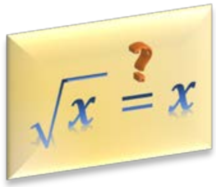
RD.5 Radical Equations



In this section, we discuss techniques for solving radical equations. These are equations containing at least one radical expression with a variable, such as $\sqrt{3x-2} = x$, or a variable expression raised to a fractional exponent, such as $(2x)^{\frac{1}{3}} + 1 = 5$.
At the end of this section, we revisit working with formulas involving radicals as well as application problems that can be solved with the use of radical equations.

Radical Equations

Definition 5.1 ► A **radical equation** is an equation in which a variable appears in one or more radicands. This includes radicands 'hidden' under fractional exponents.
For example, since $(x-1)^{\frac{1}{2}} = \sqrt{x-1}$, then the base $x - 1$ is, in fact, the 'hidden' radicand.

Some examples of radical equations are

$$x = \sqrt{2x}, \quad \sqrt{x} + \sqrt{x-2} = 5, \quad (x-4)^{\frac{3}{2}} = 8, \quad \sqrt[3]{3+x} = 5$$

Note that $x = \sqrt{2}$ is not a radical equation since there is no variable under the radical sign.

The process of solving radical equations involves clearing radicals by raising both sides of an equation to an appropriate power. This method is based on the following property of equality.

Power Rule: For any **odd** natural number $\boldsymbol{n}$, the equation $\boldsymbol{a} = \boldsymbol{b}$ is equivalent to the equation $\boldsymbol{a^n} = \boldsymbol{b^n}$.

For any **even** natural number $\boldsymbol{n}$, if an equation $\boldsymbol{a} = \boldsymbol{b}$ is true, then $\boldsymbol{a^n} = \boldsymbol{b^n}$ is true.

When rephrased, the power rule for odd powers states that the solution sets to both equations, $a = b$ and $a^n = b^n$, are exactly the same.

However, the power rule for even powers states that the solutions to the original equation $a = b$ are among the solutions to the 'power' equation $a^n = b^n$.

Unfortunately, the reverse implication does not hold for even numbers n. We cannot conclude that $a = b$ from the fact that $a^n = b^n$ is true. For instance, $3^2 = (-3)^2$ is true but $3 \neq -3$. This means that not all solutions of the equation $a^n = b^n$ are in fact true solutions to the original equation $a = b$. Solutions that do not satisfy the original equation are called **extraneous solutions** or **extraneous roots**. Such solutions must be rejected.

For example, to solve $\sqrt{2-x} = x$, we may square both sides of the equation to obtain the quadratic equation

$$2 - x = x^2.$$

Then, we solve it via factoring and the zero-product property:

$$x^2 + x - 2 = 0$$

$$(x+2)(x-1) = 0$$

So, the possible solutions are $x=-2$ and $x=1$.

Notice that $x=1$ satisfies the original equation, as $\sqrt{2-1}=1$ is true. However, $x=-2$ does not satisfy the original equation as its left side equals to $\sqrt{2-(-2)}=\sqrt{4}=2$, while the right side equals to -2. Thus, $x=-2$ is the extraneous root and as such, it does not belong to the solution set of the original equation. So, the solution set of the original equation is $\{1\}$.

Caution: When the power rule for **even powers** is used to solve an equation, **every solution** of the 'power' equation **must be checked in the original equation**.

Example 1 ► **Solving Equations with One Radical**

Solve each equation.

a. $\sqrt{3x+4}=4$

b. $\sqrt{2x-5}+4=0$

c. $2\sqrt{x+1}=x-7$

d. $\sqrt[3]{x-8}+2=0$

Solution ►

a. Since the radical in $\sqrt{3x+4}=4$ is isolated on one side of the equation, squaring both sides of the equation allows for clearing (reversing) the square root. Then, by solving the resulting polynomial equation, one can find the possible solution(s) to the original equation.

$$\left(\sqrt{a}\right)^2=\left(a^{\frac{1}{2}}\right)^2=a$$

$$\left(\sqrt{3x+4}\right)^2=(4)^2$$
$$3x+4=16$$
$$3x=12$$
$$x=4$$

To check if 4 is a true solution, it is enough to check whether or not $x=4$ satisfies the original equation.

$$\sqrt{3\cdot 4+4}\stackrel{?}{=}4$$
$$\sqrt{16}\stackrel{?}{=}4$$
$$4=4 \quad \checkmark$$

true

Since $x=4$ satisfies the original equation, the solution set is $\{\mathbf{4}\}$.

b. To solve $\sqrt{2x-5}+4=0$, it is useful to isolate the radical on one side of the equation. So, consider the equation

$$\sqrt{2x-5}=-4$$

Notice that the left side of the above equation is nonnegative for any x-value while the right side is constantly negative. Thus, such an equation cannot be satisfied by any x-value. Therefore, this equation has **no solution**.

c. Squaring both sides of the equation gives us

$$\left(2\sqrt{x+1}\right)^2 = (x-7)^2$$

$$4(x+1) = x^2 - 14x + 49$$

the bracket is essential here

apply the **perfect square** formula $(a-b)^2 = \boldsymbol{a^2 - 2ab + b^2}$

$$4x + 4 = x^2 - 14x + 49$$

$$x^2 - 18x + 45 = 0$$

$$(x-3)(x-15) = 0$$

So, the possible solutions are $x = 3$ or $x = 15$. We check each of them by substituting them into the original equation.

If $x = 3$, then

$$2\sqrt{3+1} \stackrel{?}{=} 3 - 7$$

$$2\sqrt{4} \stackrel{?}{=} -4$$

$$4 \neq -4$$

false

So $x = 3$ is the **extraneous** root.

If $x = 15$, then

$$2\sqrt{15+1} \stackrel{?}{=} 15 - 7$$

$$2\sqrt{16} \stackrel{?}{=} 8$$

$$8 = 8$$

true

Since only 15 satisfies the original equation, the solution set is $\{15\}$.

d. To solve $\sqrt[3]{x-8} + 2 = 0$, we first isolate the radical by subtracting 2 from both sides of the equation.

$$\sqrt[3]{x-8} = -2$$

Then, to clear the cube root, we raise both sides of the equation to the third power.

$$\left(\sqrt[3]{x-8}\right)^3 = (-2)^3$$

So, we obtain

$$x - 8 = -8$$

$$x = 0$$

Since we applied the power rule for odd powers, the obtained solution is the true solution. So the solution set is $\{0\}$.

Observation: When using the power rule for odd powers checking the obtained solutions against the original equation is not necessary. This is because there is no risk of obtaining extraneous roots when applying the power rule for odd powers.

To solve radical equations with more than one radical term, we might need to apply the power rule repeatedly until all radicals are cleared. In an efficient solution, each application of the power rule should cause clearing of at least one radical term. For that reason, it is a good idea to isolate a single radical term on one side of the equation before each application of the power rule. For example, to solve the equation

$$\sqrt{x-3}+\sqrt{x+5}=4,$$

we isolate one of the radicals before squaring both sides of the equation. So, we have

$$\left(\sqrt{x-3}\right)^2=\left(4-\sqrt{x+5}\right)^2$$

$$\cancel{x}-3=\underbrace{16}_{a^2}-\underbrace{8\sqrt{x+5}}_{2ab}+\underbrace{\cancel{x}+5}_{b^2}$$

Remember that the **perfect square** formula consists of **three terms**.

Then, we isolate the remaining radical term and simplify, if possible. This gives us

$$8\sqrt{x+5}=24 \qquad / \div 8$$

$$\sqrt{x+5}=3 \qquad / \,()^2$$

Squaring both sides of the last equation gives us

$$x+5=9 \qquad / -5$$

$$x=4$$

The reader is encouraged to check that $x=\mathbf{4}$ is the true solution to the original equation.

A general strategy for solving radical equations, including those with two radical terms, is as follows.

Summary of Solving a Radical Equation

1. **Isolate one of the radical terms**. Make sure that one radical term is alone on one side of the equation.
2. **Apply an appropriate power rule.** Raise each side of the equation to a power that is the same as the index of the isolated radical.
3. **Solve the resulting equation.** If it still contains a radical, repeat steps 1 and 2.
4. **Check** all proposed solutions in the original equation.
5. **State the solution set** to the original equation.

Example 2 ► **Solving Equations Containing Two Radical Terms**

Solve each equation.

a. $\sqrt{3x+1}-\sqrt{x+4}=1$ **b.** $\sqrt[3]{4x-5}=2\sqrt[3]{x+1}$

Solution ► **a.** We start solving the equation $\sqrt{3x+1}-\sqrt{x+4}=1$ by isolating one radical on one side of the equation. This can be done by adding $\sqrt{x+4}$ to both sides of the equation. So, we have

$$\sqrt{3x+1}=1+\sqrt{x+4}$$

which after squaring give us

$$\left(\sqrt{3x+1}\right)^2 = \left(1+\sqrt{x+4}\right)^2$$

$$3x+1 = 1+2\sqrt{x+4}+x+4$$

$$2x-4 = 2\sqrt{x+4} \qquad / \div 2$$

$$x-2 = \sqrt{x+4}$$

To clear the remaining radical, we square both sides of the above equation again.

$$(x-2)^2 = \left(\sqrt{x+4}\right)^2$$

$$x^2-4x+4 = x+4$$

$$x^2-5x = 0$$

The resulting polynomial equation can be solved by factoring and applying the zero-product property. Thus,

$$x(x-5) = 0.$$

So, the possible roots are $x = 0$ or $x = 5$.

We check each of them by substituting to the original equation.

If $x = 0$, then

$$\sqrt{3\cdot 0+1}-\sqrt{0+4} \stackrel{?}{=} 1$$

$$\sqrt{1}-\sqrt{4} \stackrel{?}{=} 1$$

$$1-2 \stackrel{?}{=} 1$$

$$-1 \neq 1$$

false

If $x = 5$, then

$$\sqrt{3\cdot 5+1}-\sqrt{5+4} \stackrel{?}{=} 1$$

$$\sqrt{16}-\sqrt{9} \stackrel{?}{=} 1$$

$$4-3 \stackrel{?}{=} 1$$

$$1 = 1$$

true

Since $x = 0$ is the **extraneous** root, it does not belong to the solution set.

Only 5 satisfies the original equation. So, the solution set is $\{\mathbf{5}\}$.

b. To solve the equation $\sqrt[3]{4x-5} = 2\sqrt[3]{x+1}$, we would like to clear the cubic roots. This can be done by cubing both of its sides, as shown below.

$$\left(\sqrt[3]{4x-5}\right)^3 = \left(2\sqrt[3]{x+1}\right)^3$$

$$4x-5 = 2^3(x+1)$$

the bracket is essential here

$$4x-5 = 8x+8 \qquad / -4x, -8$$

$$-13 = 4x \qquad / \div(-13)$$

$$x = -\frac{\mathbf{4}}{\mathbf{13}}$$

Since we applied the power rule for cubes, the obtained root is the true solution of the original equation.

Formulas Containing Radicals

Many formulas involve radicals. For example, the period T, in seconds, of a pendulum of length L, in feet, is given by the formula

$$T = 2\pi\sqrt{\frac{L}{32}}$$

Sometimes, we might need to solve a radical formula for a specified variable. In addition to all the strategies for solving formulas for a variable, discussed in *sections L2*, *F4*, and *RT6*, we may need to apply the power rule to clear the radical(s) in the formula.

Example 3 ► **Solving Radical Formulas for a Specified Variable**

Solve each formula for the indicated variable.

a. $N = \frac{1}{2\pi}\sqrt{\frac{\boldsymbol{a}}{r}}$ for $\boldsymbol{a}$ **b.** $r = \sqrt[3]{\frac{A}{\boldsymbol{P}}} - 1$ for $\boldsymbol{P}$

Solution ►

a. Since $\boldsymbol{a}$ appears in the radicand, to solve $N = \frac{1}{2\pi}\sqrt{\frac{\boldsymbol{a}}{r}}$ for $\boldsymbol{a}$, we may want to clear the radical by squaring both sides of the equation. So, we have

$$N^2 = \left(\frac{1}{2\pi}\sqrt{\frac{\boldsymbol{a}}{r}}\right)^2$$

$$N^2 = \frac{1}{(2\pi)^2} \cdot \frac{\boldsymbol{a}}{r} \qquad / \cdot 4\pi^2 r$$

$$\boldsymbol{4\pi^2 N^2 r = a}$$

Note: We could also first multiply by 2π and then square both sides of the equation.

b. First, observe the position of $\boldsymbol{P}$ in the equation $r = \sqrt[3]{\frac{A}{\boldsymbol{P}}} - 1$. It appears in the denominator of the radical. Therefore, to solve for $\boldsymbol{P}$, we may plan to isolate the cube root first, cube both sides of the equation to clear the radical, and finally bring $\boldsymbol{P}$ to the numerator. So, we have

$$r = \sqrt[3]{\frac{A}{\boldsymbol{P}}} - 1 \qquad / +1$$

$$(r+1)^3 = \left(\sqrt[3]{\frac{A}{\boldsymbol{P}}}\right)^3$$

$$(r+1)^3 = \frac{A}{P} \qquad / \cdot P, \ \div (r+1)^3$$

$$P = \frac{A}{(r+1)^3}$$

Radicals in Applications

Many application problems in sciences, engineering, or finances translate into radical equations.

Example 4 ► **Finding the Velocity of a Skydiver**

When skydivers initially fall from an airplane, their velocity v in kilometers per hour after free falling d meters can be approximated by $v = 15.9\sqrt{d}$. Approximately how far, in meters, do skydivers need to fall to attain 100 kph?

Solution ► We may substitute $v = 100$ into the equation $v = 15.9\sqrt{d}$ and solve it for d. Thus,

$$100 = 15.9\sqrt{d} \qquad / \div 15.9$$

$$6.3 \approx \sqrt{d} \qquad / \textit{ square both sides}$$

$$\mathbf{40 \approx d}$$

So, skydivers fall at 100 kph approximately after 40 meters of free falling.

RD.5 Exercises

Vocabulary Check *Complete each blank with the most appropriate term or phrase from the given list:* ***even, extraneous, original, power, radical, solution.***

1. A __________ equation contains at least one radical with a variable.

2. Radical equations are solved by applying an appropriate ________ rule to both sides of the equation.

3. The roots obtained in the process of solving a radical equation that do not satisfy the original equation are called _____________ roots. They do not belong to the ____________ set of the original equation.

4. If a power rule for _______ powers is applied, every possible solution must be checked in the ___________ equation.

Concept Check *True or false.*

5. $\sqrt{2}x = x^2 - \sqrt{5}$ is a radical equation.

6. When raising each side of a radical equation to a power, the resulting equation is equivalent to the original equation.

7. $\sqrt{3x+9} = x$ cannot have negative solutions.

8. -9 is a solution to the equation $\sqrt{x} = -3$.

Solve each equation.

9. $\sqrt{7x-3} = 6$ **10.** $\sqrt{5y+2} = 7$ **11.** $\sqrt{6x} + 1 = 3$ **12.** $\sqrt{2k} - 4 = 6$

13. $\sqrt{x+2} = -6$ **14.** $\sqrt{y-3} = -2$ **15.** $\sqrt[3]{x} = -3$ **16.** $\sqrt[3]{a} = -1$

17. $\sqrt[4]{y-3} = 2$ **18.** $\sqrt[4]{n+1} = 3$ **19.** $5 = \frac{1}{\sqrt{a}}$ **20.** $\frac{1}{\sqrt{y}} = 3$

21. $\sqrt{3r+1} - 4 = 0$ **22.** $\sqrt{5x-4} - 9 = 0$ **23.** $4 - \sqrt{y-2} = 0$

24. $9 - \sqrt{4a+1} = 0$ **25.** $x - 7 = \sqrt{x-5}$ **26.** $x + 2 = \sqrt{2x+7}$

27. $2\sqrt{x+1} - 1 = x$ **28.** $3\sqrt{x-1} - 1 = x$ **29.** $y - 4 = \sqrt{4-y}$

30. $x + 3 = \sqrt{9-x}$ **31.** $x = \sqrt{x^2+4x-20}$ **32.** $x = \sqrt{x^2+3x+9}$

Concept Check

33. When solving the equation $\sqrt{3x+4} = 8 - x$, a student wrote the following for the first step.

$$3x + 4 = 64 + x^2$$

Is this correct? Justify your answer.

34. When solving the equation $\sqrt{5x+6} - \sqrt{x+3} = 3$, a student wrote the following for the first step.

$$(5x+6) + (x+3) = 9$$

Is this correct? Justify your answer.

Solve each equation.

35. $\sqrt{5x+1} = \sqrt{2x+7}$ **36.** $\sqrt{5y-3} = \sqrt{2y+3}$ **37.** $\sqrt[3]{p+5} = \sqrt[3]{2p-4}$

38. $\sqrt[3]{x^2+5x+1} = \sqrt[3]{x^2+4x}$ **39.** $2\sqrt{x-3} = \sqrt{7x+15}$ **40.** $\sqrt{6x-11} = 3\sqrt{x-7}$

41. $3\sqrt{2t+3} - \sqrt{t+10} = 0$ **42.** $2\sqrt{y-1} - \sqrt{3y-1} = 0$ **43.** $\sqrt{x-9} + \sqrt{x} = 1$

44. $\sqrt{y-5} + \sqrt{y} = 5$ **45.** $\sqrt{3n} + \sqrt{n-2} = 4$ **46.** $\sqrt{x+5} - 2 = \sqrt{x-1}$

47. $\sqrt{14-n} = \sqrt{n+3} + 3$ **48.** $\sqrt{p+15} - \sqrt{2p+7} = 1$ **49.** $\sqrt{4a+1} - \sqrt{a-2} = 3$

50. $4-\sqrt{a+6}=\sqrt{a-2}$

51. $\sqrt{x-5}+1=-\sqrt{x+3}$

52. $\sqrt{3x-5}+\sqrt{2x+3}+1=0$

53. $\sqrt{2m-3}+2-\sqrt{m+7}=0$

54. $\sqrt{x+2}+\sqrt{3x+4}=2$

55. $\sqrt{6x+7}-\sqrt{3x+3}=1$

56. $\sqrt{4x+7}-4=\sqrt{4x-1}$

57. $\sqrt{5y+4}-3=\sqrt{2y-2}$

58. $\sqrt{2\sqrt{x+11}}=\sqrt{4x+2}$

59. $\sqrt{1+\sqrt{24+10x}}=\sqrt{3x+5}$

60. $(2x-9)^{\frac{1}{2}}=2+(x-8)^{\frac{1}{2}}$

61. $(3k+7)^{\frac{1}{2}}=1+(k+2)^{\frac{1}{2}}$

62. $(x+1)^{\frac{1}{2}}-(x-6)^{\frac{1}{2}}=1$

63. $\sqrt{(x^2-9)^{\frac{1}{2}}}=2$

64. $\sqrt{\sqrt{x}+4}=\sqrt{x}-2$

65. $\sqrt{a^2+30a}=a+\sqrt{5a}$

Discussion Point

66. Can the expression $\sqrt{7+4\sqrt{3}}-\sqrt{7-4\sqrt{3}}$ be evaluated without the use of a calculator?

Solve each formula for the indicated variable.

67. $Z=\sqrt{\frac{L}{C}}$ for L

68. $V=\sqrt{\frac{2K}{m}}$ for K

69. $V=\sqrt{\frac{2K}{m}}$ for m

70. $r=\sqrt{\frac{Mm}{F}}$ for M

71. $r=\sqrt{\frac{Mm}{F}}$ for F

72. $Z=\sqrt{L^2+R^2}$ for R

73. $F=\frac{1}{2\pi\sqrt{LC}}$ for C

74. $N=\frac{1}{2\pi}\sqrt{\frac{a}{r}}$ for a

75. $N=\frac{1}{2\pi}\sqrt{\frac{a}{r}}$ for r

Analytic Skills *Solve each problem.*

76. According to Einstein's theory of relativity, time passes more quickly for bodies that travel very close to the speed of light. The aging rate compared to the time spent on earth is given by the formula $\boldsymbol{r}=\frac{\sqrt{c^2-v^2}}{\sqrt{c^2}}$, where c is the speed of light, and v is the speed of the traveling body. For example, the aging rate of 0.5 means that one year for the person travelling at the speed v corresponds to two years spent on earth.

a. Find the aging rate for a person traveling at 90% of the speed of light.

b. Find the elapsed time on earth for one year of travelling time at 90% of the speed of light.

77. Before determining the dosage of a drug for a patient, doctors will sometimes calculate the patient's **Body Surface Area** (or BSA). One way to determine a person's BSA, in square meters, is to use the formula $\boldsymbol{BSA}=\sqrt{\frac{wh}{3600}}$, where w is the weight, in pounds, and h is the height, in centimeters, of the patient. Jason weighs 160 pounds and has a BSA of about $2\sqrt{2}$ m^2. How tall is he? *Round the answer to the nearest centimeter*.

78. The distance, d, to the horizon for an object h miles above the Earth's surface is given by the equation $d = \sqrt{8000h + h^2}$. How many miles above the Earth's surface is a satellite if the distance to the horizon is 900 miles?

79. To calculate the minimum speed S, in miles per hour, that a car was traveling before skidding to a stop, traffic accident investigators use the formula $S = \sqrt{30fL}$, where f is the drag factor of the road surface and L is the length of a skid mark, in feet. Calculate the length of the skid marks for a car traveling at a speed of 30 mph that skids to a stop on a road surface with a drag factor of 0.5.