Radicals and Radical Functions

So far we have discussed polynomial and rational expressions and functions. In this chapter, we study algebraic expressions that contain radicals. For example, $3+\sqrt{2}$, $\sqrt[3]{x}-1$, or $\frac{1}{\sqrt{5x-1}}$. Such expressions are called ***radical expressions***. Familiarity with radical expressions is essential when solving a wide variety of problems. For instance, in algebra, some polynomial or rational equations have radical solutions that need to be simplified. In geometry, due to the frequent use of the Pythagorean equation, $a^2+b^2=c^2$, the exact distances are often radical expressions. In sciences, many formulas involve radicals.

We begin the study of radical expressions with defining radicals of various degrees and discussing their properties. Then, we show how to simplify radicals and radical expressions, and introduce operations on radical expressions. Finally, we study the methods of solving radical equations. In addition, similarly as in earlier chapters where we looked at the related polynomial and rational functions, we will also define and look at properties of radical functions.

RD.1 Radical Expressions, Functions, and Graphs

Roots and Radicals

The operation of taking a **square root** of a number is the **reverse operation of squaring** a number. For example, a square root of 25 is 5 because raising 5 to the second power gives us 25.

Note: Observe that raising -5 to the second power also gives us 25. So, the square root of 25 could have two answers, 5 or -5. To avoid this duality, we choose the **nonnegative value**, called the **principal square root**, for the value of a square root of a number.

The operation of taking a square root is denoted by the symbol $\sqrt{}$. So, we have

$$\sqrt{25}=5,\quad \sqrt{0}=0,\quad \sqrt{1}=1,\quad \sqrt{9}=3, \text{ etc.}$$

What about $\sqrt{-4}=?$ Is there a number such that when it is squared, it gives us -4 ?

Since the square of any real number is nonnegative, the square root of a negative number is not a real number. So, when working in the set of real numbers, we can conclude that

$$\sqrt{\textit{positive}}=\textit{positive},\quad \sqrt{0}=0,\quad \text{and} \quad \sqrt{\textit{negative}}=\textit{DNE}$$

does not exist

The operation of taking a **cube root** of a number is the **reverse operation of cubing** a number. For example, a cube root of 8 is 2 because raising 2 to the third power gives us 8.

This operation is denoted by the symbol $\sqrt[3]{}$. So, we have

$$\sqrt[3]{8}=2,\quad \sqrt[3]{0}=0,\quad \sqrt[3]{1}=1,\quad \sqrt[3]{27}=3, \text{ etc.}$$

Note: Observe that $\sqrt[3]{-8}$ exists and is equal to -2. This is because $(-2)^3 = -8$. Generally, a cube root can be applied to any real number and the **sign** of the resulting value **is the same** as the sign of the original number.

Thus, we have

$$\sqrt[3]{\textit{positive}} = \textit{positive}, \quad \sqrt[3]{0} = 0, \quad \text{and} \quad \sqrt[3]{\textit{negative}} = \textit{negative}$$

The square or cube roots are special cases of n-th degree radicals.

Definition 1.1 ► The ***n*-th degree radical** of a number a is a number b such that $\boldsymbol{b^n = a}$.

Notation:

degree
(index, order)
radical

$$\sqrt[n]{a} = b \Leftrightarrow b^n = a$$

radical symbol
radicand

For example,
$\sqrt[4]{16} = 2$ because $2^4 = 16$,
$\sqrt[5]{-32} = -2$ because $(-2)^5 = -32$,
$\sqrt[3]{0.027} = 0.3$ because $(0.3)^3 = 0.027$.

Notice: A square root is a second degree radical, customarily denoted by $\sqrt{\ }$ rather than $\sqrt[2]{\ }$.

Example 1 ► Evaluating Radicals

Evaluate each radical, if possible.

a. $\sqrt{0.64}$ **b.** $\sqrt[3]{125}$ **c.** $\sqrt[4]{-16}$ **d.** $\sqrt[5]{-\frac{1}{32}}$

Solution ► **a.** Since $0.64 = (0.8)^2$, then $\sqrt{0.64} = 0.8$.

take half of the decimal places

Advice: To become fluent in evaluating square roots, it is helpful to be familiar with the following perfect square numbers:
1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, …, 400, …, 625, …

b. $\sqrt[3]{125} = 5$ as $5^3 = 125$

Advice: To become fluent in evaluating cube roots, it is helpful to be familiar with the following cubic numbers:
1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, …

c. $\sqrt[4]{-16}$ is not a real number as there is no real number which raised to the 4-th power becomes negative.

d. $\sqrt[5]{-\frac{1}{32}} = -\frac{1}{2}$ as $\left(-\frac{1}{2}\right)^5 = -\frac{1^5}{2^5} = -\frac{1}{32}$

Note: Observe that $\frac{\sqrt[5]{-1}}{\sqrt[5]{32}} = \frac{-1}{2}$, so $\sqrt[5]{-\frac{1}{32}} = \frac{\sqrt[5]{-1}}{\sqrt[5]{32}}$.

Generally, to take a radical of a quotient, $\sqrt[n]{\frac{a}{b}}$, it is the same as to take the quotient of radicals, $\frac{\sqrt[n]{a}}{\sqrt[n]{b}}$.

Example 2 ► **Evaluating Radical Expressions**

Evaluate each radical expression.

a. $-\sqrt{121}$ **b.** $-\sqrt[3]{-64}$ **c.** $\sqrt[4]{(-3)^4}$ **d.** $\sqrt[3]{(-6)^3}$

Solution ► **a.** $-\sqrt{121} = -11$

b. $-\sqrt[3]{-64} = -(-4) = 4$

c. $\sqrt[4]{(-3)^4} = \sqrt[4]{81} = 3$

the result is positive

Note: If n is **even**, then $\sqrt[n]{\boldsymbol{a^n}} = \begin{cases} a, & if\ a \geq 0 \\ -a, & if\ a < 0 \end{cases} = |\boldsymbol{a}|$.

For example, $\sqrt{7^2} = 7$ and $\sqrt{(-7)^2} = 7$.

d. $\sqrt[3]{(-6)^3} = \sqrt[3]{-216} = -6$

the result has the same sign

Note: If n is **odd**, then $\sqrt[n]{\boldsymbol{a^n}} = \boldsymbol{a}$. For example, $\sqrt[3]{5^3} = 5$ but $\sqrt[3]{(-5)^3} = -5$.

Summary of Properties of n-th Degree Radicals

- If n is **EVEN**, then

$$\sqrt[n]{\text{positive}} = \text{positive}, \quad \sqrt[n]{\text{negative}} = \text{DNE}, \quad \text{and} \quad \sqrt[n]{a^n} = |a|$$

- If n is **ODD**, then

$$\sqrt[n]{\text{positive}} = \text{positive}, \quad \sqrt[n]{\text{negative}} = \text{negative}, \quad \text{and} \quad \sqrt[n]{a^n} = a$$

- For any natural $n \geq 0$, $\sqrt[n]{\mathbf{0}} = \mathbf{0}$ and $\sqrt[n]{\mathbf{1}} = \mathbf{1}$.

Example 3 ► **Simplifying Radical Expressions Using Absolute Value Where Appropriate**

Simplify each radical, assuming that all variables represent any real number.

a. $\sqrt{9x^2y^4}$ **b.** $\sqrt[3]{-27y^3}$ **c.** $\sqrt[4]{a^{20}}$ **d.** $-\sqrt[4]{(k-1)^4}$

Solution ►

a. $\sqrt{9x^2y^4} = \sqrt{(3xy^2)^2} = |3xy^2| = \mathbf{3|x|y^2}$

An **even** degree radical is **nonnegative**, so we must use the **absolute value** operator.

Recall: As discussed in *section L6*, the absolute value operator has the following properties:

$$|xy| = |x||y|$$

$$\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$$

Note: $|y^2| = y^2$ as y^2 is already nonnegative.

b. $\sqrt[3]{-27y^3} = \sqrt[3]{(-3y)^3} = \mathbf{-3y}$

An **odd** degree radical assumes the sign of the radicand, so we **do not apply the absolute value** operator.

c. $\sqrt[4]{a^{20}} = \sqrt[4]{(a^5)^4} = |a^5| = \mathbf{|a|^5}$

Note: To simplify an expression with an absolute value, we keep the absolute value operator as close as possible to the variable(s).

d. $-\sqrt[4]{(k-1)^4} = \mathbf{-|k-1|}$

Radical Functions

Since each nonnegative real number x has exactly one principal square root, we can define the **square root** function, $\boldsymbol{f(x) = \sqrt{x}}$. The **domain** D_f of this function is the set of nonnegative real numbers, $[\mathbf{0}, \infty)$, and so is its **range** (as indicated in *Figure 1*).

To graph the square root function, we create a table of values. The easiest x-values for calculation of the corresponding y-values are the perfect square numbers. However, sometimes we want to use additional x-values that are not perfect squares. Since a square root of such a number, for example $\sqrt{2}$, $\sqrt{3}$, $\sqrt{6}$, etc., is an irrational number, we approximate these values using a calculator.

x	y
0	0
$\frac{1}{4}$	$\frac{1}{2}$
1	1
4	2
6	$\sqrt{6} \approx 2.4$

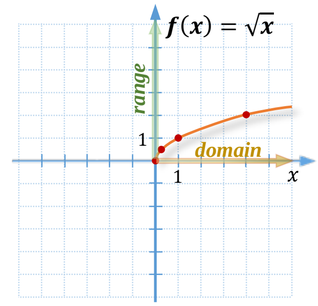


Figure 1

For example, to approximate $\sqrt{6}$, we use the sequence of keying: $\sqrt{\ }$ 6 *ENTER* or 6 ^ (1 / 2) *ENTER*. This is because a square root operator works the same way as the exponent of $\frac{1}{2}$.

Note: When graphing an even degree radical function, it is essential that we find its domain first. The end-point of the domain indicates the starting point of the graph, often called the vertex.

For example, since the domain of $f(x) = \sqrt{x}$ is $[0, \infty)$, the graph starts from the point $\left(0, f(0)\right) = (0,0)$, as in *Figure 1*.

Since the cube root can be evaluated for any real number, the **domain** D_f of the related **cube root** function, $\boldsymbol{f(x) = \sqrt[3]{x}}$, is the set of **all real numbers**, $\mathbb{R}$. The **range** can be observed in the graph (see *Figure 2*) or by inspecting the expression $\sqrt[3]{x}$. It is also $\mathbb{R}$.

To graph the cube root function, we create a table of values. The easiest x-values for calculation of the corresponding y-values are the perfect cube numbers. As before, sometimes we might need to estimate additional x-values. For example, to approximate $\sqrt[3]{6}$, we use the sequence of keying:
$\sqrt[3]{\ }$ 6 *ENTER* or
6 ^ (1 / 3) *ENTER*.

x	y
−8	−2
−6	$-\sqrt[3]{6} \approx -1.8$
−1	−1
$-\frac{1}{8}$	$-\frac{1}{2}$
0	0
$\frac{1}{8}$	$\frac{1}{2}$
1	1
6	$\sqrt[3]{6} \approx 1.8$
8	2

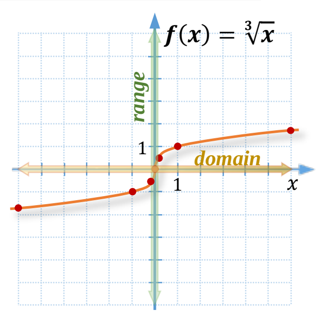


Figure 2

Example 4 ► **Finding a Calculator Approximations of Roots**

Use a calculator to approximate the given root up to three decimal places.

a. $\sqrt{3}$ **b.** $\sqrt[3]{5}$ **c.** $\sqrt[5]{100}$

Solution ►

a. $\sqrt{3} \approx 1.732$

b. $\sqrt[3]{5} \approx 1.710$

5 ^ (1 / 3) *ENTER* or *MATH* 4 5 *ENTER* on a graphing calculator

c. $\sqrt[5]{100} \approx 2.512$

100 ^ (1 / 5) *ENTER* or 5 *MATH* 5 100 *ENTER* on a graphing calculator

Example 5 ► **Finding the Best Integer Approximation of a Square Root**

Without the use of a calculator, determine the best integer approximation of the given root.

a. $\sqrt{68}$ **b.** $\sqrt{140}$

Solution ►

a. Observe that 68 lies between the following two consecutive perfect square numbers, 64 and 81. Also, 68 lies closer to 64 than to 81. Therefore, $\sqrt{68} \approx \sqrt{64} = 8$.

b. 140 lies between the following two consecutive perfect square numbers, 121 and 144. In addition, 140 is closer to 144 than to 121. Therefore, $\sqrt{140} \approx \sqrt{144} = 12$.

Example 6 ► **Finding the Domain of a Radical Function**

Find the domain of each of the following functions.

a. $f(x) = \sqrt{2x+3}$ **b.** $g(x) = 2 - \sqrt{1-x}$

Solution ►

The **domain** of an **even degree** radical is the solution set of the inequality ***radicand*** $\boldsymbol{\geq 0}$

The **domain** of an **odd degree** radical is $\mathbb{R}$.

a. When finding domain D_f of function $f(x) = \sqrt{2x+3}$, we need to protect the radicand $2x+3$ from becoming negative. So, an x-value belongs to the domain D_f if it satisfies the condition

$$2x + 3 \geq 0. \qquad / -3, \quad \div 2$$

This happens for $x \geq -\frac{3}{2}$. Therefore, $\boldsymbol{D_f} = \left[-\frac{3}{2}, \infty\right)$.

b. To find the domain D_g of function $g(x) = 2 - \sqrt{1-x}$, we solve the condition

$$1 - x \geq 0 \qquad / +x$$
$$1 \geq x$$

Thus, $\boldsymbol{D_g} = (-\infty, \mathbf{1}]$.

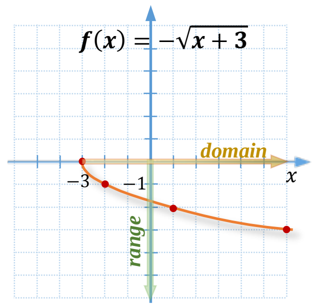
Example 7 ► **Graphing Radical Functions**

For each function, find its domain, graph it, and find its range. Then, observe what transformation(s) of a basic root function result(s) in the obtained graph.

a. $f(x) = -\sqrt{x+3}$ **b.** $g(x) = \sqrt[3]{x} - 2$

Solution ► **a.** The **domain** D_f is the solution set of the inequality $x + 3 \geq 0$, which is equivalent to $x \geq -3$. Hence, $\boldsymbol{D_f = [-3, \infty)}$.

x	y
−3	0
−2	−1
1	−2
6	−3



The projection of the graph onto the y-axis indicates the **range** of this function, which is $(-\infty, \mathbf{0}]$.

The graph of $f(x) = -\sqrt{x+3}$ has the same shape as the graph of the basic square root function $f(x) = \sqrt{x}$, except that it is flipped over the x-axis and moved to the left by three units. These transformations are illustrated in *Figure 3*.

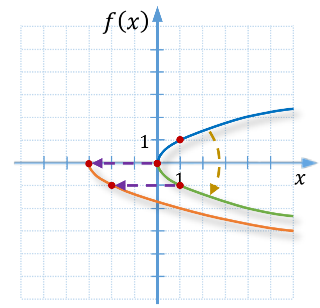
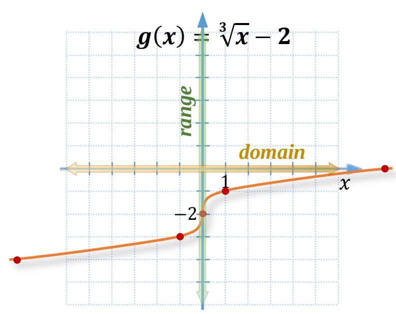


Figure 3

b. The **domain** and **range** of any odd degree radical are both the set of all real numbers. So, $\boldsymbol{D_g} = \mathbb{R}$ and $\boldsymbol{range_g} = \mathbb{R}$.

x	y
−8	−4
−1	−3
0	−2
1	−1
8	0



The graph of $g(x) = \sqrt[3]{x} - 2$ has the same shape as the graph of the basic cube root function $f(x) = \sqrt[3]{x}$, except that it is moved down by two units. This transformation is illustrated in *Figure 4*.

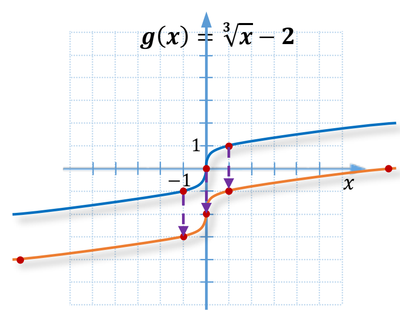


Figure 4

Radicals in Application Problems

Some application problems require evaluation of formulas that involve radicals. For example, the formula $c = \sqrt{a^2 + b^2}$ allows for finding the hypotenuse in a right angle triangle (see *section RD3*), **Heron's** formula $A = \sqrt{s(s-a)(s-b)(s-c)}$ allows for finding the area of any triangle given the lengths of its sides (see *section T5*), the formula $T = 2\pi\sqrt{\frac{d^3}{Gm}}$ allows for finding the time needed for a planet to make a complete orbit around the Sun, and so on.

Example 8 ► **Using a Radical Formula in an Application Problem**

The time t, in seconds, for one complete swing of a simple pendulum is given by the formula

$$t = 2\pi\sqrt{\frac{L}{g}},$$

where L is the length of the pendulum in feet, and g represents the acceleration due to gravity, which is about 32 ft per sec^2. Find the time of a complete swing of a 2-ft pendulum to the nearest tenth of a second.

Solution ► Since $L = 2$ ft and $g = 32$ ft/sec^2, then

$$t = 2\pi\sqrt{\frac{2}{32}} = 2\pi\sqrt{\frac{1}{16}} = 2\pi \cdot \frac{1}{4} = \frac{\pi}{2} \approx 1.6$$

So, the approximate time of a complete swing of a 2-ft pendulum is **1.6 seconds**.

RD.1 Exercises

Vocabulary Check *Complete each blank with one of the suggested words, or the most appropriate term or phrase from the given list:* ***even, index, irrational, nonnegative, odd, principal, radicand, square root****.*

1. The symbol $\sqrt{\ }$ stands for the ____________ square root.

2. For any real numbers $a \geq 0$ and b, if $b^2 = a$, then b is the __________ _________ of a.

3. The expression under the radical sign is called the ___________.

4. The square root of a negative number $\underset{is\ /is\ not}{\underline{\hspace{3em}}}$ a real number.

5. The square root of a number that is not a perfect square is a(n) _______________ number.

6. The cube root of a is written $\sqrt[3]{a}$, where 3 is called the __________ of the radical.

7. The domain of an even degree radical function is the set of all values that would make the radicand ________________.

8. The domain and range of an _____ degree radical functions is the set of all real numbers.

9. If n is ________ then $\sqrt[n]{a^n} = |a|$.

Concept Check *Evaluate each radical, if possible.*

10. $\sqrt{49}$
11. $-\sqrt{81}$
12. $\sqrt{-400}$
13. $\sqrt{0.09}$
14. $\sqrt{0.0016}$
15. $\sqrt{\frac{64}{225}}$
16. $\sqrt[3]{64}$
17. $\sqrt[3]{-125}$
18. $\sqrt[3]{0.008}$
19. $-\sqrt[3]{-1000}$
20. $\sqrt[3]{\frac{1}{0.000027}}$
21. $\sqrt[4]{16}$
22. $\sqrt[5]{0.00032}$
23. $\sqrt[7]{-1}$
24. $\sqrt[8]{-256}$
25. $-\sqrt[6]{\frac{1}{64}}$

Concept Check

26. Decide whether the expression $-\sqrt{-a}$ is positive, negative, 0, or not a real number, given that

a. $a < 0$ **b.** $a > 0$ **c.** $a = 0$

27. Assuming that n is odd, decide whether the expression $-\sqrt[n]{a}$ is positive, negative, or 0, given that

a. $a < 0$ **b.** $a > 0$ **c.** $a = 0$

Simplify each radical. Assume that letters can represent any real number.

28. $\sqrt{15^2}$
29. $\sqrt{(-15)^2}$
30. $\sqrt{x^2}$
31. $\sqrt{(-x)^2}$
32. $\sqrt{81x^2}$
33. $\sqrt{(-12y)^2}$
34. $\sqrt{(a+3)^2}$
35. $\sqrt{(2-x)^2y^4}$
36. $\sqrt{x^2-4x+4}$
37. $\sqrt{9y^2+30y+25}$
38. $\sqrt[3]{(-5)^3}$
39. $\sqrt[3]{x^3}$
40. $\sqrt[3]{-125a^3}$
41. $-\sqrt[3]{0.008(x-1)^3}$
42. $\sqrt[4]{(5x)^4}$
43. $\sqrt[8]{(-10)^8}$
44. $\sqrt[5]{(y-3)^5}$
45. $\sqrt[2017]{(a+b)^{2017}}$
46. $\sqrt[2018]{(2a-b)^{2018}}$
47. $\sqrt[6]{x^{18}}$
48. $\sqrt[4]{(a+1)^{12}}$
49. $\sqrt[5]{(-a)^{20}}$
50. $\sqrt[7]{(-k)^{35}}$
51. $\sqrt[4]{x^4(-y)^8}$

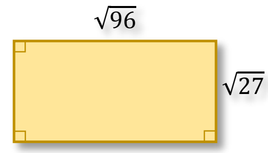
Find a decimal approximation for each radical. Round the answer to three decimal places.

52. $\sqrt{350}$
53. $-\sqrt{0.859}$
54. $\sqrt[3]{5}$
55. $\sqrt[5]{3}$

Concept Check *Without the use of a calculator, give the best integer approximation of each square root.*

56. $\sqrt{67}$ **57.** $\sqrt{95}$ **58.** $\sqrt{115}$ **59.** $\sqrt{87}$

Questions in Exercises 60 and 61 refer to the accompanying rectangle. Answer these questions without the use of a calculator.



60. Give the best integer estimation of the area of the rectangle.

61. Give the best integer estimation of the perimeter of the rectangle.

Solve each problem. Do not use any calculator.

62. A rectangular yard has a length of $\sqrt{189}$ m and a width of $\sqrt{48}$ m. Choose the best estimate of its dimensions. Then estimate the perimeter.

A. 13 m by 7 m **B.** 14 m by 7 m **C.** 14 m by 8 m **D.** 15 m by 7 m

63. If the sides of a triangle are $\sqrt{65}$ cm, $\sqrt{34}$ cm, and $\sqrt{27}$ cm, which one of the following is the best estimate of its perimeter?

A. 20 cm **B.** 26 cm **C.** 19 cm **D.** 24 cm

Graph *each function and give its* ***domain*** *and* ***range****. Then, discuss the* ***transformations*** *of a basic root function needed to obtain the graph of the given function.*

64. $f(x) = \sqrt{x+1}$ **65.** $g(x) = \sqrt{x} + 1$ **66.** $h(x) = -\sqrt{x}$

67. $f(x) = \sqrt{x-3}$ **68.** $g(x) = \sqrt{x} - 3$ **69.** $h(x) = 2 - \sqrt{x}$

70. $f(x) = \sqrt[3]{x-2}$ **71.** $g(x) = \sqrt[3]{x} + 2$ **72.** $h(x) = -\sqrt[3]{x} + 2$

Graph *each function and give its* ***domain*** *and* ***range****.*

73. $f(x) = 2 + \sqrt{x-1}$ **74.** $g(x) = 2\sqrt{x}$ **75.** $h(x) = -\sqrt{x+3}$

76. $f(x) = \sqrt{3x+9}$ **77.** $g(x) = \sqrt{3x-6}$ **78.** $h(x) = -\sqrt{2x-4}$

79. $f(x) = \sqrt{12-3x}$ **80.** $g(x) = \sqrt{8-4x}$ **81.** $h(x) = -2\sqrt{-x}$

Analytic Skills *Graph the three given functions on the same grid and discuss the relationship between them.*

82. $f(x) = 2x + 1;\ \ g(x) = \sqrt{2x+1};\ \ h(x) = \sqrt[3]{2x+1}$

83. $f(x) = -x + 2;\ \ g(x) = \sqrt{-x+2};\ \ h(x) = \sqrt[3]{-x+2}$

84. $f(x) = \frac{1}{2}x + 1;\ \ g(x) = \sqrt{\frac{1}{2}x + 1};\ \ h(x) = \sqrt[3]{\frac{1}{2}x + 1}$

Solve each problem.

85. The distance D, in miles, to the horizon from an observer's point of view over water or "flat" earth is given by $D = \sqrt{2H}$, where H is the height of the point of view, in feet. If a man whose eyes are 6 ft above ground level is standing at the top of a hill 105 ft above "flat" earth, approximately how far to the horizon he will be able to see? *Round the answer to the nearest mile.*

86. The threshold body weight T for a person is the weight above which the risk of death increases greatly. The threshold weight in pounds for men aged 40–49 is related to their height h, in inches, by the formula $h = 12.3\sqrt[3]{T}$. What height corresponds to a threshold weight of 216 lb for a 43-year-old man? *Round your answer to the nearest inch.*

87. The time needed for a planet to make a complete orbit around the Sun is given by the formula $T = 2\pi\sqrt{\frac{d^3}{GM}}$,

where d is the average distance of the planet from the Sun, G is the universal gravitational constant, and M is the mass of the Sun. To the nearest day, find the orbital period of Mercury, knowing that its average distance from the Sun is $5.791 \cdot 10^7$ km, the mass of the Sun is $1.989 \cdot 10^{30}$ kg, and $G = 6.67408 \cdot 10^{-11}$ $m^3/(kg{\cdot}s^2)$.

88. The time it takes for an object to fall a certain distance is given by the equation $t = \sqrt{\frac{2d}{g}}$, where t is the time in seconds, d is the distance in feet, and g is the acceleration due to gravity. If an astronaut above the moon's surface drops an object, how far will it have fallen in 3 seconds? The acceleration on the moon's surface is 5.5 feet per second per second.

*Half of the perimeter (**semiperimeter**) of a triangle with sides a, b, and c is* $\boldsymbol{s = \frac{1}{2}(a + b + c)}$. *The area of such a triangle is given by the **Heron's Formula**:* $\boldsymbol{A = \sqrt{s(s-a)(s-b)(s-c)}}$.
Find the area of a triangular piece of land with the following sides.

89. $a = 3$ m, $b = 4$ m, $c = 5$ m

90. $a = 80$ m, $b = 80$ m, $c = 140$ m