RD.3 Simplifying Radical Expressions and the Distance Formula

In the previous section, we simplified some radical expressions by replacing radical signs with rational exponents, applying the rules of exponents, and then converting the resulting expressions back into radical notation. In this section, we broaden the above method of simplifying radicals by examining products and quotients of radicals with the same indexes, as well as explore the possibilities of decreasing the index of a radical.

In the second part of this section, we will apply the skills of simplifying radicals in problems involving the Pythagorean Theorem. In particular, we will develop the distance formula and apply it to calculate distances between two given points in a plane.

Multiplication, Division, and Simplification of Radicals

Suppose we wish to multiply radicals with the same indexes. This can be done by converting each radical to a rational exponent and then using properties of exponents as follows:

PRODUCT RULE

$$\sqrt[n]{\boldsymbol{a}} \cdot \sqrt[n]{\boldsymbol{b}} = a^{\frac{1}{n}} \cdot b^{\frac{1}{n}} = (ab)^{\frac{1}{n}} = \sqrt[n]{\boldsymbol{ab}}$$

This shows that the **product of same index radicals** is the **radical of the product** of their radicands.

Similarly, the **quotient of same index radicals** is the **radical of the quotient** of their radicands, as we have

QUOTIENT RULE

$$\frac{\sqrt[n]{\boldsymbol{a}}}{\sqrt[n]{\boldsymbol{b}}} = \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \left(\frac{a}{b}\right)^{\frac{1}{n}} = \sqrt[n]{\frac{\boldsymbol{a}}{\boldsymbol{b}}}$$

So, $\sqrt{2} \cdot \sqrt{8} = \sqrt{2 \cdot 8} = \sqrt{16} = \mathbf{4}$. Similarly, $\frac{\sqrt[3]{16}}{\sqrt[3]{2}} = \sqrt[3]{\frac{16}{2}} = \sqrt[3]{8} = \mathbf{2}$.

Attention! There is no such rule for addition or subtraction of terms. For instance,

$$\sqrt{\boldsymbol{a}+\boldsymbol{b}} \neq \sqrt{\boldsymbol{a}} \pm \sqrt{\boldsymbol{b}},$$

and generally

$$\sqrt[n]{\boldsymbol{a} \pm \boldsymbol{b}} \neq \sqrt[n]{\boldsymbol{a}} \pm \sqrt[n]{\boldsymbol{b}}.$$

Here is a counterexample: $\sqrt[n]{2} = \sqrt[n]{1+1} \neq \sqrt[n]{1} + \sqrt[n]{1} = 1 + 1 = 2$

Example 1 ► **Multiplying and Dividing Radicals of the Same Indexes**

Perform the indicated operations and simplify, if possible. Assume that all variables are positive.

a. $\sqrt{10} \cdot \sqrt{15}$

b. $\sqrt{2x^3}\sqrt{6xy}$

c. $\frac{\sqrt{10x}}{\sqrt{5}}$

d. $\frac{\sqrt[4]{32x^3}}{\sqrt[4]{2x}}$

Solution ►

a. $\sqrt{10}\cdot\sqrt{15}=\sqrt{10\cdot 15}=\sqrt{2\cdot 5\cdot 5\cdot 3}=\sqrt{5\cdot 5\cdot 2\cdot 3}=\sqrt{25}\cdot\sqrt{6}=\mathbf{5\sqrt{6}}$

product rule; prime factorization; commutativity of multiplication; product rule

b. $\sqrt{2x^3}\,\sqrt{6xy}=\sqrt{2\cdot 2\cdot 3x^4y}=\sqrt{4x^4}\cdot\sqrt{3y}=\mathbf{2x^2\sqrt{3y}}$

use commutativity of multiplication to isolate perfect square factors

Here the multiplication sign is assumed, even if it is not indicated.

c. $\dfrac{\sqrt{10x}}{\sqrt{5}}=\sqrt{\dfrac{10x}{5}}=\mathbf{\sqrt{2x}}$

quotient rule

d. $\dfrac{\sqrt[4]{32x^3}}{\sqrt[4]{2x}}=\sqrt[4]{\dfrac{32x^3}{2x}}=\sqrt[4]{16x^2}=\sqrt[4]{16}\cdot\sqrt[4]{x^2}=\mathbf{2\sqrt{x}}$

Recall that $\sqrt[4]{x^2}=x^{\frac{2}{4}}=x^{\frac{1}{2}}=\sqrt{x}$.

Caution! Remember to indicate the index of the radical for indexes higher than two.

The product and quotient rules are essential when simplifying radicals.

To **simplify a radical** means to:

1. Make sure that all **power factors of the radicand have exponents smaller than the index of the radical**.

   For example, $\sqrt[3]{2^4x^8y}=\sqrt[3]{2^3x^6}\cdot\sqrt[3]{2x^2y}=2x\sqrt[3]{2x^2y}$.

2. Leave the radicand with **no fractions**.

   For example, $\sqrt{\dfrac{2x}{25}}=\dfrac{\sqrt{2x}}{\sqrt{25}}=\dfrac{\sqrt{2x}}{5}$.

3. **Rationalize** any **denominator**. (Make sure that denominators are **free from radicals**.)

   For example, $\sqrt{\dfrac{4}{x}}=\dfrac{\sqrt{4}}{\sqrt{x}}=\dfrac{2\cdot\sqrt{x}}{\sqrt{x}\cdot\sqrt{x}}=\dfrac{2\sqrt{x}}{x}$, providing that $x>0$.

4. **Reduce the power** of the radicand **with the index** of the radical, if possible.

   For example, $\sqrt[4]{x^2}=x^{\frac{2}{4}}=x^{\frac{1}{2}}=\sqrt{x}$.

Example 2 ► **Simplifying Radicals**

Simplify each radical. Assume that all variables are positive.

a. $\sqrt[5]{96x^7y^{15}}$ **b.** $\sqrt[4]{\dfrac{a^{12}}{16b^4}}$ **c.** $\sqrt{\dfrac{25x^2}{8x^3}}$ **d.** $\sqrt[6]{27a^{15}}$

Solution ►

a. $\sqrt[5]{96x^7y^{15}} = \sqrt[5]{2^5 \cdot 3x^7y^{15}} = \boldsymbol{2xy^3\sqrt[5]{3x^2}}$

$\sqrt[5]{y^{15}} = y^3$

$\sqrt[5]{x^7} = x\sqrt[5]{x^2}$

Generally, to simplify $\sqrt[d]{x^a}$, we perform the division

$$a \div d = quotient\ \boldsymbol{q} + remainder\ \boldsymbol{r},$$

and then pull the $\boldsymbol{q}$-th power of x out of the radical, leaving the $\boldsymbol{r}$-th power of x under the radical. So, we obtain

$$\sqrt[d]{x^a} = x^q\sqrt[d]{x^r}$$

b. $\sqrt[4]{\dfrac{a^{12}}{16b^4}} = \dfrac{\sqrt[4]{a^{12}}}{\sqrt[4]{2^4b^4}} = \boldsymbol{\dfrac{a^3}{2b}}$

c. $\sqrt{\dfrac{25x^2}{8x^3}} = \sqrt{\dfrac{25}{2^3x}} = \dfrac{\sqrt{25}}{\sqrt{2^3x}} = \dfrac{5}{2\sqrt{2x}} \cdot \dfrac{\sqrt{2x}}{\sqrt{2x}} = \dfrac{5\sqrt{2x}}{2 \cdot 2x} = \boldsymbol{\dfrac{5\sqrt{2x}}{4x}}$

d. $\sqrt[6]{27a^{15}} = \sqrt[6]{3^3a^{15}} = a^2\sqrt[6]{3^3a^3} = a^2 \cdot \sqrt[6]{(3a)^3} = \boldsymbol{a^2\sqrt{3a}}$

Example 3 ► **Simplifying Expressions Involving Multiplication, Division, or Composition of Radicals with Different Indexes**

Simplify each expression. Leave your answer in simplified single radical form. Assume that all variables are positive.

a. $\sqrt{xy^5} \cdot \sqrt[3]{x^4y}$ **b.** $\dfrac{\sqrt[4]{a^2b^3}}{\sqrt[3]{ab}}$ **c.** $\sqrt[3]{x^2\sqrt{2x}}$

Solution ►

a. $\sqrt{xy^5} \cdot \sqrt[3]{x^2y} = x^{\frac{1}{2}}y^{\frac{5}{2}} \cdot x^{\frac{2}{3}}y^{\frac{1}{3}} = x^{\frac{1\cdot3}{2\cdot3}+\frac{2\cdot2}{3\cdot2}}y^{\frac{5\cdot3}{2\cdot3}+\frac{1\cdot2}{3\cdot2}} = x^{\frac{7}{6}}y^{\frac{17}{6}} = (x^7y^{17})^{\frac{1}{6}}$

$= \sqrt[6]{x^7y^{17}} = \boldsymbol{xy^2\sqrt[6]{xy^5}}$

If radicals are of different indexes, convert them to exponential form.

b. $\dfrac{\sqrt[4]{a^2b^3}}{\sqrt[3]{ab}} = \dfrac{a^{\frac{2}{4}}b^{\frac{3}{4}}}{a^{\frac{1}{3}}b^{\frac{1}{3}}} = a^{\frac{1\cdot3}{2\cdot3}-\frac{1\cdot2}{3\cdot2}}b^{\frac{3\cdot3}{4\cdot3}-\frac{1\cdot4}{3\cdot4}} = a^{\frac{1\cdot2}{6\cdot2}}b^{\frac{5}{12}} = (a^2b^5)^{\frac{1}{12}} = \boldsymbol{\sqrt[12]{a^2b^5}}$

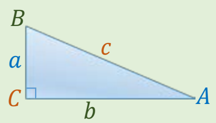
Bring the exponents to the LCD in order to leave the answer as a single radical.

c. $\sqrt[3]{x^2\sqrt{2x}} = x^{\frac{2}{3}} \cdot \left((2x)^{\frac{1}{2}}\right)^{\frac{1}{3}} = x^{\frac{2}{3}} \cdot 2^{\frac{1}{6}} \cdot x^{\frac{1}{6}} = x^{\frac{2\cdot2}{3\cdot2}+\frac{1}{6}} \cdot 2^{\frac{1}{6}} = 2^{\frac{1}{6}}x^{\frac{5}{6}} = (2x^5)^{\frac{1}{6}} = \boldsymbol{\sqrt[6]{2x^5}}$

Pythagorean Theorem and Distance Formula

One of the most famous theorems in mathematics is the Pythagorean Theorem.

Pythagorean Theorem



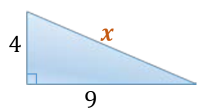
Suppose angle C in a triangle ABC is a $90°$ angle.
Then the **sum of the squares** of the lengths of the two **legs**, $\boldsymbol{a}$ and $\boldsymbol{b}$, equals to the **square** of the length of the **hypotenuse** $\boldsymbol{c}$:

$$\boldsymbol{a^2 + b^2 = c^2}$$

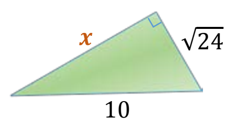
Example 4 **Using The Pythagorean Equation**

For the first two triangles, find the exact length x of the unknown side. For triangle **c.**, express length x in terms of the unknown n.

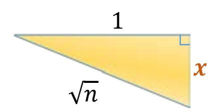
a.



b.



c.



Solution

Caution: Generally, $\sqrt{x^2} = |x|$ However, the length of a side of a triangle is positive. So, we can write $\sqrt{x^2} = x$

a. The length of the hypotenuse of the given right triangle is equal to x. So, the Pythagorean equation takes the form

$$x^2 = 4^2 + 9^2.$$

To solve it for x, we take a square root of each side of the equation. This gives us

$$\sqrt{x^2} = \sqrt{4^2 + 9^2}$$
$$x = \sqrt{16 + 81}$$
$$x = \boldsymbol{\sqrt{97}}$$

b. Since 10 is the length of the hypotenuse, we form the Pythagorean equation

$$10^2 = x^2 + \sqrt{24}^2.$$

To solve it for x, we isolate the x^2 term and then apply the square root operator to both sides of the equation. So, we have

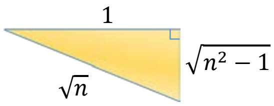
$$10^2 - \sqrt{24}^2 = x^2$$
$$100 - 24 = x^2$$
$$x^2 = 76$$
$$x = \sqrt{76} = \sqrt{4 \cdot 19} = \boldsymbol{2\sqrt{19}}$$

Customary, we simplify each root, if possible.

c. The length of the hypotenuse is $\sqrt{n}$, so we form the Pythagorean equation as below.

$$\left(\sqrt{n}\right)^2 = 1^2 + x^2$$

To solve this equation for x, we isolate the x^2 term and then apply the square root operator to both sides of the equation. So, we obtain



$$n^2 = 1 + x^2$$
$$n^2 - 1 = x^2$$
$$x = \boldsymbol{\sqrt{n^2-1}}$$

Note: Since the hypotenuse of length $\sqrt{n}$ must be longer than the leg of length 1, then $n > 1$. This means that $n^2 - 1 > 0$, and therefore $\sqrt{n^2-1}$ is a positive real number.

The Pythagorean Theorem allows us to find the distance between any two given points in a plane.

Suppose $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points in a coordinate plane. Then $|x_2 - x_1|$ represents the horizontal distance between A and B and $|y_2 - y_1|$ represents the vertical distance between A and B, as shown in *Figure 1*.

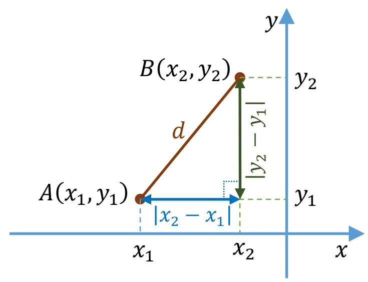


Figure 1

Notice that by applying the absolute value operator to each difference of the coordinates we guarantee that the resulting horizontal and vertical distance is indeed a nonnegative number.

Applying the Pythagorean Theorem to the right triangle shown in *Figure 1*, we form the equation

$$d^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2,$$

where d is the distance between A and B.

Notice that $|x_2 - x_1|^2 = (x_2 - x_1)^2$ as a perfect square automatically makes the expression nonnegative. Similarly, $|y_2 - y_1|^2 = (y_2 - y_1)^2$. So, the Pythagorean equation takes the form

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

After solving this equation for d, we obtain the **distance formula**:

$$\boldsymbol{d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

Note: Observe that due to squaring the difference of the corresponding coordinates, **the distance between two points is the same regardless of which point is chosen as first**, (x_1, y_1), and second, (x_2, y_2).

Example 5 ► **Finding the Distance Between Two Points**

Find the exact distance between the points $(-2,4)$ and $(5,3)$.

Solution ► Let $(-2,4) = (x_1, y_1)$ and $(5,3) = (x_2, y_2)$. To find the distance d between the two points, we follow the distance formula:

$$d=\sqrt{\left(5-(-2)\right)^2+(3-4)^2}=\sqrt{7^2+(-1)^2}=\sqrt{49+1}=\sqrt{50}=\mathbf{5\sqrt{2}}$$

So, the points $(-2,4)$ and $(5,3)$ are $5\sqrt{2}$ units apart.

RD.3 Exercises

Vocabulary Check *Complete each blank with the most appropriate term or phrase from the given list:* ***distance, index, lower, product, Pythagorean, right, simplified.***

1. The product of the same __________ radicals is the radical of the ___________ of the radicands.
2. The radicand of a simplified radical contains only powers with ________ exponents than the index of the radical.
3. The radical $\sqrt[6]{x^4}$ can still be ______________.
4. The _______________ between two points (x_1, y_1) and (x_2, y_2) on a coordinate plane is equal to $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$.
5. The _________________ equation applies to __________ triangles only.

Concept Check *Multiply and simplify, if possible. Assume that all variables are positive.*

6. $\sqrt{5}\cdot\sqrt{5}$
7. $\sqrt{18}\cdot\sqrt{2}$
8. $\sqrt{6}\cdot\sqrt{3}$
9. $\sqrt{15}\cdot\sqrt{6}$
10. $\sqrt{45}\cdot\sqrt{60}$
11. $\sqrt{24}\cdot\sqrt{75}$
12. $\sqrt{3x^3}\cdot\sqrt{6x^5}$
13. $\sqrt{5y^7}\cdot\sqrt{15a^3}$
14. $\sqrt{12x^3y}\sqrt{8x^4y^2}$
15. $\sqrt{30a^3b^4}\sqrt{18a^2b^5}$
16. $\sqrt[3]{4x^2}\sqrt[3]{2x^4}$
17. $\sqrt[4]{20a^3}\sqrt[4]{4a^5}$

Concept Check *Divide and simplify, if possible. Assume that all variables are positive.*

18. $\frac{\sqrt{90}}{\sqrt{5}}$
19. $\frac{\sqrt{48}}{\sqrt{6}}$
20. $\frac{\sqrt{42a}}{\sqrt{7a}}$
21. $\frac{\sqrt{30x^3}}{\sqrt{10x}}$
22. $\frac{\sqrt{52ab^3}}{\sqrt{13a}}$
23. $\frac{\sqrt{56xy^3}}{\sqrt{8x}}$
24. $\frac{\sqrt{128x^2y}}{2\sqrt{2}}$
25. $\frac{\sqrt{48a^3b}}{2\sqrt{3}}$
26. $\frac{\sqrt[4]{80}}{\sqrt[4]{5}}$
27. $\frac{\sqrt[3]{108}}{\sqrt[3]{4}}$
28. $\frac{\sqrt[3]{96a^5b^2}}{\sqrt[3]{12a^2b}}$
29. $\frac{\sqrt[4]{48x^9y^{13}}}{\sqrt[4]{3xy^5}}$

Concept Check *Simplify each expression. Assume that all variables are positive.*

30. $\sqrt{144x^4y^9}$
31. $-\sqrt{81m^8n^5}$
32. $\sqrt[3]{-125a^6b^9c^{12}}$
33. $\sqrt{50x^3y^4}$
34. $\sqrt[4]{\frac{1}{16}m^8n^{20}}$
35. $-\sqrt[3]{-\frac{1}{27}x^2y^7}$
36. $\sqrt{7a^7b^6}$
37. $\sqrt{75p^3q^4}$

38. $\sqrt[5]{64x^{12}y^{15}}$ **39.** $\sqrt[5]{p^{14}q^7r^{23}}$ **40.** $-\sqrt[4]{162a^{15}b^{10}}$ **41.** $-\sqrt[4]{32x^5y^{10}}$

42. $\sqrt{\frac{16}{49}}$ **43.** $\sqrt[3]{\frac{27}{125}}$ **44.** $\sqrt{\frac{121}{y^2}}$ **45.** $\sqrt{\frac{64}{x^4}}$

46. $\sqrt[3]{\frac{81a^5}{64}}$ **47.** $\sqrt{\frac{36x^5}{y^6}}$ **48.** $\sqrt[4]{\frac{16x^{12}}{y^4z^{16}}}$ **49.** $\sqrt[5]{\frac{32y^8}{x^{10}}}$

50. $\sqrt[4]{36}$ **51.** $\sqrt[6]{27}$ **52.** $-\sqrt[10]{x^{25}}$ **53.** $\sqrt[12]{x^{44}}$

54. $-\sqrt{\frac{1}{x^3y}}$ **55.** $\sqrt[3]{\frac{64x^{15}}{y^4z^5}}$ **56.** $\sqrt[6]{\frac{x^{13}}{y^6z^{12}}}$ **57.** $\sqrt[6]{\frac{p^9q^{24}}{r^{18}}}$

Discussion Point

58. When asked to simplify the radical $\sqrt{x^3+x^2}$, a student wrote $\sqrt{x^3+x^2} = x\sqrt{x} + x = x(\sqrt{x}+1)$. Is this correct? If *yes*, explain the student's reasoning. If *not*, discuss how such radical could be simplified.

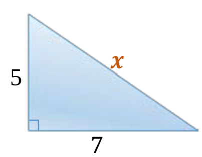
Perform operations. Leave the answer in simplified ***single radical*** *form. Assume that all variables are positive.*

59. $\sqrt{3}\cdot\sqrt[3]{4}$ **60.** $\sqrt{x}\cdot\sqrt[5]{x}$ **61.** $\sqrt[3]{x^2}\cdot\sqrt[4]{x}$ **62.** $\sqrt[3]{4}\cdot\sqrt[5]{8}$

63. $\frac{\sqrt[3]{a^2}}{\sqrt{a}}$ **64.** $\frac{\sqrt{x}}{\sqrt[4]{x}}$ **65.** $\frac{\sqrt[4]{x^2y^3}}{\sqrt[3]{xy}}$ **66.** $\frac{\sqrt[5]{16a^2}}{\sqrt[3]{2a^2}}$

67. $\sqrt[3]{2\sqrt{x}}$ **68.** $\sqrt{x\sqrt[3]{2x^2}}$ **69.** $\sqrt[4]{3\sqrt[3]{9}}$ **70.** $\sqrt[3]{x^2\sqrt[4]{x^3}}$

Concept Check *For each right triangle, find length x. Simplify the answer if possible. In problems* ***73*** *and* ***74,*** *expect the length x to be an expression in terms of n.*

71.


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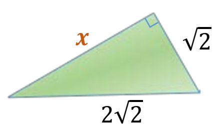

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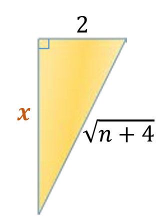

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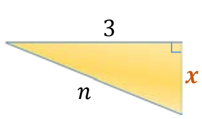

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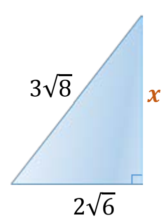

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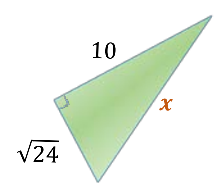


Concept Check *Find the exact distance between each pair of points.*

77. $(8,13)$ and $(2,5)$ **78.** $(-8,3)$ and $(-4,1)$ **79.** $(-6,5)$ and $(3,-4)$

80. $\left(\frac{5}{7},\frac{1}{14}\right)$ and $\left(\frac{1}{7},\frac{11}{14}\right)$ **81.** $(0,\sqrt{6})$ and $(\sqrt{7},0)$ **82.** $(\sqrt{2},\sqrt{6})$ and $(2\sqrt{2},-4\sqrt{6})$

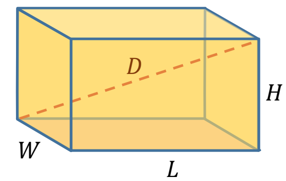
83. $\left(-\sqrt{5}, 6\sqrt{3}\right)$ and $\left(\sqrt{5}, \sqrt{3}\right)$

84. $(0,0)$ and (p, q)

85. $(x+h, y+h)$ and (x, y) (*assume that* $h > 0$)

Analytic Skills *Solve each problem.*

86. The length of the diagonal of a box is given by the formula $D = \sqrt{W^2 + L^2 + H^2}$, where W, L, and H are, respectively, the width, length, and height of the box. Find the length of the diagonal D of a box that is 4 ft long, 2 ft wide, and 3 ft high. Give the exact value, and then round to the nearest tenth of a foot.



87. The screen of a 32-inch television is 27.9-inch wide. To the nearest tenth of an inch, what is the measure of its height? *TVs are measured diagonally, so a 32-inch television means that its screen measures diagonally 32 inches.*

88. Find all ordered pairs on the x-axis of a Cartesian coordinate system that are 5 units from the point (0, 4).

89. Find all ordered pairs on the y-axis of a Cartesian coordinate system that are 5 units from the point (3, 0).

90. During the summer heat, a 2-mi bridge expands 2 ft in length. If we assume that the bulge occurs in the middle of the bridge, how high is the bulge? *The answer may surprise you. In reality, bridges are built with expansion spaces to avoid such buckling.*