect the shape of the parabola to be **narrower** than that of the basic parabola.

To graph  $g(x) = -2x^2$ , first we plot the vertex at  $(0,0)$  and then points  $(\pm 1, -2)$  and  $(\pm 2, -2 \cdot 4) = (\pm 2, -8)$ . After connecting these points with a curve, we obtain the graph of the parabola.

By projecting the graph onto the  $y$ -axis, we observe that the range of the function is  $(-\infty, 0]$ .


**Properties and Graphs of the Basic Parabola with Shifts**


**Figure 3.6**

Suppose we would like to graph the function  $f(x) = x^2 - 2$ . We could do this via a table of values, but there is an easier way if we already know the shape of the basic parabola  $y = x^2$ .

Observe that for every  $x$ -value, the value of  $x^2 - 2$  is obtained by subtracting 2 from the value of  $x^2$ . So, to graph  $f(x) = x^2 - 2$ , it is enough to **move each point**  $(x, x^2)$  of the basic parabola by **two units down**, as indicated in *Figure 3.6*.

The shift of  $y$ -values by 2 units down causes the **range** of the new function,  $f(x) = x^2 - 2$ , to become  $[-2, \infty)$ . Observe that this vertical shift also changes the minimum value of this function, from 0 to  $-2$ .

The **axis of symmetry** remains unchanged, and it is  $x = 0$ .

Generally, the graph of a quadratic function of the form  $f(x) = x^2 + q$  can be obtained by

- **shifting** the graph of the basic parabola  $q$  steps **up**, if  $q > 0$ ;
- **shifting** the graph of the basic parabola  $|q|$  steps **down**, if  $q < 0$ .

The **vertex** of such parabola is at  $(0, q)$ . The **range** of it is  $[q, \infty)$ .

The **minimum** (lowest) **value** of the function is  $q$ .

The **axis of symmetry** is  $x = 0$ .

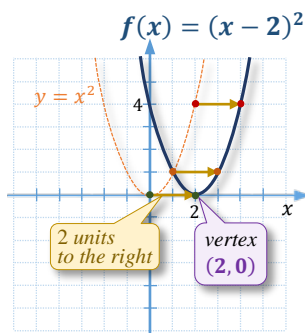


Figure 3.7

Now, suppose we wish to graph the function  $f(x) = (x - 2)^2$ . We can graph it by joining the points calculated in the table below.

$x$	$(x - 2)^2$
0	4
1	1
2	0
3	1
4	4

Observe that the parabola  $f(x) = (x - 2)^2$  assumes its lowest value at the vertex. The lowest value of the perfect square  $(x - 2)^2$  is zero, and it is attained at the  $x$ -value of 2. Thus, the vertex of this parabola is  $(2, 0)$ .

Notice that the **vertex**  $(2, 0)$  of  $f(x) = (x - 2)^2$  is positioned 2 units to the right from the vertex  $(0, 0)$  of the basic parabola.

This suggests that the graph of the function  $f(x) = (x - 2)^2$  can be obtained without the aid of a table of values. It is enough to shift the graph of the basic parabola **2 units** to the **right**, as shown in *Figure 3.7*.

Observe that the horizontal shift does not influence the **range** of the new parabola  $f(x) = (x - 2)^2$ . It is still  $[0, \infty)$ , the same as for the basic parabola. However, the **axis of symmetry** has changed to  $x = 2$ .

Generally, the graph of a quadratic function of the form  $f(x) = (x - p)^2$  can be obtained by

- **shifting** the graph of the basic parabola  $p$  steps to the **right**, if  $p > 0$ ;
- **shifting** the graph of the basic parabola  $|p|$  steps to the **left**, if  $p < 0$ .

The **vertex** of such a parabola is at  $(p, 0)$ . The **range** of it is  $[0, \infty)$ .

The **minimum value** of the function is **0**.

The **axis of symmetry** is  $x = p$ .

### Example 2



### Graphing Parabolas and Observing Transformations of the Basic Parabola

Graph each parabola by plotting its vertex and following the appropriate opening and shape. Then describe transformations of the basic parabola that would lead to the obtained graph. Finally, state the range and the equation of the axis of symmetry.

a.  $f(x) = (x + 3)^2$

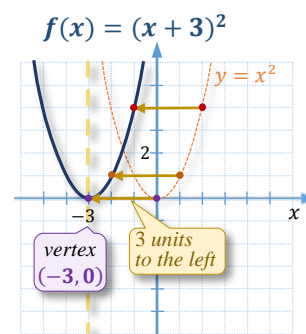
b.  $g(x) = -x^2 + 1$

**Solution**

- a.** The perfect square  $(x + 3)^2$  attains its lowest value at  $x = -3$ . So, the **vertex** of the parabola  $f(x) = (x + 3)^2$  is  $(-3, 0)$ . Since the leading coefficient is 1, the parabola takes the shape of  $y = x^2$ , and its **arms open up**.

The graph of the function  $f$  can be obtained by **shifting** the graph of the basic parabola **3 units to the left**, as shown in *Figure 3.8*.

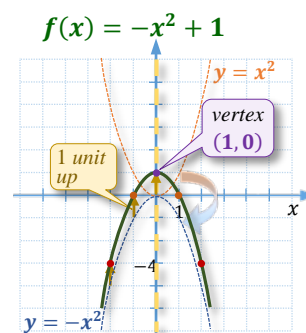
The **range** of function  $f$  is  $[0, \infty)$ , and the equation of the **axis of symmetry** is  $x = -3$ .


**Figure 3.8**

- b.** The expression  $-x^2 + 1$  attains its highest value at  $x = 0$ . So, the **vertex** of the parabola  $g(x) = -x^2 + 1$  is  $(0, 1)$ . Since the leading coefficient is  $-1$ , the parabola takes the shape of  $y = x^2$ , but its **arms open down**.

The graph of the function  $g$  can be obtained by:

- first, **flipping the graph** of the basic parabola **over the  $x$ -axis**, and then
- **shifting** the graph of  $y = -x^2$  **1 unit up**, as shown in *Figure 3.9*.


**Figure 3.9**

The **range** of the function  $g$  is  $(-\infty, 1]$ , and the equation of the **axis of symmetry** is  $x = 0$ .

**Note:** The order of transformations in the above example is essential. The reader is encouraged to check that **shifting** the graph of  $y = x^2$  by 1 unit up first and then **flipping** it over the  $x$ -axis results in a different graph than the one in *Figure 3.9*.

**Properties and Graphs of Quadratic Functions Given in the Vertex Form  $f(x) = a(x - p)^2 + q$** 

So far, we have discussed properties and graphs of quadratic functions that can be obtained from the graph of the basic parabola by applying mainly a single transformation. These transformations were: dilations (including flips over the  $x$ -axis), and horizontal and vertical shifts. Sometimes, however, we need to apply more than one transformation. We have already encountered such a situation in *Example 2b*, where a flip and a horizontal shift was applied. Now, we will look at properties and graphs of any function of the form  $f(x) = a(x - p)^2 + q$ , referred to as the **vertex form** of a quadratic function.

Suppose we wish to graph  $f(x) = 2(x + 1)^2 - 3$ . This can be accomplished by connecting the points calculated in a table of values, such as the one below, or by observing the coordinates of the vertex and following the shape of the graph of  $y = 2x^2$ . Notice that the vertex of our parabola is at  $(-1, -3)$ . This information can be taken directly from the equation  $f(x) = 2(x + 1)^2 - 3 = 2(x - (-1))^2 - 3$ ,

$x$	$2(x + 1)^2 - 3$
-3	5
-2	-1
-1	-3
0	-1
1	5

1 unit apart  
from zero,  
2 units up

vertex

$f(x) = 2(x + 1)^2 - 3 = 2(x - (-1))^2 - 3$ ,  
opposite to the  
number in the bracket      the same last  
number

without the aid of a table of values.

The rest of the points follow the pattern of the shape for the  $y = 2x^2$  parabola: 1 across, 2 up; 2 across,  $4 \cdot 2 = 8$  up. So, we connect the points as in Figure 3.10.

Notice that the graph of function  $f$  could also be obtained as a result of translating the graph of  $y = 2x^2$  by 1 unit left and 3 units down, as indicated in Figure 3.10 by the blue vectors.

Here are the main properties of the graph of function  $f$ :

- It has a **shape** of  $y = 2x^2$ ;
- It is a parabola that **opens up**;
- It has a **vertex** at  $(-1, -3)$ ;
- It is **symmetrical** about the line  $x = -1$ ;
- Its **minimum value** is  $-3$ , and this minimum is attained at  $x = -1$ ;
- Its domain is the set of all real numbers, and its **range** is the interval  $[-3, \infty)$ ;
- It **decreases** for  $x \in (-\infty, -1]$  and **increases** for  $x \in [-1, \infty)$ .

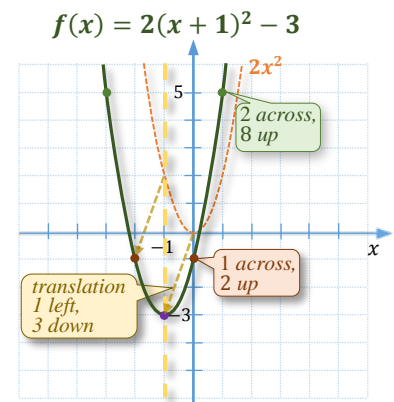


Figure 3.10

The above discussion of properties and graphs of a quadratic function given in vertex form leads us to the following general observations:

### Characteristics of Quadratic Functions Given in Vertex Form $f(x) = a(x - p)^2 + q$

1. The graph of a quadratic function given in **vertex form**

$$f(x) = a(x - p)^2 + q, \text{ where } a \neq 0,$$

is a **parabola** with **vertex**  $(p, q)$  and **axis of symmetry**  $x = p$ .

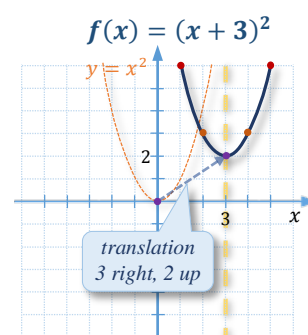
2. The graph **opens up** if  $a$  is **positive** and **down** if  $a$  is **negative**.
3. If  $a > 0$ ,  $q$  is the **minimum value**. If  $a < 0$ ,  $q$  is the **maximum value**.
3. The graph is **narrower** than that of  $y = x^2$  if  $|a| > 1$ .  
The graph is **wider** than that of  $y = x^2$  if  $0 < |a| < 1$ .
4. The **domain** of function  $f$  is the set of real numbers,  $\mathbb{R}$ .  
The **range** of function  $f$  is  $[q, \infty)$  if  $a$  is **positive** and  $(-\infty, q]$  if  $a$  is **negative**.

**Example 3** ▶ **Identifying Properties and Graphing Quadratic Functions Given in Vertex Form**  
 $f(x) = a(x - p)^2 + q$

For each function, identify its **vertex**, **opening**, **axis of symmetry**, and **shape**. Then graph the function and state its **domain** and **range**. Finally, describe **transformations** of the basic parabola that would lead to the obtained graph.

a.  $f(x) = (x - 3)^2 + 2$                       b.  $g(x) = -\frac{1}{2}(x + 1)^2 + 3$

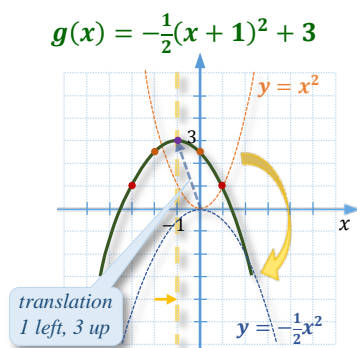
**Solution** ▶ a. The vertex of the parabola  $f(x) = (x - 3)^2 + 2$  is **(3, 2)**; the graph **opens up**, and the equation of the axis of symmetry is  $x = 3$ . To graph this function, we can plot the vertex first and then follow the shape of the basic parabola  $y = x^2$ .



The domain of function  $f$  is  $\mathbb{R}$ , and the range is  $[2, \infty)$ .

The graph of  $f$  can be obtained by shifting the graph of the basic parabola **3 units to the right** and **2 units up**.

b. The vertex of the parabola  $g(x) = -\frac{1}{2}(x + 1)^2 + 3$  is **(-1, 3)**; the graph **opens down**, and the equation of the axis of symmetry is  $x = -1$ . To graph this function, we can plot the vertex first and then follow the shape of the parabola  $y = -\frac{1}{2}x^2$ . This means that starting from the vertex, we move the pen one unit across (both ways) and drop half a unit to plot the next two points,  $(0, \frac{5}{2})$  and symmetrically  $(-2, \frac{5}{2})$ . To plot the following two points, again, we start from the vertex and move our pen two units across and 2 units down (as  $-\frac{1}{2} \cdot 4 = -2$ ). So, the next two points are  $(1, 1)$  and symmetrically  $(-4, 1)$ , as indicated in *Figure 3.11*.



The domain of function  $g$  is  $\mathbb{R}$ , and the range is  $(-\infty, 3]$ .

**Figure 3.11**

The graph of  $g$  can be obtained from the graph of the basic parabola in two steps:

1. **Dilate** the basic parabola by multiplying its  $y$ -values by the factor of  $-\frac{1}{2}$ .
2. Shift the graph of the dilated parabola  $y = -\frac{1}{2}x^2$ , **1 unit to the left** and **3 units up**, as indicated in *Figure 3.11*.

### Q.3 Exercises

**Vocabulary Check** Complete each blank with the most appropriate term or phrase from the given list: **axis of symmetry, horizontal, maximum, minimum, opens, shape, up, vertex, vertical**.

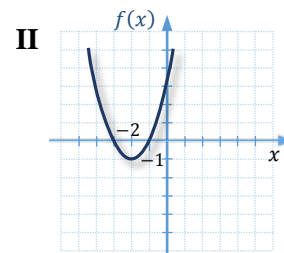
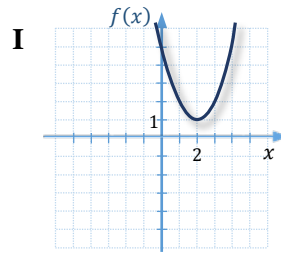
1. A quadratic function can be stated in a form  $f(x) = a(x - p)^2 + q$ . This form is referred to as the \_\_\_\_\_ form because it allows for quick identification of the vertex of the corresponding parabola.

2. The line  $x = p$  is the \_\_\_\_\_ of the parabola  $f(x) = a(x - p)^2 + q$ .
3. The \_\_\_\_\_ of the function  $f(x) = a(x - p)^2 + q$  is the same as the shape of the parabola  $y = ax^2$ .
4. If  $a > 0$ , the parabola  $f(x) = a(x - p)^2 + q$  opens \_\_\_\_\_. If  $a < 0$ , the parabola  $f(x) = a(x - p)^2 + q$  \_\_\_\_\_ down.
5. The value  $p$  in the formula  $f(x) = a(x - p)^2 + q$ , represents the \_\_\_\_\_ shift of the parabola  $y = ax^2$  needed to obtain the graph of function  $f$ .
6. The value  $q$  in the formula  $f(x) = a(x - p)^2 + q$ , represents the \_\_\_\_\_ shift of the parabola  $y = ax^2$  needed to obtain the graph of function  $f$ .
7. If  $a > 0$ , then the vertex  $(p, q)$  of the parabola  $f(x) = a(x - p)^2 + q$  is its \_\_\_\_\_. If  $a < 0$ , then the vertex  $(p, q)$  of the parabola  $f(x) = a(x - p)^2 + q$  is its \_\_\_\_\_.

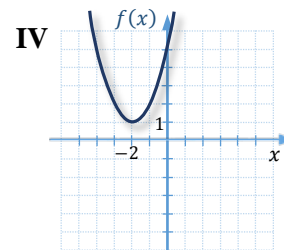
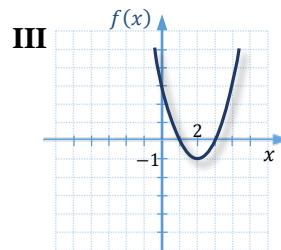
**Concept Check**

8. Match each quadratic function **a.-d.** with its graph **I-IV**.

a.  $f(x) = (x - 2)^2 - 1$



b.  $f(x) = (x - 2)^2 + 1$

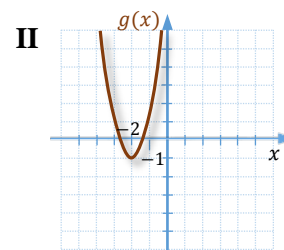
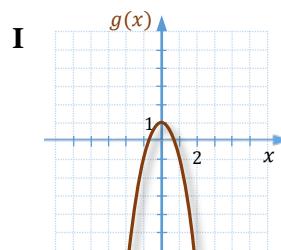


c.  $f(x) = (x + 2)^2 + 1$

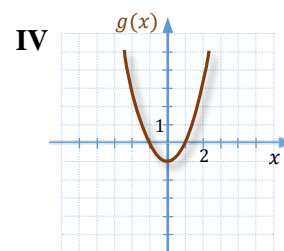
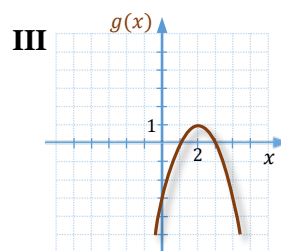
d.  $f(x) = (x + 2)^2 - 1$

9. Match each quadratic function **a.-d.** with its graph **I-IV**.

a.  $g(x) = -(x - 2)^2 + 1$



b.  $g(x) = x^2 - 1$



c.  $g(x) = -2x^2 + 1$

d.  $g(x) = 2(x + 2)^2 - 1$



### Concept Check

10. Match each quadratic function with the characteristics of its parabolic graph.

- |                                       |                                 |
|---------------------------------------|---------------------------------|
| a. $f(x) = 5(x - 3)^2 + 2$            | I vertex (3,2), opens down      |
| b. $f(x) = -4(x + 2)^2 - 3$           | II vertex (3,2), opens up       |
| c. $f(x) = -\frac{1}{2}(x - 3)^2 + 2$ | III vertex (-2, -3), opens down |
| d. $f(x) = \frac{1}{4}(x + 2)^2 - 3$  | IV vertex (-2, -3), opens up    |

For each quadratic function, describe the **shape** (as **wider**, **narrower**, or the **same** as the shape of  $y = x^2$ ) and **opening** (**up** or **down**) of its graph. Then **graph it** and determine its **range**.

- |                             |                              |                              |
|-----------------------------|------------------------------|------------------------------|
| 11. $f(x) = 3x^2$           | 12. $f(x) = -\frac{1}{2}x^2$ | 13. $f(x) = -\frac{3}{2}x^2$ |
| 14. $f(x) = \frac{5}{2}x^2$ | 15. $f(x) = -x^2$            | 16. $f(x) = \frac{1}{3}x^2$  |

**Graph** each parabola by plotting its vertex, and following its shape and opening. Then, **describe transformations** of the basic parabola that would lead to the obtained graph. Finally, state the **domain** and **range**, and the equation of the **axis of symmetry**.

- |                         |                        |                                   |
|-------------------------|------------------------|-----------------------------------|
| 17. $f(x) = (x - 3)^2$  | 18. $f(x) = -x^2 + 2$  | 19. $f(x) = x^2 - 5$              |
| 20. $f(x) = -(x + 2)^2$ | 21. $f(x) = -2x^2 - 1$ | 22. $f(x) = \frac{1}{2}(x + 2)^2$ |

For each parabola, state its **vertex**, **shape**, **opening**, and the equation of its **axis of symmetry**. Then, **graph** the function and describe **transformations** of the basic parabola that would lead to the obtained graph.

- |  |                                       |
|--|---------------------------------------|
| 23. $f(x) = 3x^2 - 1$                  | 24. $f(x) = -\frac{3}{4}x^2 + 3$      |
| 25. $f(x) = -\frac{1}{2}(x + 4)^2 + 2$ | 26. $f(x) = \frac{5}{2}(x - 2)^2 - 4$ |
| 27. $f(x) = 2(x - 3)^2 + \frac{3}{2}$  | 28. $f(x) = -3(x + 1)^2 + 5$          |
| 29. $f(x) = -\frac{2}{3}(x + 2)^2 + 4$ | 30. $f(x) = \frac{4}{3}(x - 3)^2 - 2$ |

### Discussion Point

31. Four students, **A**, **B**, **C**, and **D**, tried to graph the function  $f(x) = -2(x + 1)^2 - 3$  by transforming the graph of the basic parabola,  $y = x^2$ . Here are the transformations that each student applied

*Student A:*

- shift 1 unit left and 3 units down
- dilation of y-values by the factor of  $-2$

*Student B:*

- dilation of y-values by the factor of  $-2$
- shift 1 unit left
- shift 3 units down

*Student C:*

- flip over the  $x$ -axis
- shift 1 unit left and 3 units down
- dilation of  $y$ -values by the factor of 2

*Student D:*

- shift 1 unit left
- dilation of  $y$ -values by the factor of 2
- shift 3 units down
- flip over the  $x$ -axis

With the assumption that all transformations were properly applied, discuss whose graph was correct and what went wrong with the rest of the graphs. Is there any other sequence of transformations that would result in a correct graph?

For each parabola, state the coordinates of its **vertex** and then **graph** it. Finally, state the **extreme value** (**maximum** or **minimum**, whichever applies) and the **range** of the function.

32.  $f(x) = 3(x - 1)^2$

33.  $f(x) = -\frac{5}{2}(x + 3)^2$

34.  $f(x) = (x + 2)^2 - 3$

35.  $f(x) = -3(x + 4)^2 + 5$

36.  $f(x) = -2(x - 5)^2 - 2$

37.  $f(x) = 2(x - 4)^2 + 1$

38.  $f(x) = \frac{1}{2}(x + 1)^2 + \frac{3}{2}$

39.  $f(x) = -\frac{1}{2}(x - 1)^2 - 3$

40.  $f(x) = -\frac{1}{4}(x - 3)^2 + 4$

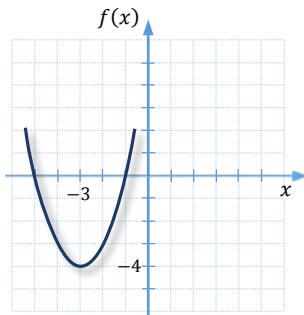
41.  $f(x) = \frac{3}{4}\left(x + \frac{5}{2}\right)^2 - \frac{3}{2}$



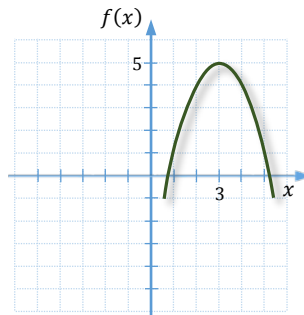
**Analytic Skills**

Given the graph of a parabola, state the most probable **equation** of the corresponding function. Hint: Use the vertex form of a quadratic function.

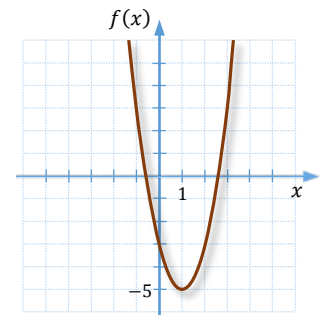
42.



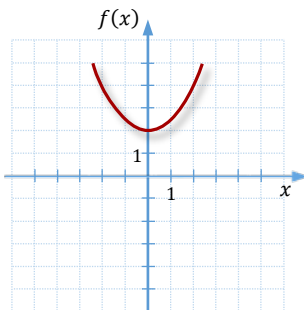
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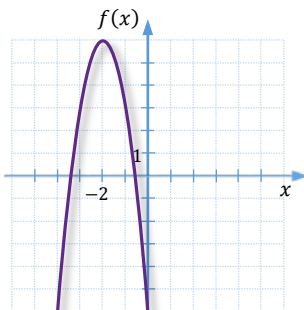
44.



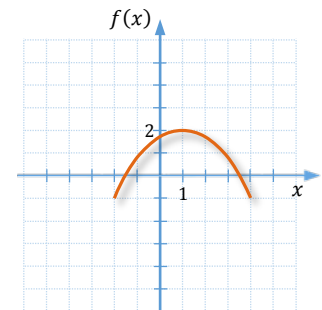
45.



46.



47.



## Q.4

## Properties of Quadratic Function and Optimization Problems



In the previous section, we examined how to graph and read the characteristics of the graph of a quadratic function given in vertex form,  $f(x) = a(x - p)^2 + q$ . In this section, we discuss the ways of **graphing** and reading the **characteristics** of the graph of a quadratic function given in **standard form**,  $f(x) = ax^2 + bx + c$ . One of these ways is to convert standard form of the function to vertex form by **completing the square** so that the information from the vertex form may be used for graphing. The other handy way of graphing and reading properties of a quadratic function is to **factor** the defining trinomial and use the **symmetry** of a parabolic function.

At the end of this section, we apply properties of quadratic functions to solve certain **optimization problems**. To solve these problems, we look for the **maximum** or **minimum** of a particular quadratic function satisfying specified conditions called **constraints**. Optimization problems often appear in geometry, calculus, business, computer science, etc.

 Graphing Quadratic Functions Given in the Standard Form  $f(x) = ax^2 + bx + c$ 

To graph a quadratic function given in standard form,  $f(x) = ax^2 + bx + c$ , we can use one of the following methods:

1. constructing a **table of values** (this would always work, but it could be cumbersome);
2. converting to **vertex form** by using the technique of completing the square (see *Example 1-3*);
3. **factoring** and employing the properties of a parabolic function. (this is a handy method if the function can be easily factored – see *Example 3 and 4*)

The table of values approach can be used for any function, and it was already discussed on various occasions throughout this textbook.

Converting to **vertex form** involves completing the square. For example, to convert the function  $f(x) = 2x^2 + x - 5$  to its vertex form, we might want to start by dividing both sides of the equation by the leading coefficient 2, and then complete the square for the polynomial on the right side of the equation, as below.

$$\frac{f(x)}{2} = x^2 + \frac{1}{2}x - \frac{5}{2}$$

$$\frac{f(x)}{2} = \left(x + \frac{1}{4}\right)^2 - \frac{1}{16} - \frac{5 \cdot 8}{2 \cdot 8}$$

$$\frac{f(x)}{2} = \left(x + \frac{1}{4}\right)^2 - \frac{41}{16}$$

Finally, the vertex form is obtained by multiplying both sides of the equation back by 2. So, we have

$$f(x) = 2\left(x + \frac{1}{4}\right)^2 - \frac{41}{8}$$

This form lets us identify the vertex,  $\left(-\frac{1}{4}, -\frac{41}{8}\right)$ , and the shape,  $y = 2x^2$ , of the parabola, which is essential for graphing it. To create an approximate graph of

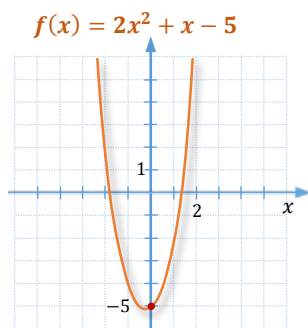


Figure 4.1

function  $f$ , we may want to round the vertex to approximately  $(-0.25, -5.1)$  and evaluate  $f(0) = 2 \cdot 0^2 + 0 - 5 = -5$ . So, the graph is as in *Figure 4.1*.

### Example 1 ▶ Converting the Standard Form of a Quadratic Function to the Vertex Form

Rewrite each function in its vertex form. Then, identify the vertex.

a.  $f(x) = -3x^2 + 2x$  b.  $g(x) = \frac{1}{2}x^2 + x + 3$

**Solution** ▶ a. To convert  $f$  to its vertex form, we follow the completing the square procedure. After dividing the equation by the leading coefficient,

$$f(x) = -3x^2 + 2x, \quad / \div (-3)$$

we have

$$\frac{f(x)}{-3} = x^2 - \frac{2}{3}x$$

Then, we complete the square for the right side of the equation,

$$\frac{f(x)}{-3} = \left(x - \frac{1}{3}\right)^2 - \frac{1}{9}, \quad / \cdot (-3)$$

and finally, multiply back by the leading coefficient,

$$f(x) = -3\left(x - \frac{1}{3}\right)^2 + \frac{1}{3}.$$

Therefore, the vertex of this parabola is at the point  $\left(\frac{1}{3}, \frac{1}{3}\right)$ .

b. As in the previous example, to convert  $g$  to its vertex form, we first wish to get rid of the leading coefficient. This can be achieved by multiplying both sides of the equation  $g(x) = \frac{1}{2}x^2 - x + 3$  by 2. So, we obtain

$$\begin{aligned} 2g(x) &= x^2 + 2x + 6 \\ 2g(x) &= (x + 1)^2 - 1 + 6 \\ 2g(x) &= (x + 1)^2 + 5, \quad / \div 2 \end{aligned}$$

which can be solved back for  $g$ ,

$$g(x) = \frac{1}{2}(x + 1)^2 + \frac{5}{2}.$$

Therefore, the vertex of this parabola is at the point  $\left(-1, \frac{5}{2}\right)$ .

Completing the square allows us to derive a formula for the vertex of the graph of any quadratic function given in its standard form,  $f(x) = ax^2 + bx + c$ , where  $a \neq 0$ . Applying the same procedure as in *Example 1*, we calculate

$$\begin{aligned}
 f(x) &= ax^2 + bx + c && / \div a \\
 \frac{f(x)}{a} &= x^2 + \frac{b}{a}x + \frac{c}{a} \\
 \frac{f(x)}{a} &= \left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} \\
 \frac{f(x)}{a} &= \left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2} && \text{Recall: This is the discriminant } \Delta! \\
 f(x) &= a \left(x - \left(-\frac{b}{2a}\right)\right)^2 + \frac{-(b^2 - 4ac)}{4a} && / \cdot a
 \end{aligned}$$

Thus, the coordinates of the vertex  $(p, q)$  are  $p = -\frac{b}{2a}$  and  $q = \frac{-(b^2 - 4ac)}{4a} = \frac{-\Delta}{4a}$ .

**Observation:** Notice that the expression for  $q$  can also be found by evaluating  $f$  at  $x = -\frac{b}{2a}$ .

So, the vertex of the parabola can also be expressed as  $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$ .

Summarizing, the **vertex** of a parabola defined by  $f(x) = ax^2 + bx + c$ , where  $a \neq 0$ , can be calculated by following one of the formulas:

$$\left(-\frac{b}{2a}, \frac{-(b^2 - 4ac)}{4a}\right) = \left(-\frac{b}{2a}, \frac{-\Delta}{4a}\right) = \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$$

## VERTEX FORMULA

### Example 2 ▶ Using the Vertex Formula to Find the Vertex of a Parabola

Use the vertex formula to find the vertex of the graph of  $f(x) = -x^2 - x + 1$ .

**Solution** ▶ The first coordinate of the vertex is equal to  $-\frac{b}{2a} = -\frac{-1}{2 \cdot (-1)} = -\frac{1}{2}$ .

The second coordinate can be calculated by following the formula

$$\frac{-\Delta}{4a} = \frac{-((-1)^2 - 4 \cdot (-1) \cdot 1)}{4 \cdot (-1)} = \frac{5}{4},$$

or by evaluating  $f\left(-\frac{1}{2}\right) = -\left(-\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right) + 1 = -\frac{1}{4} + \frac{1}{2} + 1 = \frac{5}{4}$ .

So, the vertex is  $\left(-\frac{1}{2}, \frac{5}{4}\right)$ .

**Example 3** ▶ **Graphing a Quadratic Function Given in the Standard Form**

Graph each function.

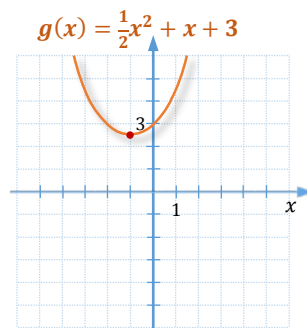
a.  $g(x) = \frac{1}{2}x^2 + x + 3$

b.  $f(x) = -x^2 - x + 1$

**Solution** ▶

- a. The shape of the graph of function  $g$  is the same as this of  $y = \frac{1}{2}x^2$ . Since the leading coefficient is positive, the arms of the parabola **open up**.

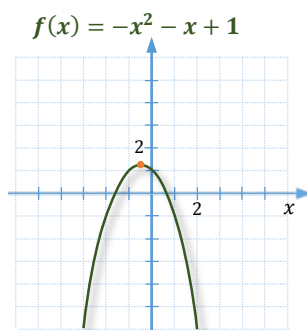
The **vertex**,  $(-1, \frac{5}{2})$ , was found in *Example 1b* as a result of completing the square. Since the vertex is in quadrant II and the graph opens up, we do not expect any  $x$ -intercepts. However, without much effort, we can find the  $y$ -intercept by evaluating  $g(0) = 3$ . Furthermore, since  $(0, 3)$  belongs to the graph, then by symmetry,  $(-2, 3)$  must also belong to the graph. So, we graph function  $g$  is as in *Figure 4.2*.

**Figure 4.2**

When plotting points with fractional coordinates, round the values to one place value.

- b. The graph of function  $f$  has the shape of the basic parabola. Since the leading coefficient is negative, the arms of the parabola **open down**.

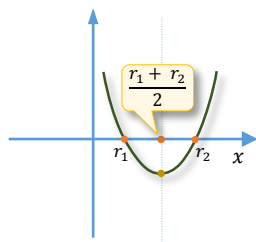
The **vertex**,  $(-\frac{1}{2}, \frac{5}{4})$ , was found in *Example 2* by using the vertex formula. Since the vertex is in quadrant II and the graph opens down, we expect two  $x$ -intercepts. Their values can be found via the quadratic formula applied to the equation  $-x^2 - x + 1 = 0$ . So, the  $x$ -intercepts are  $x_{1,2} = \frac{1 \pm \sqrt{5}}{-2} \approx -1.6$  or  $0.6$ . In addition, the  $y$ -intercept of the graph is  $f(0) = 1$ .

**Figure 4.3**

Using all this information, we graph function  $f$ , as in *Figure 4.3*.

**Graphing Quadratic Functions Given in the Factored Form  $f(x) = a(x - r_1)(x - r_2)$** 

$f(x) = a(x - r_1)(x - r_2)$

**Figure 4.4**

What if a quadratic function is given in factored form? Do we have to change it to vertex or standard form in order to find the vertex and graph it?

The factored form,  $f(x) = a(x - r_1)(x - r_2)$ , allows us to find the roots (or  $x$ -intercepts) of such a function. These are  $r_1$  and  $r_2$ . A parabola is symmetrical about the axis of symmetry, which is the vertical line passing through its vertex. So, the first coordinate of the vertex is the same as the first coordinate of the midpoint of the line segment connecting the roots,  $r_1$  with  $r_2$ , as indicated in *Figure 4.4*. Thus, the first coordinate of the vertex is the average of the two roots,  $\frac{r_1 + r_2}{2}$ . Then, the second coordinate of the vertex can be found by evaluating  $f\left(\frac{r_1 + r_2}{2}\right)$ .

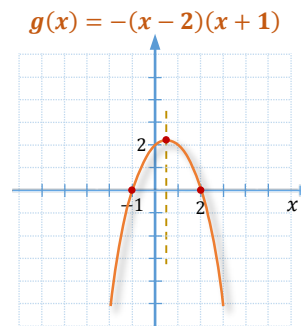
**Example 4** ▶ **Graphing a Quadratic Function Given in a Factored Form**

Graph function  $g(x) = -(x - 2)(x + 1)$ .

**Solution** ▶ First, observe that the graph of function  $g$  has the same shape as the graph of the basic parabola,  $f(x) = x^2$ . Since the leading coefficient is negative, the arms of the parabola **open down**. Also, the graph intersects the  $x$ -axis at 2 and  $-1$ . So, the first coordinate of the vertex is the average of 2 and  $-1$ , which is  $\frac{1}{2}$ . The second coordinate is

$$g\left(\frac{1}{2}\right) = -\left(\frac{1}{2} - 2\right)\left(\frac{1}{2} + 1\right) = -\left(-\frac{3}{2}\right)\left(\frac{3}{2}\right) = \frac{9}{4}$$

Therefore, function  $g$  can be graphed by connecting the vertex,  $\left(\frac{1}{2}, \frac{9}{4}\right)$ , and the  $x$ -intercepts,  $(-1, 0)$  and  $(2, 0)$ , with a parabolic curve, as in *Figure 4.5*. For a more precise graph, we may additionally plot the  $y$ -intercept,  $g(0) = 2$ , and the symmetrical point  $g(1) = 2$ .



**Figure 4.5**

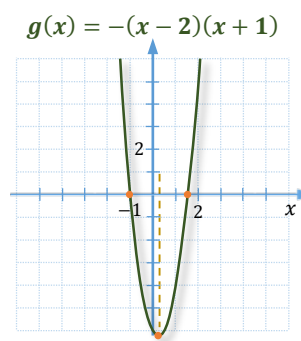
**Example 5** ▶ **Using Complete Factorization to Graph a Quadratic Function**

Graph function  $f(x) = 4x^2 - 2x - 6$ .

**Solution** ▶ Since the discriminant  $\Delta = (-2)^2 - 4 \cdot 4 \cdot (-6) = 4 + 96 = 100$  is a perfect square number, the defining trinomial is factorable. So, to graph function  $f$ , we may want to factor it first. Notice that the GCF of all the terms is 2. So,  $f(x) = 2(2x^2 - x - 3)$ . Then, using factoring techniques discussed in *section F2*, we obtain  $f(x) = 2(2x - 3)(x + 1)$ . This form allows us to identify the roots (or zeros) of function  $f$ , which are  $\frac{3}{2}$  and  $-1$ . So, the first coordinate of the vertex is the average of  $\frac{3}{2} = 1.5$  and  $-1$ , which is  $\frac{1.5 + (-1)}{2} = \frac{0.5}{2} = 0.25$ . The second coordinate can be calculated by evaluating

$$f(0.25) = 2(2 \cdot 0.25 - 3)(0.25 + 1) = 2(0.5 - 3)(1.25) = 2(-2.5)(1.25) = -6.25$$

So, we can graph function  $f$  by connecting its vertex,  $(0.25, -6.25)$ , and its  $x$ -intercepts,  $(-1, 0)$  and  $(1.5, 0)$ , with a parabolic curve, as in *Figure 4.6*. For a more precise graph, we may additionally plot the  $y$ -intercept,  $f(0) = -6$ , and by symmetry,  $f(0.5) = -6$ .



**Figure 4.6**

**Optimization Problems**

In many applied problems we are interested in **maximizing** or **minimizing** some quantity under specific conditions, called **constraints**. For example, we might be interested in

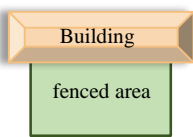
finding the greatest area that can be fenced in by a given length of fence, or minimizing the cost of producing a container of a given shape and volume. These types of problems are called **optimization problems**.

Since the vertex of the graph of a quadratic function is either the highest or the lowest point of the parabola, it can be used in solving optimization problems that can be modeled by a quadratic function.

The vertex of a parabola provides the following information.

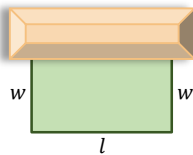
- The  $y$ -value of the vertex gives the maximum or minimum value of  $y$ .
- The  $x$ -value tells where the maximum or minimum occurs.

### Example 6 ▶ Maximizing Area of a Rectangular Region



A farmer has 60 meters of fencing to enclose a rectangular field next to a building. Assuming that the attached building forms one side of the rectangle, find the maximum area he can enclose and the dimensions of the enclosed field that yields this area.

#### Solution ▶



Let  $l$  and  $w$  represent the length and width of the enclosed area correspondingly, as indicated in *Figure 4.7*. The 60 meters of fencing is used to cover the distance of twice along the width and once along the length. So, we can form the constraint equation

$$2W + l = 60 \quad (1)$$

To analyse the area of the field,

$$A = lw, \quad (2)$$

we would like to express it as a function of one variable, for example  $w$ . To do this, we can solve the constraint equation (1) for  $l$  and substitute the obtained expression into the equation of area, (2). Since  $l = 60 - 2w$ , then

$$A = lw = (60 - 2w)w$$

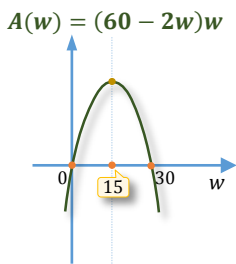


Figure 4.8

Observe that the graph of the function  $A(w) = (60 - 2w)w$  is a parabola that opens down and intersects the  $x$ -axis at 0 and 30. This is because the leading coefficient of  $(60 - 2w)w$  is negative and the roots to the equation  $(60 - 2w)w = 0$  are 0 and 30. These roots are symmetrical in the axis of symmetry, which also passes through the vertex of the parabola, as illustrated in *Figure 4.8*. So, the first coordinate of the vertex is the average of the two roots, which is  $\frac{0+30}{2} = 15$ . Thus, the width that would maximize the enclosed area is  $w_{max} = 15$  meters. Consequently, the length that would maximize the enclosed area is  $l_{max} = 60 - 2w_{max} = 60 - 2 \cdot 15 = 30$  meters. The maximum area represented by the second coordinate of the vertex can be obtained by evaluating the function of area at the width of 15 meters.

$$A(15) = (60 - 2 \cdot 15)15 = 30 \cdot 15 = 450 \text{ m}^2$$

So, the maximum area that can be enclosed by 60 meters of fencing is **450 square meters**, and the dimensions of this rectangular area are **15 by 30 meters**.



**Example 7** ▶ **Minimizing Average Unit Cost**


A company producing bicycles has determined that when  $x$  hundred bicycles are produced, the average cost of producing one bicycle is given by the function

$$C(x) = 0.1x^2 - 0.7x + 2.425,$$

where  $C(x)$  is in hundreds of dollars. How many bicycles should be produced to minimize the average cost of producing one bicycle? What would this cost be?

**Solution** ▶

Since  $C(x)$  is a quadratic function, to find its minimum, it is enough to find the vertex of the parabola  $C(x) = 0.1x^2 - 0.7x + 2.425$ . This can be done either by completing the square method or by using the formula for the vertex,  $\left(\frac{-b}{2a}, \frac{-\Delta}{4a}\right)$ . We will do the latter. So, the vertex is

$$\begin{aligned} \left(\frac{-b}{2a}, \frac{-\Delta}{4a}\right) &= \left(\frac{0.7}{0.2}, \frac{-(0.7^2 - 4 \cdot 0.1 \cdot 2.425)}{0.4}\right) = \left(3.5, \frac{-(0.49 - 0.97)}{0.4}\right) = \left(3.5, \frac{0.48}{0.4}\right) \\ &= (3.5, 1.2). \end{aligned}$$

This means that the lowest average unit cost can be achieved when 350 bicycles are produced, and then the average cost of a bicycle would be \$120.

## Q.4 Exercises

**Vocabulary Check** Complete each blank with the most appropriate term or phrase from the given list: **average**, **completing**, **down**, **first**, **minimum**, **optimization**, **second**, **standard**, **vertex**.

- The highest or lowest point of a parabola is called a \_\_\_\_\_.
- When  $a$  is negative, the parabola given by the function  $f(x) = ax^2 + bx + c$  opens \_\_\_\_\_.
- When  $a$  is positive, the parabola given by the function  $f(x) = ax^2 + bx + c$  has a \_\_\_\_\_ point.
- When converting the \_\_\_\_\_ form of a quadratic function  $f(x) = ax^2 + bx + c$  to its vertex form  $f(x) = a(x - p)^2 + q$ , the process of \_\_\_\_\_ the square is used.
- The **extreme** value (**minimum** or **maximum**) of a quadratic function is the \_\_\_\_\_ coordinate of the vertex of its graph. This extreme value occurs at the \_\_\_\_\_ coordinate of the vertex.
- Application problems that involve finding the minimum or maximum points on a graph of a function are often referred to as \_\_\_\_\_ problems.
- The first coordinate of the vertex of the parabola given by the function  $f(x) = a(x - r_1)(x - r_2)$  is the \_\_\_\_\_ of the roots  $r_1$  and  $r_2$ .

**Concept Check**

Convert each quadratic function to its **vertex form**. Then, state the coordinates of the **vertex**.

8.  $f(x) = x^2 + 6x + 10$

9.  $f(x) = x^2 - 4x - 5$

10.  $f(x) = x^2 + x - 3$

11.  $f(x) = x^2 - x + 7$

12.  $f(x) = -x^2 + 7x + 3$

13.  $f(x) = 2x^2 - 4x + 1$

14.  $f(x) = -3x^2 + 6x + 12$

15.  $f(x) = -2x^2 - 8x + 10$

16.  $f(x) = \frac{1}{2}x^2 + 3x - 1$

**Concept Check**

Use the vertex formula,  $\left(-\frac{b}{2a}, \frac{-\Delta}{4a}\right)$ , to find the coordinates of the **vertex** of each parabola.

17.  $f(x) = x^2 + 6x + 3$

18.  $f(x) = -x^2 + 3x - 5$

19.  $f(x) = \frac{1}{2}x^2 - 4x - 7$

20.  $f(x) = -3x^2 + 6x + 5$

21.  $f(x) = 5x^2 - 7x$

22.  $f(x) = 3x^2 + 6x - 20$

For each parabola, state its **vertex**, **opening** and **shape**. Then **graph** it and state the **domain** and **range**.

23.  $f(x) = x^2 - 5x$

24.  $f(x) = x^2 + 3x$

25.  $f(x) = x^2 - 2x - 5$

26.  $f(x) = -x^2 + 6x - 3$

27.  $f(x) = -x^2 - 3x + 2$

28.  $f(x) = 2x^2 + 12x + 18$

29.  $f(x) = -2x^2 + 3x - 1$

30.  $f(x) = -2x^2 + 4x + 1$

31.  $f(x) = 3x^2 + 4x + 2$

For each quadratic function, state its **zeros** (roots), coordinates of the **vertex**, **opening** and **shape**. Then **graph** it and identify its **extreme** (minimum or maximum) **value** as well as where it occurs.

32.  $f(x) = (x - 2)(x + 2)$

33.  $f(x) = -(x + 3)(x - 1)$

34.  $f(x) = x^2 - 4x$

35.  $f(x) = x^2 + 5x$

36.  $f(x) = x^2 - 8x + 16$

37.  $f(x) = -x^2 - 4x - 4$

38.  $f(x) = -3(x^2 - 1)$

39.  $f(x) = \frac{1}{2}(x + 3)(x - 4)$

40.  $f(x) = -\frac{3}{2}(x - 1)(x - 5)$

**Discussion Point**

41. Suppose the  $x$ -intercepts of the graph of a parabola are  $(x_1, 0)$  and  $(x_2, 0)$ . What is the equation of the axis of symmetry of this graph?

**Discussion Point**



42. How can we determine the number of  $x$ -intercepts of the graph of a quadratic function without graphing the function?

**Concept Check** True or false? Explain.

43. The domain and range of a quadratic function are both the set of real numbers.
44. The graph of every quadratic function has exactly one  $y$ -intercept.
45. The graph of  $y = -2(x - 1)^2 - 5$  has no  $x$ -intercepts.

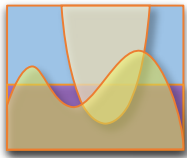
46. The maximum value of  $y$  in the function  $y = -4(x - 1)^2 + 9$  is 9.
47. The value of the function  $f(x) = x^2 - 2x + 1$  is at its minimum when  $x = 0$ .
48. The graph of  $f(x) = 9x^2 + 12x + 4$  has one  $x$ -intercept and one  $y$ -intercept.
49. If a parabola opens down, it has two  $x$ -intercepts.

**Analytic Skills** Solve each problem.

50. If an object on Earth is projected upward with an initial velocity of 32 ft per sec, then its height after  $t$  seconds is given by  $s(t) = -16t^2 + 32t$ . Find the maximum height attained by the object and the number of seconds it takes to hit the ground.
51. A projectile on Earth is fired straight upward so that its distance (in feet) above the ground  $t$  seconds after firing is given by  $s(t) = -16t^2 + 400t$ . Find the maximum height it reaches and the number of seconds it takes to reach that height.
52. John has a frozen yogurt cart. His daily costs are approximated by  $C(x) = x^2 - 70x + 1500$ , where  $C(x)$  is the cost, in dollars, to sell  $x$  units of frozen yogurt. Find the number of units of frozen yogurt he must sell to minimize his costs. What is the minimum cost? 
53. Chris has a taco stand. He has found that his daily costs are approximated by  $C(x) = x^2 - 40x + 610$ , where  $C(x)$  is the cost, in dollars, to sell  $x$  units of tacos. Find the number of units of tacos he should sell to minimize his costs. What is the minimum cost?
54. Find two positive real numbers whose sum is 40 and whose product is a maximum.
55. Find two real numbers whose difference is 40 and whose product is a minimum.
56. Mike wants to enclose a rectangular area for his rabbits alongside his large barn using 30 feet of fencing. What dimensions will maximize the area fenced if the barn is used for one side of the rectangle?
57. Kevin wants to enclose a rectangular garden using 14 eight-foot railroad ties, which he cannot cut. What are the dimensions of the rectangle that maximize the area enclosed?
58. Martin plans to construct a rectangular kennel for two dogs using 120 feet of chain-link fencing. He plans to fence all four sides and down the middle to keep the dogs separate. What overall dimensions will maximize the total area fenced?
59. Mona gives a walking tour of Honolulu to one person for \$49. To increase her business, she advertised that she would lower the price by \$1 per person for each additional person, up to 49 people.
- Write the price per person  $p$  as a function of the number of people  $n$ .
  - Write her revenue as a function of the number of people on the tour.
  - How many people on the tour would maximize Mona's revenue and what would this revenue be?
- 
60. At \$10 per ticket, Willie Williams and the Wranglers will fill all 8000 seats in the Assembly Center. The manager knows that for every \$1 increase in the price, 500 tickets will go unsold.
- Write the number of tickets sold  $n$  as a function of ticket price  $p$ .
  - Write the total revenue as a function of the ticket price.
  - What ticket price will maximize the revenue and what would this revenue be?

## Q.5

## Polynomial and Rational Inequalities



In sections L4 and L5, we discussed solving linear inequalities in one variable as well as solving systems of such inequalities. In this section, we examine polynomial and rational inequalities in one variable. Such inequalities can be solved using either graphical or analytic methods. Below, we discuss both types of methods with a particular interest in the analytic one.

## Solving Quadratic Inequalities by Graphing

**Definition 5.1** ▶ A **quadratic inequality** is any inequality that can be written in one of the forms

$$ax^2 + bx + c > (\geq) 0, \text{ or } ax^2 + bx + c < (\leq) 0, \text{ or } ax^2 + bx + c \neq 0$$

where  $a$ ,  $b$ , and  $c$  are real numbers, with  $a \neq 0$ .

To solve a quadratic inequality, it is useful to solve the related quadratic equation first. For example, to solve  $x^2 + 2x - 3 > (\geq) 0$ , or  $x^2 + 2x - 3 < (\leq) 0$ , we may consider solving the related equation:

$$x^2 + 2x - 3 = 0$$

$$(x + 3)(x - 1) = 0$$

$$x = -3 \text{ or } x = 1.$$

This helps us to sketch an approximate graph of the related function  $f(x) = x^2 + x - 2$ , as in Figure 1.1. The graph of function  $f$  is a parabola that crosses the  $x$ -axis at  $x = -3$  and  $x = 1$ , and is directed upwards. Observe that the graph extends below the  $x$ -axis for  $x$ -values from the interval  $(-3, 1)$  and above the  $x$ -axis for  $x$ -values from the set  $(-\infty, -3) \cup (1, \infty)$ . This allows us to read solution sets of several inequalities, as listed below.

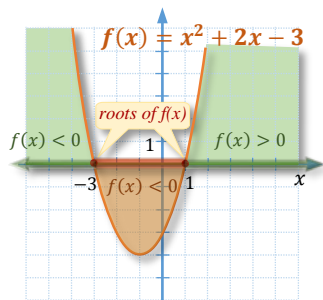


Figure 1.1

Inequality	Solution Set
$x^2 + 2x - 3 > 0$	$(-\infty, -3) \cup (1, \infty)$
$x^2 + 2x - 3 \geq 0$	$(-\infty, -3] \cup [1, \infty)$
$x^2 + 2x - 3 < 0$	$(-3, 1)$
$x^2 + 2x - 3 \leq 0$	$[-3, 1]$
$x^2 + 2x - 3 \neq 0$	$\mathbb{R} \setminus \{-3, 1\}$

**Note:** If the inequalities contain equations ( $\geq, \leq$ ), the  $x$ -values of the intercepts are included in the solution sets. Otherwise, the  $x$ -values of the intercepts are excluded from the solution sets.

## Solving Polynomial Inequalities

**Definition 5.2** ▶ A **polynomial inequality** is any inequality that can be written in one of the forms

$$P(x) > (\geq) 0, \text{ or } P(x) < (\leq) 0, \text{ or } P(x) \neq 0$$

where  $P(x)$  is a polynomial with real coefficients.

**Note:** Linear or quadratic inequalities are special cases of polynomial inequalities.

Polynomial inequalities can be solved graphically or analytically, without the use of a graph. The analytic method involves determining the sign of the polynomial by analysing signs of the polynomial factors for various  $x$ -values, as in the following example.

### Example 1 ▶ Solving Polynomial Inequalities Using Sign Analysis

Solve each inequality using sign analysis.

a.  $(x + 1)(x - 2)x > 0$

b.  $2x^4 + 8 \leq 10x^2$

**Solution** ▶ a. The solution set of  $(x + 1)(x - 2)x > 0$  consists of all  $x$ -values that make the product  $(x + 1)(x - 2)x$  positive. To analyse how the sign of this product depends on the  $x$ -values, we can visualise the sign behaviour of each factor with respect to the  $x$ -value, by recording applicable signs in particular sections of a number line.

For example, the expression  $x + 1$  changes its sign at  $x = -1$ .

If  $x > -1$ , the expression  $x + 1$  is positive, so we mark “+” in the interval  $(-1, \infty)$ .

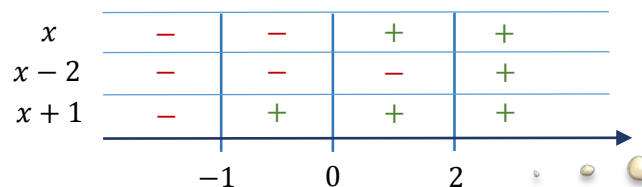
If  $x < -1$ , the expression  $x + 1$  is negative, so we mark “-” in the interval  $(-\infty, -1)$ .

So, the sign behaviour of the expression  $x + 1$  can be recorded on a number line, as below.



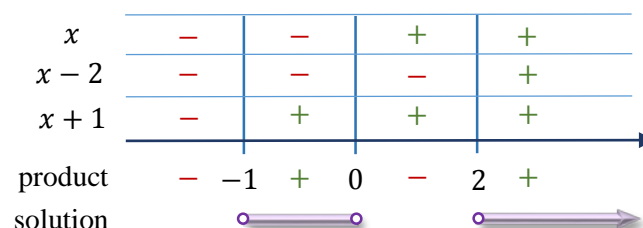
Here, the **zeros** of polynomials are referred to as **critical numbers**. This is because the polynomials may change their signs at these numbers.

A similar analysis can be conducted for the remaining factors,  $x - 2$  and  $x$ . These expressions change their signs at  $x = 2$  and  $x = 0$ , correspondingly. The sign behaviour of all the factors can be visualised by reserving one line of signs per each factor, as shown below.



Remember to write the critical numbers in increasing order!

The sign of the product  $(x + 1)(x - 2)x$  is obtained by multiplying signs in each column. The result is marked beneath the number line, as below.

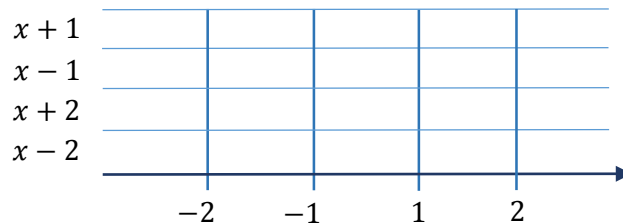


The signs in the “product” row give us the grounds to state the solution set for the original inequality,  $(x + 1)(x - 2)x > 0$ . Before we write the final answer though, it is helpful to visualize the solution set by graphing it in the “solution” row. To satisfy the original inequality, we need the product to be positive, so we look for the intervals within which the product is positive. Since the inequality does not include an equation, the intervals are open. Therefore, the solution set is  $(-1, 0) \cup (2, \infty)$ .

- b. To solve  $2x^4 + 8 \leq 10x^2$  by sign analysis, we need to keep one side of the inequality equal to zero and factor the other side so that we may identify the critical numbers. To do this, we may change the inequality as below.

$$\begin{aligned}
 2x^4 + 8 &\leq 10x^2 && / -10x^2 \\
 2x^4 - 10x^2 + 8 &\leq 0 && / \div 2 \\
 x^4 - 5x^2 + 4 &\leq 0 \\
 (x^2 - 4)(x^2 - 1) &\leq 0 \\
 (x - 2)(x + 2)(x - 1)(x + 1) &\leq 0
 \end{aligned}$$

The critical numbers (the  $x$ -values that make the factors equal to zero) are 2,  $-2$ , 1, and  $-1$ . To create a table of signs, we arrange these numbers on a number line in increasing order and list all the factors in the left column.



Then, we can fill in the table with signs that each factor assumes for the  $x$ -values from the corresponding section of the number line.

$x + 1$	-	-	+	+	+
$x - 1$	-	-	-	+	+
$x + 2$	-	+	+	+	+
$x - 2$	-	-	-	-	+
product	+	-	+	-	+
solution		○ ——— ○		○ ——— ○	

SIGN ANALYSIS

The signs in each part of the number line can be determined either by:

- **analysing** the behaviour of each factor with respect to its critical number (For example,  $x - 2 > 0$  for  $x > 2$ . So, the row of signs assumed by  $x - 2$  consists of negative signs until the critical number 2 and positive signs after this number.)

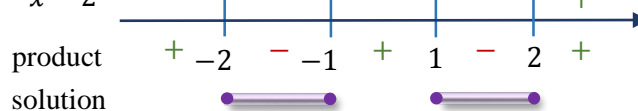
or

- **testing** an  $x$ -value from each section of the number line (For example, since the expression  $x - 2$  does not change its sign inside the interval  $(-1, 1)$ , to determine

its sign, it is enough to evaluate it for an easy to calculate **test number** from this interval. For instance, when  $x = 0 \in (-1, 1)$ , the value of  $x - 2$  is negative. This means that all the values of  $x - 2$  are negative between  $-1$  and  $1$ .)

Finally, underneath each column, we record the sign of the product and graph the solution set to the inequality  $(x - 2)(x + 2)(x - 1)(x + 1) \leq 0$

$x + 1$	-	-	+	+	+
$x - 1$	-	-	-	+	+
$x + 2$	-	+	+	+	+
$x - 2$	-	-	-	-	+
product	+	-	+	-	+
solution	-2	-1	1	2	



**Note:** Since the inequality contains an equation, the endpoints of the resulting intervals belong to the solution set as well. Hence, they are marked by **filled-in circles** and notated with **square brackets** in interval notation.

So, the solution set is  $[-2, -1] \cup [1, 2]$ .

## Solving Rational Inequalities

**Definition 5.3** ▶ A **rational inequality** is any inequality that can be written in one of the forms

$$\frac{P(x)}{Q(x)} > (\geq) 0, \text{ or } \frac{P(x)}{Q(x)} < (\leq) 0, \text{ or } \frac{P(x)}{Q(x)} \neq 0$$

where  $P(x)$  and  $Q(x)$  are polynomials with real coefficients.

Rational inequalities can be solved similarly as polynomial inequalities. To solve a rational inequality using the sign analysis method, we need to make sure that one side of the inequality is **zero** and the other side is expressed as a **single algebraic fraction** with a completely factored numerator and denominator.

### Example 2 ▶ Solving Rational Inequalities Using Sign Analysis

Solve each inequality using sign analysis.

a.  $\frac{(x-2)x}{x+1} \geq 0$

b.  $\frac{4-x}{x+2} \geq x$

**Solution** ▶ a. The right side of the inequality  $\frac{(x-2)x}{x+1} > 0$  is zero, and the left side is a single fraction with both numerator and denominator in factored form. So, to solve this inequality, it is enough to analyse signs of the expression  $\frac{(x-2)x}{x+1}$  at different intervals of the domain.

These intervals are determined by the **critical numbers** (the zeros of the numerator and denominator), which are  $-1$ ,  $0$ , and  $2$ .

$x$	$-$	$-$	$+$	$+$
$x - 2$	$-$	$-$	$-$	$+$
$x + 1$	$-$	$+$	$+$	$+$
product	$-$	$+$	$-$	$+$
solution		$-1$	$0$	$2$

As indicated in the above table of signs, the solution set to the inequality  $\frac{(x-2)x}{x+1} \geq 0$  contains numbers between  $-1$  and  $0$  and numbers higher than  $2$ . In addition, since the inequality includes an equation,  $x = 0$  and  $x = 2$  are also solutions. However,  $x = -1$  is not a solution because  $-1$  does not belong to the domain of the expression  $\frac{(x-2)x}{x+1}$  since it would make the denominator  $0$ . So, the solution set is  $(-1, 0] \cup [2, \infty)$ .

**Attention:** **Solutions** to a rational inequality **must belong to the domain** of the inequality. This means that any number that makes the denominator  $0$  must be excluded from the solution set.

- b. To solve  $\frac{4-x}{x+2} \geq x$  by the sign analysis method, first, we would like to keep the right side equal to zero. So, we rearrange the inequality as below.

When working with inequalities, **avoid multiplying by the denominator** as it can be positive or negative for different  $x$ -values!

$$\begin{aligned} \frac{4-x}{x+2} &\geq x && / -x \\ \frac{4-x}{x+2} - x &\geq 0 \\ \frac{4-x-x(x+2)}{x+2} &\geq 0 \\ \frac{4-x-x^2-2x}{x+2} &\geq 0 \\ \frac{-x^2-3x+4}{x+2} &\geq 0 \\ \frac{-(x^2+3x-4)}{x+2} &\geq 0 \\ \frac{-(x+4)(x-1)}{x+2} &\geq 0 && / \cdot (-1) \\ \frac{(x+4)(x-1)}{x+2} &\leq 0 \end{aligned}$$

When multiplying by a **negative** number, remember to **reverse** the inequality sign!



Now, we can analyse the signs of the expression  $\frac{(x+4)(x-1)}{x+2}$ , using the table of signs with the critical numbers  $-4$ ,  $-2$ , and  $1$ .

$x + 4$	-	+	+	+			
$x - 1$	-	-	-	+			
$x + 2$	-	-	+	+			
product	-	-4	+	-2	-	1	+
solution	←						→

So, the solution set for the inequality  $\frac{(x+4)(x-1)}{x+2} \leq 0$ , which is equivalent to  $\frac{4-x}{x+2} \geq x$ , contains numbers lower than  $-4$  and numbers between  $-2$  and  $1$ . Since the inequality includes an equation,  $x = -4$  and  $x = 1$  are also solutions. However,  $x = -2$  is not in the domain of  $\frac{(x+4)(x-1)}{x+2}$ , and therefore it is not a solution.

Thus, the solution set to the original inequality is  $(-\infty, -4] \cup (-2, 1]$ .

### Summary of Solving Polynomial or Rational Inequalities

1. **Write the inequality so that one of its sides is zero** and the other side is expressed as the **product or quotient of prime polynomials**.
2. **Determine the critical numbers**, which are the roots of all the prime polynomials appearing in the inequality.
3. **Divide the number line into intervals** formed by the set of critical numbers.
4. **Create a table of signs** for all prime factors in all intervals formed by the set of critical numbers. This can be done by analysing the sign of each factor, or by testing a number from each interval.
5. **Determine the sign of the overall expression** in each of the intervals.
6. **Graph the intervals of numbers that satisfy the inequality**. Make sure to **exclude endpoints that are not in the domain** of the inequality.
7. **State the solution set** to the original inequality **in interval notation**.

### Solving Special Cases of Polynomial or Rational Inequalities

Some inequalities can be solved without the use of a graph or a table of signs.

#### Example 3 ▶ Solving Special Cases of Inequalities

Solve each inequality.

a.  $(3x + 2)^2 > -1$

b.  $\frac{(x-4)^2}{x^2} \leq 0$

**Solution**

- ▶ a. First, notice that the left side of the inequality  $(3x + 2)^2 > -1$  is a perfect square and as such, it assumes a nonnegative value for any input  $x$ . Since a nonnegative quantity is always bigger than  $-1$ , the inequality is satisfied by any real number  $x$ . So, the solution set is  $\mathbb{R}$ .

**Note:** The solution set of an inequality that is **always true** is the set of all real numbers,  $\mathbb{R}$ . For example, inequalities that take one of the following forms

$$\text{nonnegative} > \text{negative}$$

$$\text{positive} \geq \text{negative}$$

$$\text{positive} > \text{nonpositive}$$

$$\text{positive} > 0$$

$$\text{negative} < 0$$

are **always true**. So their solution set is  $\mathbb{R}$ .

- b. Since the left side of the inequality  $\frac{(x-4)^2}{x^2} \leq 0$  is a perfect square, it is bigger or equal to zero for all  $x$ -values. So, we have

$$0 \leq \frac{(x-4)^2}{x^2} \leq 0,$$

which can be true only if  $\frac{(x-4)^2}{x^2} = 0$ . Since a fraction equals to zero only when its numerator equals to zero, the solution to the last equation is  $x = 4$ . Thus, the solution set for the original inequality is  $\{4\}$ .

**Observation:** Notice that the inequality  $\frac{(x-4)^2}{x^2} < 0$  has no solution as a perfect square is never negative.

**Note:** The solution set of an inequality that is **never true** is the empty set,  $\emptyset$ . For example, inequalities that take one of the following forms

$$\text{positive (or nonnegative)} \leq \text{negative}$$

$$\text{nonnegative} < \text{nonpositive}$$

$$\text{positive} \leq \text{nonpositive}$$

$$\text{positive} \leq 0 \text{ or } \text{negative} \geq 0$$

$$\text{nonnegative} < 0 \text{ or } \text{nonpositive} > 0$$

are **never true**. So, their solution sets are  $\emptyset$ .

## Polynomial and Rational Inequalities in Application Problems

Some application problems involve solving polynomial or rational inequalities.

### Example 4 ▶ Finding the Range of Values Satisfying the Given Condition

An outerwear manufacturer determines that its weekly revenue,  $R$ , for selling a rain parka at a price of  $p$  dollars is modeled by the function  $R(p) = 150p - p^2$ . What range of prices for the parka will generate a weekly revenue of at least \$5000?

**Solution** ▶ Since the revenue must be at least \$5000, we can set up the inequality

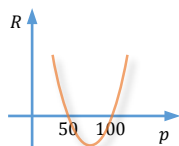
$$R(p) = 150p - p^2 \geq 5000,$$

and solve it for  $p$ . So, we have

$$-p^2 + 150p - 5000 \geq 0 \quad / \cdot (-1)$$

$$p^2 - 150p + 5000 \leq 0$$

$$(p - 50)(p - 100) \leq 0$$



Since the left-hand side expression represents a directed upwards parabola with roots at  $p = 50$  and  $p = 100$ , its graph looks like in the accompanying figure. The graph extends below the  $p$ -axis for  $p$ -values between 50 and 100. So, to generate weekly revenue of at least \$5000, the price  $p$  of a parka must take a value within the interval  $[50, 100]$ .

## Q.5 Exercises

**Concept Check** *True or False.*

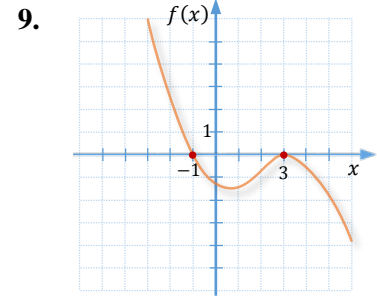
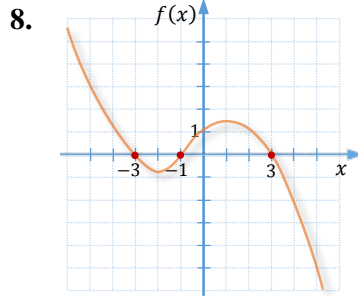
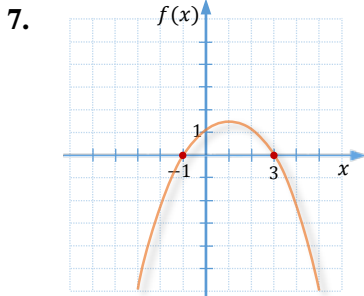
- To determine if the value of an expression is greater than or less than 0 in a given interval, a test number can be used.
- If the solution to the inequality  $P(x) \geq 0$ , where  $P(x)$  is a polynomial with real coefficients, is  $[2, 5)$ , then the solution to the inequality  $P(x) < 0$  is  $(-\infty, 2) \cup [5, \infty)$ .
- The inequalities  $(x - 1)(x + 3) \leq 0$  and  $\frac{(x-1)}{(x+3)} \leq 0$  have the same solutions.
- The solution set of the inequality  $(x - 1)^2 > 0$  is the set of all real numbers.
- The inequality  $x^2 + 1 \leq 0$  has no solution.
- The solution set of the inequality  $\frac{(x-1)^2}{(x+1)^2} \geq 0$  is the set of all real numbers.

**Concept Check**

Given the graph of function  $f$ , state the solution set for each inequality

a.  $f(x) \geq 0$

b.  $f(x) < 0$



Solve each inequality by sketching an approximate graph for the related equation.

10.  $(x + 4)(x - 2) > 0$

11.  $(x + 1)(x - 2) < 0$

12.  $x^2 - 4x + 3 \geq 0$

13.  $\frac{1}{2}(x^2 - 3x - 10) \leq 0$

14.  $4 - 9x^2 > 0$

15.  $-x^2 - 2x < 0$

Solve each inequality using sign analysis.

16.  $(x - 3)(x + 2) > 0$

17.  $(x + 4)(x - 5) < 0$

18.  $x^2 + 2x - 7 \leq 8$

19.  $x^2 - x - 2 \geq 10$

20.  $3x^2 + 10x > 8$

21.  $2x^2 + 5x < -2$

22.  $x^2 + 9 > -6x$

23.  $x^2 + 4 \leq 4x$

24.  $6 + x - x^2 \leq 0$

25.  $20 - x - x^2 < 0$

26.  $(x - 1)(x + 2)(x - 3) \geq 0$

27.  $(x + 3)(x - 2)(x - 5) \leq 0$

28.  $x(x + 3)(2x - 1) > 0$

29.  $x^2(x - 2)(2x - 1) \geq 0$

30.  $x^4 - 13x^2 + 36 \leq 0$

Solve each inequality using sign analysis.

31.  $\frac{x}{x+1} > 0$

32.  $\frac{x+1}{x-2} < 0$

33.  $\frac{2x-1}{x+3} \leq 0$

34.  $\frac{2x-3}{x+1} \geq 0$

35.  $\frac{3}{y+5} > 1$

36.  $\frac{5}{t-1} \leq 2$

37.  $\frac{x-1}{x+2} \leq 3$

38.  $\frac{a+4}{a+3} \geq 2$

39.  $\frac{2t-3}{t+3} < 4$

40.  $\frac{3y+9}{2y-3} < 3$

41.  $\frac{1-2x}{2x+5} \leq 2$

42.  $\frac{2x+3}{1-x} \leq 1$

43.  $\frac{4x}{2x-1} \leq x$

44.  $\frac{-x}{x+2} > 2x$

45.  $\frac{2x-3}{(x+1)^2} \leq 0$

46.  $\frac{2x-3}{(x-2)^2} \geq 0$

47.  $\frac{x^2+1}{5-x^2} > 0$

48.  $x < \frac{3x-8}{5-x}$

49.  $\frac{1}{x+2} \geq \frac{1}{x-3}$

50.  $\frac{2}{x+3} \leq \frac{1}{x-1}$

51.  $\frac{(x-3)(x+1)}{4-x} \geq 0$

52.  $\frac{(x+2)(x-1)}{(x+4)^2} \geq 0$

53.  $\frac{x^2-2x-8}{x^2+10x+25} > 0$

54.  $\frac{x^2-4x}{x^2-x-6} \leq 0$

Solve each inequality.

55.  $(4 - 3x)^2 \geq -2$

56.  $(5 + 2x)^2 < -1$

57.  $\frac{(1-2x)^2}{2x^4} \leq 0$

58.  $\frac{(1-2x)^2}{(x+2)^2} > -3$

59.  $\frac{-2x^2}{(x+2)^2} \geq 0$

60.  $\frac{-x^2}{(x-3)^2} < 0$

**Analytic Skills** Solve each problem.

60. A penny is tossed upwards with an initial velocity of 48 ft/sec. The height,  $h$ , of the penny relative to the point of release  $t$  seconds after it is tossed is modeled by the function  $h(t) = -16t^2 + 48t$ . For what interval of time is the penny above the point of release?
61. A furniture maker determines that the weekly cost,  $C$ , for producing  $x$  accent tables is given by the function  $C(x) = 2x^2 - 60x + 900$ . How many accent tables can be produced to keep the weekly cost under \$500?
62. A gardener wants to enclose a rectangular flower bed using 90 ft of fencing. For what range of lengths will the area exceed 450 ft<sup>2</sup>?
63. A company determines that its average cost  $C$ , in dollars, for selling  $x$  units of a product is modeled by the function  $C(x) = \frac{870+2x}{x}$ . For what number of units  $x$  will the average cost be less than \$8?
64. A publishing company had a revenue of \$18 million last year. The company's financial analyst uses the formula  $P(R) = \frac{100R-1800}{R}$  to determine the percent growth,  $P$ , in the company's revenue this year over last year, where  $R$  is in millions of dollars. For what revenues  $R$  will the company's revenue grow by more than 10%?

