

Radicals and Radical Functions



So far we have discussed polynomial and rational expressions and functions. In this chapter, we study algebraic expressions that contain radicals. For example, $3 + \sqrt{2}$, $\sqrt[3]{x} - 1$, or $\frac{1}{\sqrt{5x-1}}$. Such expressions are called **radical expressions**. Familiarity with radical expressions is essential when solving a wide variety of problems. For instance, in algebra, some polynomial or rational equations have radical solutions that need to be simplified. In geometry, due to the frequent use of the Pythagorean equation, $a^2 + b^2 = c^2$, the exact distances are often radical expressions. In sciences, many formulas involve radicals.

We begin the study of radical expressions with defining radicals of various degrees and discussing their properties. Then, we show how to simplify radicals and radical expressions, and introduce operations on radical expressions. Finally, we study the methods of solving radical equations. In addition, similarly as in earlier chapters where we looked at the related polynomial and rational functions, we will also define and look at properties of radical functions.

RD.1

Radical Expressions, Functions, and Graphs

Roots and Radicals

The operation of taking a **square root** of a number is the **reverse operation of squaring** a number. For example, a square root of 25 is 5 because raising 5 to the second power gives us 25.

Note: Observe that raising -5 to the second power also gives us 25. So, the square root of 25 could have two answers, 5 or -5 . To avoid this duality, we choose the **nonnegative value**, called the **principal square root**, for the value of a square root of a number.

The operation of taking a square root is denoted by the symbol $\sqrt{\quad}$. So, we have

$$\sqrt{25} = 5, \quad \sqrt{0} = 0, \quad \sqrt{1} = 1, \quad \sqrt{9} = 3, \text{ etc.}$$

What about $\sqrt{-4} = ?$ Is there a number such that when it is squared, it gives us -4 ?

Since the square of any real number is nonnegative, the square root of a negative number is not a real number. So, when working in the set of real numbers, we can conclude that

$$\sqrt{\text{positive}} = \text{positive}, \quad \sqrt{0} = 0, \quad \text{and} \quad \sqrt{\text{negative}} = \text{DNE}$$

does not exist

The operation of taking a **cube root** of a number is the **reverse operation of cubing** a number. For example, a cube root of 8 is 2 because raising 2 to the third power gives us 8.

This operation is denoted by the symbol $\sqrt[3]{\quad}$. So, we have

$$\sqrt[3]{8} = 2, \quad \sqrt[3]{0} = 0, \quad \sqrt[3]{1} = 1, \quad \sqrt[3]{27} = 3, \text{ etc.}$$

Note: Observe that $\sqrt[3]{-8}$ exists and is equal to -2 . This is because $(-2)^3 = -8$. Generally, a cube root can be applied to any real number and the **sign** of the resulting value **is the same** as the sign of the original number.

Thus, we have

$$\sqrt[3]{\text{positive}} = \text{positive}, \quad \sqrt[3]{0} = 0, \quad \text{and} \quad \sqrt[3]{\text{negative}} = \text{negative}$$

The square or cube roots are special cases of n -th degree radicals.

Definition 1.1 ▶ The n -th degree radical of a number a is a number b such that $b^n = a$.

Notation:

$$\sqrt[n]{a} = b \Leftrightarrow b^n = a$$

For example, $\sqrt[4]{16} = 2$ because $2^4 = 16$,
 $\sqrt[5]{-32} = -2$ because $(-2)^5 = -32$,
 $\sqrt[3]{0.027} = 0.3$ because $(0.3)^3 = 0.027$.

Note: A square root is a second degree radical, customarily denoted by $\sqrt{\quad}$ rather than $\sqrt[2]{\quad}$.

Example 1 ▶ Evaluating Radicals

Evaluate each radical, if possible.

a. $\sqrt{0.64}$ b. $\sqrt[3]{125}$ c. $\sqrt[4]{-16}$ d. $\sqrt[5]{-\frac{1}{32}}$

Solution ▶ a. Since $0.64 = (0.8)^2$, then $\sqrt{0.64} = 0.8$.

take half of the decimal places

Advice: To become fluent in evaluating square roots, it is helpful to be familiar with the following perfect square numbers:

1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, ..., 400, ..., 625, ...

b. $\sqrt[3]{125} = 5$ as $5^3 = 125$

Advice: To become fluent in evaluating cube roots, it is helpful to be familiar with the following cubic numbers:
1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, ...

c. $\sqrt[4]{-16}$ is not a real number as there is no real number which raised to the 4-th power becomes negative.

d. $\sqrt[5]{-\frac{1}{32}} = -\frac{1}{2}$ as $\left(-\frac{1}{2}\right)^5 = -\frac{1^5}{2^5} = -\frac{1}{32}$

Note: Observe that $\frac{\sqrt[5]{-1}}{\sqrt[5]{32}} = \frac{-1}{2}$, so $\sqrt[5]{-\frac{1}{32}} = \frac{\sqrt[5]{-1}}{\sqrt[5]{32}}$.

Generally, to take a radical of a quotient, $\sqrt[n]{\frac{a}{b}}$, it is the same as to take the quotient of radicals, $\frac{\sqrt[n]{a}}{\sqrt[n]{b}}$.

Example 2 ▶ Evaluating Radical Expressions

Evaluate each radical expression.

a. $-\sqrt{121}$

b. $-\sqrt[3]{-64}$

c. $\sqrt[4]{(-3)^4}$

d. $\sqrt[3]{(-6)^3}$

Solution ▶

a. $-\sqrt{121} = -11$

b. $-\sqrt[3]{-64} = -(-4) = 4$

c. $\sqrt[4]{(-3)^4} = \sqrt[4]{81} = 3$
 the result is positive

Note: If n is **even**, then $\sqrt[n]{a^n} = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases} = |a|$.

For example, $\sqrt{7^2} = 7$ and $\sqrt{(-7)^2} = 7$.

d. $\sqrt[3]{(-6)^3} = \sqrt[3]{-216} = -6$
 the result has the same sign

Note: If n is **odd**, then $\sqrt[n]{a^n} = a$. For example, $\sqrt[3]{5^3} = 5$ but $\sqrt[3]{(-5)^3} = -5$.

Summary of Properties of n -th Degree Radicals

➤ If n is **EVEN**, then

$$\sqrt[n]{\text{positive}} = \text{positive}, \quad \sqrt[n]{\text{negative}} = \text{DNE}, \quad \text{and} \quad \sqrt[n]{a^n} = |a|$$

➤ If n is **ODD**, then

$$\sqrt[n]{\text{positive}} = \text{positive}, \quad \sqrt[n]{\text{negative}} = \text{negative}, \quad \text{and} \quad \sqrt[n]{a^n} = a$$

➤ For any natural $n \geq 0$, $\sqrt[n]{0} = 0$ and $\sqrt[n]{1} = 1$.

Example 3 ▶ **Simplifying Radical Expressions Using Absolute Value Where Appropriate**

Simplify each radical, assuming that all variables represent any real number.

a. $\sqrt{9x^2y^4}$ b. $\sqrt[3]{-27y^3}$ c. $\sqrt[4]{a^{20}}$ d. $-\sqrt[4]{(k-1)^4}$

Solution

a. $\sqrt{9x^2y^4} = \sqrt{(3xy^2)^2} = |3xy^2| = 3|x|y^2$

An even degree radical is nonnegative, so we must use the absolute value operator.

Recall: As discussed in *section L6*, the absolute value operator has the following properties:

$$|xy| = |x||y|$$

$$\left| \frac{x}{y} \right| = \frac{|x|}{|y|}$$

Note: $|y^2| = y^2$ as y^2 is already nonnegative.

b. $\sqrt[3]{-27y^3} = \sqrt[3]{(-3y)^3} = -3y$

An odd degree radical assumes the sign of the radicand, so we do not apply the absolute value operator.

c. $\sqrt[4]{a^{20}} = \sqrt[4]{(a^5)^4} = |a^5| = |a|^5$

Note: To simplify an expression with an absolute value, we keep the absolute value operator as close as possible to the variable(s).

d. $-\sqrt[4]{(k-1)^4} = -|k-1|$

Radical Functions

Since each nonnegative real number x has exactly one principal square root, we can define the **square root** function, $f(x) = \sqrt{x}$. The **domain** D_f of this function is the set of nonnegative real numbers, $[0, \infty)$, and so is its **range** (as indicated in *Figure 1*).

To graph the square root function, we create a table of values. The easiest x -values for calculation of the corresponding y -values are the perfect square numbers. However, sometimes we want to use additional x -values that are not perfect squares. Since a square root of such a number, for example $\sqrt{2}$, $\sqrt{3}$, $\sqrt{6}$, etc., is an irrational number, we approximate these values using a calculator.

x	y
0	0
$\frac{1}{4}$	$\frac{1}{2}$
1	1
4	2
6	$\sqrt{6} \approx 2.4$

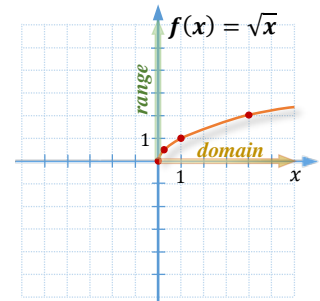


Figure 1

For example, to approximate $\sqrt{6}$, we use the sequence of keying: $\sqrt{\quad}$ 6 ENTER or $\sqrt{\quad}$ (1 / 2) ENTER. This is because a square root operator works the same way as the exponent of $\frac{1}{2}$.

Note: When graphing an even degree radical function, it is essential that we find its domain first. The end-point of the domain indicates the starting point of the graph, often called the vertex.

For example, since the domain of $f(x) = \sqrt{x}$ is $[0, \infty)$, the graph starts from the point $(0, f(0)) = (0, 0)$, as in *Figure 1*.

Since the cube root can be evaluated for any real number, the **domain** D_f of the related **cube root** function, $f(x) = \sqrt[3]{x}$, is the set of **all real numbers**, \mathbb{R} . The **range** can be observed in the graph (see *Figure 2*) or by inspecting the expression $\sqrt[3]{x}$. It is also \mathbb{R} .

To graph the cube root function, we create a table of values. The easiest x -values for calculation of the corresponding y -values are the perfect cube numbers. As before, sometimes we might need to estimate additional x -values. For example, to approximate $\sqrt[3]{6}$, we use the sequence of keying:

$\sqrt[3]{\quad}$ 6 ENTER or

6 ^ (1 / 3) ENTER.

x	y
-8	-2
-6	$-\sqrt[3]{6} \approx -1.8$
-1	-1
$-\frac{1}{8}$	$-\frac{1}{2}$
0	0
$\frac{1}{8}$	$\frac{1}{2}$
1	1
6	$\sqrt[3]{6} \approx 1.8$
8	2

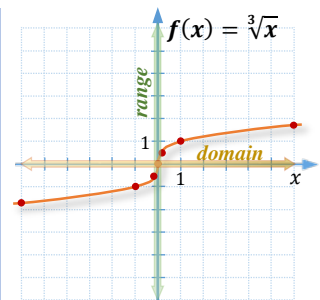


Figure 2

Example 4 ▶ **Finding a Calculator Approximations of Roots**

Use a calculator to approximate the given root up to three decimal places.

- a. $\sqrt{3}$ b. $\sqrt[3]{5}$ c. $\sqrt[5]{100}$

Solution ▶

a. $\sqrt{3} \approx 1.732$

b. $\sqrt[3]{5} \approx 1.710$

c. $\sqrt[5]{100} \approx 2.512$


Example 5 ▶ **Finding the Best Integer Approximation of a Square Root**

Without the use of a calculator, determine the best integer approximation of the given root.

- a. $\sqrt{68}$ b. $\sqrt{140}$

Solution ▶

- a. Observe that 68 lies between the following two consecutive perfect square numbers, 64 and 81. Also, 68 lies closer to 64 than to 81. Therefore, $\sqrt{68} \approx \sqrt{64} = 8$.
- b. 140 lies between the following two consecutive perfect square numbers, 121 and 144. In addition, 140 is closer to 144 than to 121. Therefore, $\sqrt{140} \approx \sqrt{144} = 12$.

Example 6 ▶ **Finding the Domain of a Radical Function**

Find the domain of each of the following functions.

- a. $f(x) = \sqrt{2x + 3}$ b. $g(x) = 2 - \sqrt{1 - x}$

Solution ▶

- a. When finding domain D_f of function $f(x) = \sqrt{2x + 3}$, we need to protect the radicand $2x + 3$ from becoming negative. So, an x -value belongs to the domain D_f if it satisfies the condition

$$2x + 3 \geq 0. \quad / -3, \div 2$$

This happens for $x \geq -\frac{3}{2}$. Therefore, $D_f = \left[-\frac{3}{2}, \infty\right)$.

- b. To find the domain D_g of function $g(x) = 2 - \sqrt{1 - x}$, we solve the condition

$$\begin{aligned} 1 - x &\geq 0 && / +x \\ 1 &\geq x \end{aligned}$$

Thus, $D_g = (-\infty, 1]$.

The **domain** of an **even degree** radical is the solution set of the inequality **radicand ≥ 0**

The **domain** of an **odd degree** radical is \mathbb{R} .

Example 7 ▶ **Graphing Radical Functions**

For each function, find its domain, graph it, and find its range. Then, observe what transformation(s) of a basic root function result(s) in the obtained graph.

a. $f(x) = -\sqrt{x+3}$

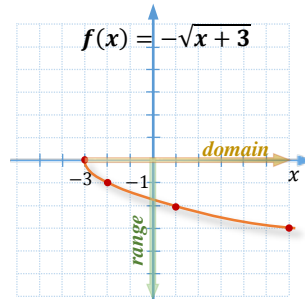
b. $g(x) = \sqrt[3]{x} - 2$

Solution ▶

- a. The **domain** D_f is the solution set of the inequality $x + 3 \geq 0$, which is equivalent to $x \geq -3$. Hence, $D_f = [-3, \infty)$.



x	y
-3	0
-2	-1
1	-2
6	-3



The projection of the graph onto the y -axis indicates the **range** of this function, which is $(-\infty, 0]$.

The graph of $f(x) = -\sqrt{x+3}$ has the same shape as the graph of the basic square root function $f(x) = \sqrt{x}$, except that it is flipped over the x -axis and moved to the left by three units. These transformations are illustrated in *Figure 3*.

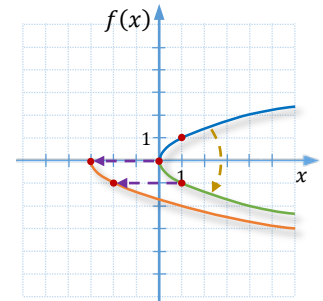
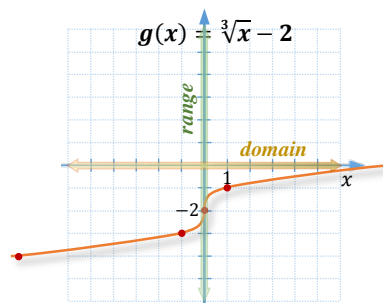


Figure 3

- b. The **domain** and **range** of any odd degree radical are both the set of all real numbers. So, $D_g = \mathbb{R}$ and $range_g = \mathbb{R}$.

x	y
-8	-4
-1	-3
0	-2
1	-1
8	0



The graph of $g(x) = \sqrt[3]{x} - 2$ has the same shape as the graph of the basic cube root function $f(x) = \sqrt[3]{x}$, except that it is moved down by two units. This transformation is illustrated in *Figure 4*.

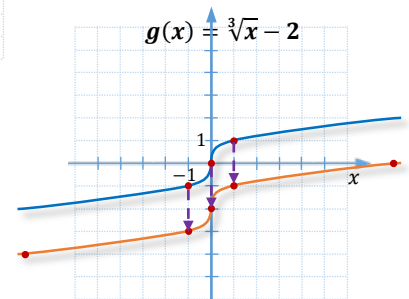


Figure 4

Radicals in Application Problems

Some application problems require evaluation of formulas that involve radicals. For example, the formula $c = \sqrt{a^2 + b^2}$ allows for finding the hypotenuse in a right angle triangle (see *section RD3*), **Heron's** formula $A = \sqrt{s(s-a)(s-b)(s-c)}$ allows for finding the area of any triangle given the lengths of its sides (see *section T5*), the formula $T = 2\pi\sqrt{\frac{d^3}{Gm}}$ allows for finding the time needed for a planet to make a complete orbit around the Sun, and so on.

Example 8 ▶ Using a Radical Formula in an Application Problem

The time t , in seconds, for one complete swing of a simple pendulum is given by the formula

$$t = 2\pi\sqrt{\frac{L}{g}},$$

where L is the length of the pendulum in feet, and g represents the acceleration due to gravity, which is about 32 ft per sec². Find the time of a complete swing of a 2-ft pendulum to the nearest tenth of a second.

Solution ▶ Since $L = 2$ ft and $g = 32$ ft/sec², then

$$t = 2\pi\sqrt{\frac{2}{32}} = 2\pi\sqrt{\frac{1}{16}} = 2\pi \cdot \frac{1}{4} = \frac{\pi}{2} \approx 1.6$$

So, the approximate time of a complete swing of a 2-ft pendulum is **1.6 seconds**.

RD.1 Exercises

Vocabulary Check Complete each blank with one of the suggested words, or the most appropriate term or phrase from the given list: *even, index, irrational, nonnegative, odd, principal, radicand, square root*.

- The symbol $\sqrt{\quad}$ stands for the _____ square root.
- For any real numbers $a \geq 0$ and b , if $b^2 = a$, then b is the _____ of a .
- The expression under the radical sign is called the _____.
- The square root of a negative number _____ a real number.
is / is not
- The square root of a number that is not a perfect square is a(n) _____ number.

6. The cube root of a is written $\sqrt[3]{a}$, where 3 is called the _____ of the radical.
7. The domain of an even degree radical function is the set of all values that would make the radicand _____.
8. The domain and range of an _____ degree radical functions is the set of all real numbers.
9. If n is _____ then $\sqrt[n]{a^n} = |a|$.

Concept Check Evaluate each radical, if possible.

- | | | | |
|-------------------------|-----------------------------|------------------------------------|-------------------------------|
| 10. $\sqrt{49}$ | 11. $-\sqrt{81}$ | 12. $\sqrt{-400}$ | 13. $\sqrt{0.09}$ |
| 14. $\sqrt{0.0016}$ | 15. $\sqrt{\frac{64}{225}}$ | 16. $\sqrt[3]{64}$ | 17. $\sqrt[3]{-125}$ |
| 18. $\sqrt[3]{0.008}$ | 19. $-\sqrt[3]{-1000}$ | 20. $\sqrt[3]{\frac{1}{0.000027}}$ | 21. $\sqrt[4]{16}$ |
| 22. $\sqrt[5]{0.00032}$ | 23. $\sqrt[7]{-1}$ | 24. $\sqrt[8]{-256}$ | 25. $-\sqrt[6]{\frac{1}{64}}$ |

Concept Check

26. Decide whether the expression $-\sqrt{-a}$ is positive, negative, 0, or not a real number, given that
- | | | |
|------------|------------|------------|
| a. $a < 0$ | b. $a > 0$ | c. $a = 0$ |
|------------|------------|------------|
27. Assuming that n is odd, decide whether the expression $-\sqrt[n]{a}$ is positive, negative, or 0, given that
- | | | |
|------------|------------|------------|
| a. $a < 0$ | b. $a > 0$ | c. $a = 0$ |
|------------|------------|------------|

Simplify each radical. Assume that letters can represent any real number.

- | | | | |
|----------------------------|---------------------------------|----------------------------------|---------------------------|
| 28. $\sqrt{15^2}$ | 29. $\sqrt{(-15)^2}$ | 30. $\sqrt{x^2}$ | 31. $\sqrt{(-x)^2}$ |
| 32. $\sqrt{81x^2}$ | 33. $\sqrt{(-12y)^2}$ | 34. $\sqrt{(a+3)^2}$ | 35. $\sqrt{(2-x)^2y^4}$ |
| 36. $\sqrt{x^2 - 4x + 4}$ | 37. $\sqrt{9y^2 + 30y + 25}$ | 38. $\sqrt[3]{(-5)^3}$ | 39. $\sqrt[3]{x^3}$ |
| 40. $\sqrt[3]{-125a^3}$ | 41. $-\sqrt[3]{0.008(x-1)^3}$ | 42. $\sqrt[4]{(5x)^4}$ | 43. $\sqrt[8]{(-10)^8}$ |
| 44. $\sqrt[5]{(y-3)^5}$ | 45. $\sqrt[2017]{(a+b)^{2017}}$ | 46. $\sqrt[2018]{(2a-b)^{2018}}$ | 47. $\sqrt[6]{x^{18}}$ |
| 48. $\sqrt[4]{(a+1)^{12}}$ | 49. $\sqrt[5]{(-a)^{20}}$ | 50. $\sqrt[7]{(-k)^{35}}$ | 51. $\sqrt[4]{x^4(-y)^8}$ |

Find a decimal approximation for each radical. Round the answer to three decimal places.

- | | | | |
|------------------|---------------------|-------------------|-------------------|
| 52. $\sqrt{350}$ | 53. $-\sqrt{0.859}$ | 54. $\sqrt[3]{5}$ | 55. $\sqrt[5]{3}$ |
|------------------|---------------------|-------------------|-------------------|

Concept Check Without the use of a calculator, give the best integer approximation of each square root.

56. $\sqrt{67}$

57. $\sqrt{95}$

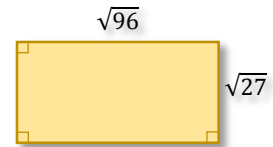
58. $\sqrt{115}$

59. $\sqrt{87}$

Questions in Exercises 60 and 61 refer to the accompanying rectangle. Answer these questions without the use of a calculator.

60. Give the best integer estimation of the area of the rectangle.

61. Give the best integer estimation of the perimeter of the rectangle.



Solve each problem. Do not use any calculator.

62. A rectangular yard has a length of $\sqrt{189}$ m and a width of $\sqrt{48}$ m. Choose the best estimate of its dimensions. Then estimate the perimeter.

A. 13 m by 7 m

B. 14 m by 7 m

C. 14 m by 8 m

D. 15 m by 7 m

63. If the sides of a triangle are $\sqrt{65}$ cm, $\sqrt{34}$ cm, and $\sqrt{27}$ cm, which one of the following is the best estimate of its perimeter?

A. 20 cm

B. 26 cm

C. 19 cm

D. 24 cm

Graph each function and give its **domain** and **range**. Then, discuss the **transformations** of a basic root function needed to obtain the graph of the given function.

64. $f(x) = \sqrt{x+1}$

65. $g(x) = \sqrt{x} + 1$

66. $h(x) = -\sqrt{x}$

67. $f(x) = \sqrt{x-3}$

68. $g(x) = \sqrt{x} - 3$

69. $h(x) = 2 - \sqrt{x}$

70. $f(x) = \sqrt[3]{x-2}$

71. $g(x) = \sqrt[3]{x} + 2$

72. $h(x) = -\sqrt[3]{x} + 2$

Graph each function and give its **domain** and **range**.

73. $f(x) = 2 + \sqrt{x-1}$

74. $g(x) = 2\sqrt{x}$

75. $h(x) = -\sqrt{x+3}$

76. $f(x) = \sqrt{3x+9}$

77. $g(x) = \sqrt{3x-6}$

78. $h(x) = -\sqrt{2x-4}$

79. $f(x) = \sqrt{12-3x}$

80. $g(x) = \sqrt{8-4x}$

81. $h(x) = -2\sqrt{-x}$

Analytic Skills Graph the three given functions on the same grid and discuss the relationship between them.

82. $f(x) = 2x + 1$; $g(x) = \sqrt{2x+1}$; $h(x) = \sqrt[3]{2x+1}$

83. $f(x) = -x + 2$; $g(x) = \sqrt{-x+2}$; $h(x) = \sqrt[3]{-x+2}$

84. $f(x) = \frac{1}{2}x + 1$; $g(x) = \sqrt{\frac{1}{2}x+1}$; $h(x) = \sqrt[3]{\frac{1}{2}x+1}$

Solve each problem.

85. The distance D , in miles, to the horizon from an observer's point of view over water or "flat" earth is given by $D = \sqrt{2H}$, where H is the height of the point of view, in feet. If a man whose eyes are 6 ft above ground level is standing at the top of a hill 105 ft above "flat" earth, approximately how far to the horizon he will be able to see? Round the answer to the nearest mile.



86. The threshold body weight T for a person is the weight above which the risk of death increases greatly. The threshold weight in pounds for men aged 40–49 is related to their height h , in inches, by the formula $h = 12.3\sqrt[3]{T}$. What height corresponds to a threshold weight of 216 lb for a 43-year-old man? Round your answer to the nearest inch.

87. The time needed for a planet to make a complete orbit around the Sun is given by the formula $T = 2\pi\sqrt{\frac{d^3}{GM}}$,



where d is the average distance of the planet from the Sun, G is the universal gravitational constant, and M is the mass of the Sun. To the nearest day, find the orbital period of Mercury, knowing that its average distance from the Sun is $5.791 \cdot 10^7$ km, the mass of the Sun is $1.989 \cdot 10^{30}$ kg, and $G = 6.67408 \cdot 10^{-11}$ m³/(kg·s²).

88. The time it takes for an object to fall a certain distance is given by the equation $t = \sqrt{\frac{2d}{g}}$, where t is the time in seconds, d is the distance in feet, and g is the acceleration due to gravity. If an astronaut above the moon's surface drops an object, how far will it have fallen in 3 seconds? The acceleration on the moon's surface is 5.5 feet per second per second.

Half of the perimeter (*semiperimeter*) of a triangle with sides a , b , and c is $s = \frac{1}{2}(a + b + c)$. The area of such a triangle is given by the **Heron's Formula**: $A = \sqrt{s(s - a)(s - b)(s - c)}$.

Find the area of a triangular piece of land with the following sides.

89. $a = 3$ m, $b = 4$ m, $c = 5$ m
 90. $a = 80$ m, $b = 80$ m, $c = 140$ m



RD.2

Rational Exponents



In sections P2 and RT1, we reviewed properties of powers with natural and integral exponents. All of these properties hold for real exponents as well. In this section, we give meaning to expressions with rational exponents, such as $a^{\frac{1}{2}}$, $8^{\frac{1}{3}}$, or $(2x)^{0.54}$, and use the rational exponent notation as an alternative way to write and simplify radical expressions.

Rational Exponents

Observe that $\sqrt{9} = 3 = 3^{2 \cdot \frac{1}{2}} = 9^{\frac{1}{2}}$. Similarly, $\sqrt[3]{8} = 2 = 2^{3 \cdot \frac{1}{3}} = 8^{\frac{1}{3}}$. This suggests the following generalization:

For any real number a and a natural number $n > 1$, we have

$$\sqrt[n]{a} = a^{\frac{1}{n}}$$

Notice: The **denominator** of the rational exponent is the **index** of the radical.

Caution! If $a < 0$ and n is an even natural number, then $a^{\frac{1}{n}}$ is not a real number.

Example 1

Converting Radical Notation to Rational Exponent Notation

Convert each radical to a power with a rational exponent and simplify, if possible. Assume that all variables represent positive real numbers.

a. $\sqrt[6]{16}$

b. $\sqrt[3]{27x^3}$

c. $\sqrt{\frac{4}{b^6}}$

Solution

a. $\sqrt[6]{16} = 16^{\frac{1}{6}} = (2^4)^{\frac{1}{6}} = 2^{\frac{4}{6}} = 2^{\frac{2}{3}}$

Observation: Expressing numbers as **powers of prime numbers** often allows for further simplification.

$$\sqrt[3]{27x^3} = (27x^3)^{\frac{1}{3}} = 27^{\frac{1}{3}} \cdot (x^3)^{\frac{1}{3}} = (3^3)^{\frac{1}{3}} \cdot x = 3x$$

distribution of exponents change into a power of a prime number

Note: The above example can also be done as follows:

$$\sqrt[3]{27x^3} = \sqrt[3]{3^3 x^3} = (3^3 x^3)^{\frac{1}{3}} = 3x$$

c. $\sqrt{\frac{9}{b^6}} = \left(\frac{9}{b^6}\right)^{\frac{1}{2}} = \frac{(3^2)^{\frac{1}{2}}}{(b^6)^{\frac{1}{2}}} = \frac{3}{b^3}, \text{ as } b > 0.$

Observation: $\sqrt{a^4} = a^{\frac{4}{2}} = a^2.$

Generally, for any real number $a \neq 0$, natural number $n > 1$, and integral number m , we have

$$\sqrt[n]{a^m} = (a^m)^{\frac{1}{n}} = a^{\frac{m}{n}}$$

Rational exponents are introduced in such a way that they automatically agree with the rules of exponents, as listed in *section RT1*.

Furthermore, the rules of exponents hold not only for rational but also for **real exponents**.

Observe that following the rules of exponents and the commutativity of multiplication, we have

$$\sqrt[n]{a^m} = (a^m)^{\frac{1}{n}} = \left(a^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{a}\right)^m,$$

provided that $\sqrt[n]{a}$ exists.

Example 2 ▶ Converting Rational Exponent Notation to the Radical Notation

Convert each power with a rational exponent to a radical and simplify, if possible.

a. $5^{\frac{3}{4}}$

b. $(-27)^{\frac{1}{3}}$

c. $3x^{-\frac{2}{5}}$

Solution ▶

a. $5^{\frac{3}{4}} = \sqrt[4]{5^3} = \sqrt[4]{125}$

b. $(-27)^{\frac{1}{3}} = \sqrt[3]{-27} = -3$

c. $3x^{-\frac{2}{5}} = \frac{3}{x^{\frac{2}{5}}} = \frac{3}{\sqrt[5]{x^2}}$

Notice that $-27^{\frac{1}{3}} = -\sqrt[3]{27} = -3$, so $(-27)^{\frac{1}{3}} = -27^{\frac{1}{3}}$.

However, $(-9)^{\frac{1}{2}} \neq -9^{\frac{1}{2}}$, as $(-9)^{\frac{1}{2}}$ is not a real number while $-9^{\frac{1}{2}} = -\sqrt{9} = -3$.

Caution: A negative exponent indicates a reciprocal not a negative number!

Also, the exponent refers to x only, so 3 remains in the numerator.

Observation: If $a^{\frac{m}{n}}$ is a real number, then

$$a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}},$$

provided that $a \neq 0$.

Caution! Make sure to distinguish between a negative exponent and a negative result. Negative exponent leads to a reciprocal of the base. The result can be either positive or negative, depending on the sign of the base. For example,

$$8^{-\frac{1}{3}} = \frac{1}{8^{\frac{1}{3}}} = \frac{1}{2}, \text{ but } (-8)^{-\frac{1}{3}} = \frac{1}{(-8)^{\frac{1}{3}}} = \frac{1}{-2} = -\frac{1}{2} \text{ and } -8^{-\frac{1}{3}} = -\frac{1}{8^{\frac{1}{3}}} = -\frac{1}{2}.$$

Example 3 ▶ Applying Rules of Exponents When Working with Rational Exponents

Simplify each expression. Write your answer with only positive exponents. Assume that all variables represent positive real numbers.

a. $a^{\frac{3}{4}} \cdot 2a^{-\frac{2}{3}}$ b. $\frac{4^{\frac{1}{3}}}{4^{\frac{5}{3}}}$ c. $\left(x^{\frac{3}{8}} \cdot y^{\frac{5}{2}}\right)^{\frac{4}{3}}$

Solution ▶ a. $a^{\frac{3}{4}} \cdot 2a^{-\frac{2}{3}} = 2a^{\frac{3}{4} + (-\frac{2}{3})} = 2a^{\frac{9}{12} - \frac{8}{12}} = 2a^{\frac{1}{12}}$

b. $\frac{4^{\frac{1}{3}}}{4^{\frac{5}{3}}} = 4^{\frac{1}{3} - \frac{5}{3}} = 4^{-\frac{4}{3}} = \frac{1}{4^{\frac{4}{3}}}$

c. $\left(x^{\frac{3}{8}} \cdot y^{\frac{5}{2}}\right)^{\frac{4}{3}} = x^{\frac{3 \cdot 4}{8 \cdot 3}} \cdot y^{\frac{5 \cdot 4}{2 \cdot 3}} = x^{\frac{1}{2}} y^{\frac{10}{3}}$

Example 4 ▶ Evaluating Powers with Rational Exponents

Evaluate each power.

a. $64^{-\frac{1}{3}}$ b. $\left(-\frac{8}{125}\right)^{\frac{2}{3}}$

Solution ▶ a. $64^{-\frac{1}{3}} = (2^6)^{-\frac{1}{3}} = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$

b. $\left(-\frac{8}{125}\right)^{\frac{2}{3}} = \left(\left(-\frac{2}{5}\right)^3\right)^{\frac{2}{3}} = \left(-\frac{2}{5}\right)^2 = \frac{4}{25}$

It is helpful to change the base into a power of prime number, if possible.

Observe that if m in $\sqrt[n]{a^m}$ is a multiple of n , that is if $m = kn$ for some integer k , then

$$\sqrt[n]{a^{kn}} = a^{\frac{kn}{n}} = a^k$$

Example 5 ▶ Simplifying Radical Expressions by Converting to Rational Exponents

Simplify. Assume that all variables represent positive real numbers. Leave your answer in simplified single radical form.

a. $\sqrt[5]{3^{20}}$ b. $\sqrt{x} \cdot \sqrt[4]{x^3}$ c. $\sqrt[3]{2\sqrt{2}}$

Solution

a. $\sqrt[5]{3^{20}} = (3^{20})^{\frac{1}{5}} = 3^4 = 81$
 divide at the exponential level

b. $\sqrt{x} \cdot \sqrt[4]{x^3} = x^{\frac{1}{2}} \cdot x^{\frac{3}{4}} = x^{\frac{1 \cdot 2}{2 \cdot 2} + \frac{3}{4}} = x^{\frac{5}{4}} = x \cdot x^{\frac{1}{4}} = x\sqrt[4]{x}$
 add exponents as $\frac{5}{4} = 1 + \frac{1}{4}$

c. $\sqrt[3]{2\sqrt{2}} = (2 \cdot 2^{\frac{1}{2}})^{\frac{1}{3}} = (2^{1 + \frac{1}{2}})^{\frac{1}{3}} = (2^{\frac{3}{2}})^{\frac{1}{3}} = 2^{\frac{1}{2}} = \sqrt{2}$

This bracket is essential!

Another solution:

$$\sqrt[3]{2\sqrt{2}} = 2^{\frac{1}{3}} \cdot (2^{\frac{1}{2}})^{\frac{1}{3}} = 2^{\frac{1}{3}} \cdot 2^{\frac{1}{6}} = 2^{\frac{1 \cdot 2}{3 \cdot 2} + \frac{1}{6}} = 2^{\frac{1}{2}} = \sqrt{2}$$

RD.2 Exercises

Concept Check Match each expression from Column I with the equivalent expression from Column II.

1. Column I	Column II	2. Column I	Column II
a. $25^{\frac{1}{2}}$	A. $\frac{1}{5}$	a. $(-32)^{\frac{2}{5}}$	A. 2
b. $25^{-\frac{1}{2}}$	B. 5	b. $-27^{\frac{2}{3}}$	B. $\frac{1}{4}$
c. $-25^{\frac{3}{2}}$	C. -125	c. $32^{\frac{1}{5}}$	C. -8
d. $-25^{-\frac{1}{2}}$	D. not a real number	d. $32^{-\frac{2}{5}}$	D. -9
e. $(-25)^{\frac{1}{2}}$	E. $\frac{1}{125}$	e. $-4^{\frac{3}{2}}$	E. not a real number
f. $25^{-\frac{3}{2}}$	F. $-\frac{1}{5}$	f. $(-4)^{\frac{3}{2}}$	F. 4

Concept Check Write the base as a power of a prime number to evaluate each expression, if possible.

3. $32^{\frac{1}{5}}$	4. $27^{\frac{4}{3}}$	5. $-49^{\frac{3}{2}}$	6. $16^{\frac{3}{4}}$
7. $-100^{-\frac{1}{2}}$	8. $125^{-\frac{1}{3}}$	9. $(\frac{64}{81})^{\frac{3}{4}}$	10. $(\frac{8}{27})^{-\frac{2}{3}}$

11. $(-36)^{\frac{1}{2}}$

12. $(-64)^{\frac{1}{3}}$

13. $\left(-\frac{1}{8}\right)^{-\frac{1}{3}}$

14. $(-625)^{-\frac{1}{4}}$

Concept Check Rewrite **with** rational exponents and simplify, if possible. Assume that all variables represent positive real numbers.

15. $\sqrt{5}$

16. $\sqrt[3]{6}$

17. $\sqrt{x^6}$

18. $\sqrt[5]{y^2}$

19. $\sqrt[3]{64x^6}$

20. $\sqrt[3]{16x^2y^3}$

21. $\sqrt{\frac{25}{x^5}}$

22. $\sqrt[4]{\frac{16}{a^6}}$

Concept Check Rewrite **without** rational exponents, and simplify, if possible. Assume that all variables represent positive real numbers.

23. $4^{\frac{5}{2}}$

24. $8^{\frac{3}{4}}$

25. $x^{\frac{3}{5}}$

26. $a^{\frac{7}{3}}$

27. $(-3)^{\frac{2}{3}}$

28. $(-2)^{\frac{3}{5}}$

29. $2x^{-\frac{1}{2}}$

30. $x^{\frac{1}{3}}y^{-\frac{1}{2}}$

Concept Check Use the **laws of exponents** to simplify. Write the answers with positive exponents. Assume that all variables represent positive real numbers.

31. $3^{\frac{3}{4}} \cdot 3^{\frac{1}{8}}$

32. $x^{\frac{2}{3}} \cdot x^{-\frac{1}{4}}$

33. $\frac{2^{\frac{5}{8}}}{2^{-\frac{1}{8}}}$

34. $\frac{a^{\frac{1}{3}}}{a^{\frac{2}{3}}}$

35. $\left(5^{\frac{15}{8}}\right)^{\frac{2}{3}}$

36. $\left(y^{\frac{2}{3}}\right)^{-\frac{3}{7}}$

37. $\left(x^{\frac{3}{8}} \cdot y^{\frac{5}{2}}\right)^{\frac{4}{3}}$

38. $\left(a^{-\frac{2}{3}} \cdot b^{\frac{5}{8}}\right)^{-4}$

39. $\left(\frac{y^{-\frac{3}{2}}}{x^{\frac{5}{3}}}\right)^{\frac{1}{3}}$

40. $\left(\frac{a^{-\frac{2}{3}}}{b^{\frac{5}{6}}}\right)^{\frac{3}{4}}$

41. $x^{\frac{2}{3}} \cdot 5x^{-\frac{2}{5}}$

42. $x^{\frac{2}{5}} \cdot \left(4x^{-\frac{4}{5}}\right)^{-\frac{1}{4}}$

Use rational exponents to **simplify**. Write the answer **in radical notation** if appropriate. Assume that all variables represent positive real numbers.

43. $\sqrt[6]{x^2}$

44. $(\sqrt[3]{ab})^{15}$

45. $\sqrt[6]{y^{-18}}$

46. $\sqrt{x^4y^{-6}}$

47. $\sqrt[6]{81}$

48. $\sqrt[4]{128}$

49. $\sqrt[3]{8y^6}$

50. $\sqrt[4]{81p^6}$

51. $\sqrt[3]{(4x^3y)^2}$

52. $\sqrt[5]{64(x+1)^{10}}$

53. $\sqrt[4]{16x^4y^2}$

54. $\sqrt[5]{32a^{10}d^{15}}$

Use rational exponents to rewrite in a **single radical expression** in a simplified form. Assume that all variables represent positive real numbers.

55. $\sqrt[3]{5} \cdot \sqrt{5}$

56. $\sqrt[3]{2} \cdot \sqrt[4]{3}$

57. $\sqrt{a} \cdot \sqrt[3]{3a}$

58. $\sqrt[3]{x} \cdot \sqrt[5]{2x}$

59. $\sqrt[6]{x^5} \cdot \sqrt[3]{x^2}$

60. $\sqrt[3]{xz} \cdot \sqrt{z}$

61. $\frac{\sqrt{x^5}}{\sqrt{x^8}}$

62. $\frac{\sqrt[3]{a^5}}{\sqrt{a^3}}$

63. $\frac{\sqrt[3]{8x}}{\sqrt[4]{x^3}}$

64. $\sqrt[3]{\sqrt{a}}$

65. $\sqrt[4]{\sqrt[3]{xy}}$

66. $\sqrt{\sqrt[3]{(3x)^2}}$

67. $\sqrt{\sqrt[3]{\sqrt[4]{x}}}$

68. $\sqrt[3]{3\sqrt{3}}$

69. $\sqrt[4]{x\sqrt{x}}$

70. $\sqrt[3]{2\sqrt{x}}$

Discussion Point

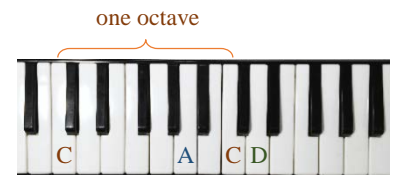
71. Suppose someone claims that $\sqrt[n]{a^n + b^n}$ must equal $a + b$, since, when $a = 1$ and $b = 0$, the two expressions are equal: $\sqrt[n]{a^n + b^n} = \sqrt[n]{1^n + 0^n} = 1 = 1 + 0 = a + b$. Explain why this is faulty reasoning.

Analytic Skills Solve each problem.

72. One octave on a piano contains 12 keys (including both the black and white keys). The frequency of each successive key increases by a factor of $2^{\frac{1}{12}}$. For example, middle C is two keys below the first D above it. Therefore, the frequency of this D is

$$2^{\frac{1}{12}} \cdot 2^{\frac{1}{12}} = 2^{\frac{1}{6}} \approx 1.12$$

times the frequency of the middle C.

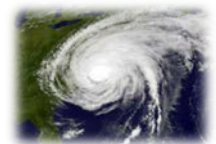


- If two tones are one octave apart, how do their frequencies compare?
- The A tone below middle C has a frequency of 220 cycles per second. Middle C is 3 keys above this A note. Estimate the frequency of middle C.



73. According to one model, an animal's heart rate varies according to its weight. The formula $N(w) = 885w^{-\frac{1}{2}}$ gives an estimate for the average number N of beats per minute for an animal that weighs w pounds. Use the formula to estimate the heart rate for a horse that weighs 800 pounds.

74. Meteorologists can determine the duration of a storm by using the function defined by $T(D) = 0.07D^{\frac{3}{2}}$, where D is the diameter of the storm in miles and T is the time in hours. Find the duration of a storm with a diameter of 16 mi. Round your answer to the nearest tenth of an hour.



RD.3

Simplifying Radical Expressions and the Distance Formula



In the previous section, we simplified some radical expressions by replacing radical signs with rational exponents, applying the rules of exponents, and then converting the resulting expressions back into radical notation. In this section, we broaden the above method of simplifying radicals by examining products and quotients of radicals with the same indexes, as well as explore the possibilities of decreasing the index of a radical.

In the second part of this section, we will apply the skills of simplifying radicals in problems involving the Pythagorean Theorem. In particular, we will develop the distance formula and apply it to calculate distances between two given points in a plane.

Multiplication, Division, and Simplification of Radicals

Suppose we wish to multiply radicals with the same indexes. This can be done by converting each radical to a rational exponent and then using properties of exponents as follows:

**PRODUCT
RULE**

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = a^{\frac{1}{n}} \cdot b^{\frac{1}{n}} = (ab)^{\frac{1}{n}} = \sqrt[n]{ab}$$

This shows that the **product of same index radicals** is the **radical of the product** of their radicands.

Similarly, the **quotient of same index radicals** is the **radical of the quotient** of their radicands, as we have

**QUOTIENT
RULE**

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \left(\frac{a}{b}\right)^{\frac{1}{n}} = \sqrt[n]{\frac{a}{b}}$$

So, $\sqrt{2} \cdot \sqrt{8} = \sqrt{2 \cdot 8} = \sqrt{16} = 4$. Similarly, $\sqrt[3]{16} = \sqrt[3]{\frac{16}{2}} = \sqrt[3]{8} = 2$.

Attention! There is no such rule for addition or subtraction of terms. For instance,

$$\sqrt{a+b} \neq \sqrt{a} \pm \sqrt{b},$$

and generally

$$\sqrt[n]{a \pm b} \neq \sqrt[n]{a} \pm \sqrt[n]{b}.$$

Here is a counterexample: $\sqrt{2} = \sqrt{1+1} \neq \sqrt{1} + \sqrt{1} = 1 + 1 = 2$

Example 1
Multiplying and Dividing Radicals of the Same Indexes

Perform the indicated operations and simplify, if possible. Assume that all variables are positive.

a. $\sqrt{10} \cdot \sqrt{15}$

b. $\sqrt{2x^3} \sqrt{6xy}$

c. $\frac{\sqrt{10x}}{\sqrt{5}}$

d. $\frac{\sqrt[4]{32x^3}}{\sqrt[4]{2x}}$

Solution

- a. $\sqrt{10} \cdot \sqrt{15} = \sqrt{10 \cdot 15} = \sqrt{2 \cdot 5 \cdot 5 \cdot 3} = \sqrt{5 \cdot 5 \cdot 2 \cdot 3} = \sqrt{25} \cdot \sqrt{6} = 5\sqrt{6}$
 product rule prime factorization commutativity of multiplication product rule
- b. $\sqrt{2x^3} \sqrt{6xy} = \sqrt{2 \cdot 2 \cdot 3x^4y} = \sqrt{4x^4} \cdot \sqrt{3y} = 2x^2\sqrt{3y}$
 use commutativity of multiplication to isolate perfect square factors
- c. $\frac{\sqrt{10x}}{\sqrt{5}} = \sqrt{\frac{10x}{5}} = \sqrt{2x}$
 quotient rule
- d. $\frac{\sqrt[4]{32x^3}}{\sqrt[4]{2x}} = \sqrt[4]{\frac{16x^3}{2x}} = \sqrt[4]{16x^2} = \sqrt[4]{16} \cdot \sqrt[4]{x^2} = 2\sqrt{x}$

Here the multiplication sign is assumed, even if it is not indicated.

Recall that $\sqrt[4]{x^2} = x^{\frac{2}{4}} = x^{\frac{1}{2}} = \sqrt{x}$.

Caution! Remember to indicate the index of the radical for indexes higher than two.

The product and quotient rules are essential when simplifying radicals.

To simplify a radical means to:

1. Make sure that all **power factors of the radicand have exponents smaller than the index of the radical**.

$$\text{For example, } \sqrt[3]{2^4x^8y} = \sqrt[3]{2^3x^6} \cdot \sqrt[3]{2x^2y} = 2x^2\sqrt[3]{2x^2y}.$$

2. Leave the radicand with **no fractions**.

$$\text{For example, } \sqrt{\frac{2x}{25}} = \frac{\sqrt{2x}}{\sqrt{25}} = \frac{\sqrt{2x}}{5}.$$

3. **Rationalize any denominator**. (Make sure that denominators are **free from radicals**.)

$$\text{For example, } \sqrt{\frac{4}{x}} = \frac{\sqrt{4}}{\sqrt{x}} = \frac{2 \cdot \sqrt{x}}{\sqrt{x} \cdot \sqrt{x}} = \frac{2\sqrt{x}}{x}, \text{ providing that } x > 0.$$

4. **Reduce the power of the radicand with the index of the radical**, if possible.

$$\text{For example, } \sqrt[4]{x^2} = x^{\frac{2}{4}} = x^{\frac{1}{2}} = \sqrt{x}.$$

Example 2 Simplifying Radicals

Simplify each radical. Assume that all variables are positive.

- a. $\sqrt[5]{96x^7y^{15}}$ b. $\sqrt[4]{\frac{a^{12}}{16b^4}}$ c. $\sqrt{\frac{25x^2}{8x^3}}$ d. $\sqrt[6]{27a^{15}}$

Solution

a. $\sqrt[5]{96x^7y^{15}} = \sqrt[5]{2^5 \cdot 3x^7y^{15}} = 2xy^3\sqrt[5]{3x^2}$

$\sqrt[5]{y^{15}} = y^3$

$\sqrt[5]{x^7} = x\sqrt[5]{x^2}$

Generally, to simplify $\sqrt[d]{x^a}$, we perform the division

$$a \div d = \text{quotient } q + \text{remainder } r,$$

and then pull the q -th power of x out of the radical, leaving the r -th power of x under the radical. So, we obtain

$$\sqrt[d]{x^a} = x^q \sqrt[d]{x^r}$$

b. $\sqrt[4]{\frac{a^{12}}{16b^4}} = \frac{\sqrt[4]{a^{12}}}{\sqrt[4]{2^4b^4}} = \frac{a^3}{2b}$

c. $\sqrt{\frac{25x^2}{8x^3}} = \sqrt{\frac{25}{2^3x}} = \frac{\sqrt{25}}{\sqrt{2^3x}} = \frac{5}{2\sqrt{2x}} \cdot \frac{\sqrt{2x}}{\sqrt{2x}} = \frac{5\sqrt{2x}}{2 \cdot 2x} = \frac{5\sqrt{2x}}{4x}$

d. $\sqrt[6]{27a^{15}} = \sqrt[6]{3^3a^{15}} = a^2\sqrt[6]{3^3a^3} = a^2 \cdot \sqrt[6]{(3a)^3} = a^2\sqrt{3a}$

Example 3**Simplifying Expressions Involving Multiplication, Division, or Composition of Radicals with Different Indexes**

Simplify each expression. Leave your answer in simplified single radical form. Assume that all variables are positive.

a. $\sqrt{xy^5} \cdot \sqrt[3]{x^4y}$

b. $\frac{\sqrt[4]{a^2b^3}}{\sqrt[3]{ab}}$

c. $\sqrt[3]{x^2\sqrt{2x}}$

Solution

a. $\sqrt{xy^5} \cdot \sqrt[3]{x^4y} = x^{\frac{1}{2}}y^{\frac{5}{2}} \cdot x^{\frac{4}{3}}y^{\frac{1}{3}} = x^{\frac{1 \cdot 3}{2 \cdot 3} + \frac{2 \cdot 2}{3 \cdot 2}}y^{\frac{5 \cdot 3}{2 \cdot 3} + \frac{1 \cdot 2}{3 \cdot 2}} = x^{\frac{7}{6}}y^{\frac{17}{6}} = (x^7y^{17})^{\frac{1}{6}} = \sqrt[6]{x^7y^{17}} = xy^2\sqrt[6]{xy^5}$

If radicals are of different indexes, convert them to exponential form.

b. $\frac{\sqrt[4]{a^2b^3}}{\sqrt[3]{ab}} = \frac{a^{\frac{2}{4}}b^{\frac{3}{4}}}{a^{\frac{1}{3}}b^{\frac{1}{3}}} = a^{\frac{1 \cdot 3}{2 \cdot 3} - \frac{1 \cdot 2}{3 \cdot 2}}b^{\frac{3 \cdot 3}{4 \cdot 3} - \frac{1 \cdot 4}{3 \cdot 4}} = a^{\frac{1 \cdot 2}{6} - \frac{2}{6}}b^{\frac{5}{4} - \frac{4}{4}} = (a^{\frac{1}{6}}b^{\frac{1}{6}})^{\frac{1}{12}} = \sqrt[12]{a^2b^5}$

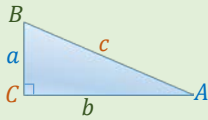
Bring the exponents to the LCD in order to leave the answer as a single radical.

c. $\sqrt[3]{x^2\sqrt{2x}} = x^{\frac{2}{3}} \cdot ((2x)^{\frac{1}{2}})^{\frac{1}{3}} = x^{\frac{2}{3}} \cdot 2^{\frac{1}{6}} \cdot x^{\frac{1}{6}} = x^{\frac{2 \cdot 2}{3 \cdot 2} + \frac{1}{6}} \cdot 2^{\frac{1}{6}} = 2^{\frac{1}{6}}x^{\frac{5}{6}} = (2x^5)^{\frac{1}{6}} = \sqrt[6]{2x^5}$

Pythagorean Theorem and Distance Formula

One of the most famous theorems in mathematics is the Pythagorean Theorem.

Pythagorean Theorem

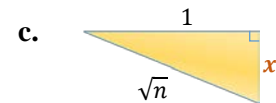
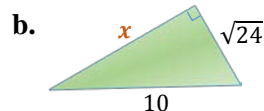
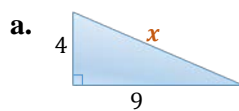


Suppose angle C in a triangle ABC is a 90° angle. Then the **sum of the squares** of the lengths of the two **legs**, a and b , equals to the **square** of the length of the **hypotenuse** c :

$$a^2 + b^2 = c^2$$

Example 4 Using The Pythagorean Equation

For the first two triangles, find the exact length x of the unknown side. For triangle **c.**, express length x in terms of the unknown n .



Solution

Caution: Generally, $\sqrt{x^2} = |x|$. However, the length of a side of a triangle is positive. So, we can write $\sqrt{x^2} = x$.

- a. The length of the hypotenuse of the given right triangle is equal to x . So, the Pythagorean equation takes the form

$$x^2 = 4^2 + 9^2.$$

To solve it for x , we take a square root of each side of the equation. This gives us

$$\begin{aligned}\sqrt{x^2} &= \sqrt{4^2 + 9^2} \\ x &= \sqrt{16 + 81} \\ x &= \sqrt{97}\end{aligned}$$

- b. Since 10 is the length of the hypotenuse, we form the Pythagorean equation

$$10^2 = x^2 + \sqrt{24}^2.$$

To solve it for x , we isolate the x^2 term and then apply the square root operator to both sides of the equation. So, we have

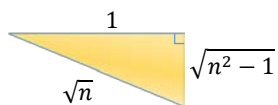
$$\begin{aligned}10^2 - \sqrt{24}^2 &= x^2 \\ 100 - 24 &= x^2 \\ x^2 &= 76 \\ x &= \sqrt{76} = \sqrt{4 \cdot 19} = 2\sqrt{19}\end{aligned}$$

Customary, we simplify each root, if possible.

- c. The length of the hypotenuse is \sqrt{n} , so we form the Pythagorean equation as below.

$$(\sqrt{n})^2 = 1^2 + x^2$$

To solve this equation for x , we isolate the x^2 term and then apply the square root operator to both sides of the equation. So, we obtain



$$\begin{aligned}n^2 &= 1 + x^2 \\n^2 - 1 &= x^2 \\x &= \sqrt{n^2 - 1}\end{aligned}$$

Note: Since the hypotenuse of length \sqrt{n} must be longer than the leg of length 1, then $n > 1$. This means that $n^2 - 1 > 0$, and therefore $\sqrt{n^2 - 1}$ is a positive real number.

The Pythagorean Theorem allows us to find the distance between any two given points in a plane.

Suppose $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points in a coordinate plane. Then $|x_2 - x_1|$ represents the horizontal distance between A and B and $|y_2 - y_1|$ represents the vertical distance between A and B , as shown in *Figure 1*. Notice that by applying the absolute value operator to each difference of the coordinates we guarantee that the resulting horizontal and vertical distance is indeed a nonnegative number.

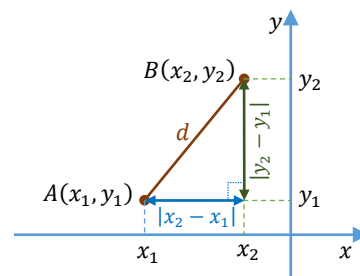


Figure 1

Applying the Pythagorean Theorem to the right triangle shown in *Figure 1*, we form the equation

$$d^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2,$$

where d is the distance between A and B .

Notice that $|x_2 - x_1|^2 = (x_2 - x_1)^2$ as a perfect square automatically makes the expression nonnegative. Similarly, $|y_2 - y_1|^2 = (y_2 - y_1)^2$. So, the Pythagorean equation takes the form

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

After solving this equation for d , we obtain the **distance formula**:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Note: Observe that due to squaring the difference of the corresponding coordinates, **the distance between two points is the same regardless of which point is chosen as first, (x_1, y_1) , and second, (x_2, y_2) .**

Example 5 ▶ Finding the Distance Between Two Points

Find the exact distance between the points $(-2, 4)$ and $(5, 3)$.

Solution ▶ Let $(-2, 4) = (x_1, y_1)$ and $(5, 3) = (x_2, y_2)$. To find the distance d between the two points, we follow the distance formula:

$$d = \sqrt{(5 - (-2))^2 + (3 - 4)^2} = \sqrt{7^2 + (-1)^2} = \sqrt{49 + 1} = \sqrt{50} = 5\sqrt{2}$$

So, the points $(-2, 4)$ and $(5, 3)$ are $5\sqrt{2}$ units apart.

RD.3 Exercises

Vocabulary Check Complete each blank with the most appropriate term or phrase from the given list: **distance**, **index**, **lower**, **product**, **Pythagorean**, **right**, **simplified**.

- The product of the same _____ radicals is the radical of the _____ of the radicands.
- The radicand of a simplified radical contains only powers with _____ exponents than the index of the radical.
- The radical $\sqrt[6]{x^4}$ can still be _____.
- The _____ between two points (x_1, y_1) and (x_2, y_2) on a coordinate plane is equal to $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
- The _____ equation applies to _____ triangles only.

Concept Check Multiply and simplify, if possible. Assume that all variables are positive.

- | | | | |
|------------------------------------|---------------------------------------|-------------------------------------|--------------------------------------|
| 6. $\sqrt{5} \cdot \sqrt{5}$ | 7. $\sqrt{18} \cdot \sqrt{2}$ | 8. $\sqrt{6} \cdot \sqrt{3}$ | 9. $\sqrt{15} \cdot \sqrt{6}$ |
| 10. $\sqrt{45} \cdot \sqrt{60}$ | 11. $\sqrt{24} \cdot \sqrt{75}$ | 12. $\sqrt{3x^3} \cdot \sqrt{6x^5}$ | 13. $\sqrt{5y^7} \cdot \sqrt{15a^3}$ |
| 14. $\sqrt{12x^3y} \sqrt{8x^4y^2}$ | 15. $\sqrt{30a^3b^4} \sqrt{18a^2b^5}$ | 16. $\sqrt[3]{4x^2} \sqrt[3]{2x^4}$ | 17. $\sqrt[4]{20a^3} \sqrt[4]{4a^5}$ |

Concept Check Divide and simplify, if possible. Assume that all variables are positive.

- | | | | |
|--|---|---|---|
| 18. $\frac{\sqrt{90}}{\sqrt{5}}$ | 19. $\frac{\sqrt{48}}{\sqrt{6}}$ | 20. $\frac{\sqrt{42a}}{\sqrt{7a}}$ | 21. $\frac{\sqrt{30x^3}}{\sqrt{10x}}$ |
| 22. $\frac{\sqrt{52ab^3}}{\sqrt{13a}}$ | 23. $\frac{\sqrt{56xy^3}}{\sqrt{8x}}$ | 24. $\frac{\sqrt{128x^2y}}{2\sqrt{2}}$ | 25. $\frac{\sqrt{48a^3b}}{2\sqrt{3}}$ |
| 26. $\frac{\sqrt[4]{80}}{\sqrt[4]{5}}$ | 27. $\frac{\sqrt[3]{108}}{\sqrt[3]{4}}$ | 28. $\frac{\sqrt[3]{96a^5b^2}}{\sqrt[3]{12a^2b}}$ | 29. $\frac{\sqrt[4]{48x^9y^{13}}}{\sqrt[4]{3xy^5}}$ |

Concept Check Simplify each expression. Assume that all variables are positive.

- | | | | |
|---------------------------------------|--------------------------------------|----------------------------------|-----------------------|
| 30. $\sqrt{144x^4y^9}$ | 31. $-\sqrt{81m^8n^5}$ | 32. $\sqrt[3]{-125a^6b^9c^{12}}$ | 33. $\sqrt{50x^3y^4}$ |
| 34. $\sqrt[4]{\frac{1}{16}m^8n^{20}}$ | 35. $-\sqrt[3]{-\frac{1}{27}x^2y^7}$ | 36. $\sqrt{7a^7b^6}$ | 37. $\sqrt{75p^3q^4}$ |

38. $\sqrt[5]{64x^{12}y^{15}}$

39. $\sqrt[5]{p^{14}q^7r^{23}}$

40. $-\sqrt[4]{162a^{15}b^{10}}$

41. $-\sqrt[4]{32x^5y^{10}}$

42. $\sqrt{\frac{16}{49}}$

43. $\sqrt[3]{\frac{27}{125}}$

44. $\sqrt{\frac{121}{y^2}}$

45. $\sqrt{\frac{64}{x^4}}$

46. $\sqrt[3]{\frac{81a^5}{64}}$

47. $\sqrt{\frac{36x^5}{y^6}}$

48. $\sqrt[4]{\frac{16x^{12}}{y^4z^{16}}}$

49. $\sqrt[5]{\frac{32y^8}{x^{10}}}$

50. $\sqrt[4]{36}$

51. $\sqrt[6]{27}$

52. $-\sqrt[10]{x^{25}}$

53. $\sqrt[12]{x^{44}}$

54. $-\sqrt{\frac{1}{x^3y}}$

55. $\sqrt[3]{\frac{64x^{15}}{y^4z^5}}$

56. $\sqrt[6]{\frac{x^{13}}{y^6z^{12}}}$

57. $\sqrt[6]{\frac{p^9q^{24}}{r^{18}}}$

Discussion Point

58. When asked to simplify the radical $\sqrt{x^3 + x^2}$, a student wrote $\sqrt{x^3 + x^2} = x\sqrt{x} + x = x(\sqrt{x} + 1)$. Is this correct? If *yes*, explain the student's reasoning. If *not*, discuss how such radical could be simplified.

Perform operations. Leave the answer in simplified **single radical** form. Assume that all variables are positive.

59. $\sqrt{3} \cdot \sqrt[3]{4}$

60. $\sqrt{x} \cdot \sqrt[5]{x}$

61. $\sqrt[3]{x^2} \cdot \sqrt[4]{x}$

62. $\sqrt[3]{4} \cdot \sqrt[5]{8}$

63. $\frac{\sqrt[3]{a^2}}{\sqrt{a}}$

64. $\frac{\sqrt{x}}{\sqrt[4]{x}}$

65. $\frac{\sqrt[4]{x^2y^3}}{\sqrt[3]{xy}}$

66. $\frac{\sqrt[5]{16a^2}}{\sqrt[3]{2a^2}}$

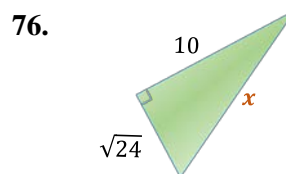
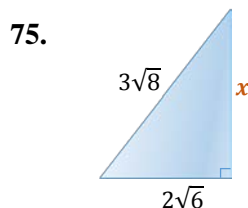
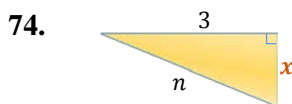
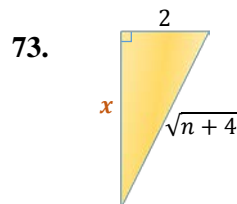
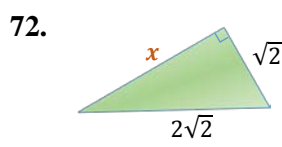
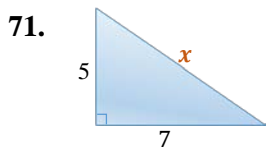
67. $\sqrt[3]{2\sqrt{x}}$

68. $\sqrt{x} \sqrt[3]{2x^2}$

69. $\sqrt[4]{3} \sqrt[3]{9}$

70. $\sqrt[3]{x^2} \sqrt[4]{x^3}$

Concept Check For each right triangle, find length x . Simplify the answer if possible. In problems 73 and 74, expect the length x to be an expression in terms of n .



Concept Check Find the exact distance between each pair of points.

77. (8,13) and (2,5)

78. (-8,3) and (-4,1)

79. (-6,5) and (3,-4)

80. $(\frac{5}{7}, \frac{1}{14})$ and $(\frac{1}{7}, \frac{11}{14})$

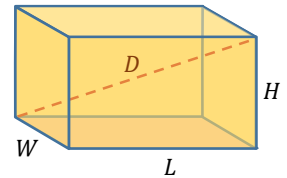
81. $(0, \sqrt{6})$ and $(\sqrt{7}, 0)$

82. $(\sqrt{2}, \sqrt{6})$ and $(2\sqrt{2}, -4\sqrt{6})$

83. $(-\sqrt{5}, 6\sqrt{3})$ and $(\sqrt{5}, \sqrt{3})$ 84. $(0,0)$ and (p, q) 85. $(x + h, y + h)$ and (x, y)
 (assume that $h > 0$)

Analytic Skills Solve each problem.

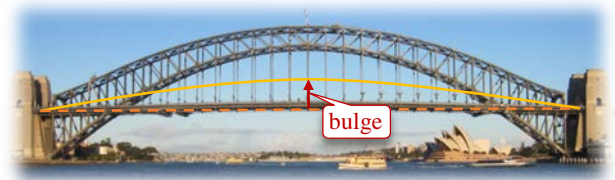
86. The length of the diagonal of a box is given by the formula $D = \sqrt{W^2 + L^2 + H^2}$, where W , L , and H are, respectively, the width, length, and height of the box. Find the length of the diagonal D of a box that is 4 ft long, 2 ft wide, and 3 ft high. Give the exact value, and then round to the nearest tenth of a foot.



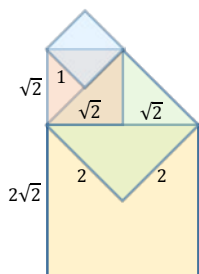
87. The screen of a 32-inch television is 27.9-inch wide. To the nearest tenth of an inch, what is the measure of its height? *TVs are measured diagonally, so a 32-inch television means that its screen measures diagonally 32 inches.*

88. Find all ordered pairs on the x -axis of a Cartesian coordinate system that are 5 units from the point $(0, 4)$.
 89. Find all ordered pairs on the y -axis of a Cartesian coordinate system that are 5 units from the point $(3, 0)$.

90. During the summer heat, a 2-mi bridge expands 2 ft in length. If we assume that the bulge occurs in the middle of the bridge, how high is the bulge? *The answer may surprise you. In reality, bridges are built with expansion spaces to avoid such buckling.*



RD.4

Operations on Radical Expressions;
Rationalization of Denominators

Unlike operations on fractions or decimals, sums and differences of many radicals cannot be simplified. For instance, we cannot combine $\sqrt{2}$ and $\sqrt{3}$, nor simplify expressions such as $\sqrt[3]{2} - 1$. These types of radical expressions can only be approximated with the aid of a calculator.

However, some radical expressions can be combined (added or subtracted) and simplified.

For example, the sum of $2\sqrt{2}$ and $\sqrt{2}$ is $3\sqrt{2}$, similarly as $2x + x = 3x$.

In this section, first, we discuss the addition and subtraction of radical expressions. Then, we show how to work with radical expressions involving a combination of the four basic operations. Finally, we examine how to rationalize denominators of radical expressions.

Addition and Subtraction of Radical Expressions

Recall that to perform addition or subtraction of two variable terms we need these terms to be **like**. This is because the addition and subtraction of terms are performed by factoring out the variable “like” part of the terms as a common factor. For example,

$$x^2 + 3x^2 = (1 + 3)x^2 = 4x^2$$

The same strategy works for addition and subtraction of the same types of radicals or **radical terms** (terms containing radicals).

Definition 4.1 ▶ Radical terms containing radicals with the same index and the same radicands are referred to as **like radicals** or **like radical terms**.

For example,

$$\sqrt{5x} \text{ and } 2\sqrt{5x} \text{ are **like** (the indexes and the radicands are the same)}$$

while

$$5\sqrt{2} \text{ and } 2\sqrt{5} \text{ are **not like** (the radicands are different)}$$

and

$$\sqrt{x} \text{ and } \sqrt[3]{x} \text{ are **not like radicals** (the indexes are different).}$$

To **add** or **subtract like radical expressions** we **factor out the common radical** and any other common factor, if applicable. For example,

$$4\sqrt{2} + 3\sqrt{2} = (4 + 3)\sqrt{2} = 7\sqrt{2},$$

and

$$4xy\sqrt{2} - 3x\sqrt{2} = (4y + 3)x\sqrt{2}.$$

Caution! Unlike radical expressions cannot be combined. For example, we are unable to perform the addition $\sqrt{6} + \sqrt{3}$. Such a sum can only be approximated using a calculator.

Notice that unlike radicals may become like if we simplify them first. For example, $\sqrt{200}$ and $\sqrt{50}$ are not like, but $\sqrt{200} = 10\sqrt{2}$ and $\sqrt{50} = 5\sqrt{2}$. Since $10\sqrt{2}$ and $5\sqrt{2}$ are like radical terms, they can be combined. So, we can perform, for example, the addition:

$$\sqrt{200} + \sqrt{50} = 10\sqrt{2} + 5\sqrt{2} = 15\sqrt{2}$$

Example 1 ▶ **Adding and Subtracting Radical Expressions**

Perform operations and simplify, if possible. Assume that all variables represent positive real numbers.

- a. $5\sqrt{3} - 8\sqrt{3}$ b. $3^5\sqrt{2} - 7x^5\sqrt{2} + 6^5\sqrt{2}$
 c. $7\sqrt{45} + \sqrt{80} - \sqrt{12}$ d. $3^3\sqrt{y^5} - 7^3\sqrt{y^2} + \sqrt[3]{8y^8}$
 e. $\sqrt{\frac{x}{16}} + 2\sqrt{\frac{x^3}{9}}$ f. $\sqrt{25x - 25} - \sqrt{9x - 9}$

Solution ▶

- a. To subtract like radicals, we combine their coefficients via factoring.

$$5\sqrt{3} - 8\sqrt{3} = (5 - 8)\sqrt{3} = -3\sqrt{3}$$

The brackets are essential here.

- b. $3^5\sqrt{2} - 7x^5\sqrt{2} + 6^5\sqrt{2} = (3 - 7x + 6)^5\sqrt{2} = (9 - 7x)^5\sqrt{2}$

Note: Even if not all coefficients are like, factoring the common radical is a useful strategy that allows us to combine like radical expressions.

- c. The expression $7\sqrt{45} + \sqrt{80} - \sqrt{12}$ consists of unlike radical terms, so they cannot be combined in this form. However, if we simplify the radicals, some of them may become like and then become possible to combine.

$$\begin{aligned} 7\sqrt{45} + \sqrt{80} - \sqrt{12} &= 7\sqrt{9 \cdot 5} + \sqrt{16 \cdot 5} - \sqrt{4 \cdot 3} = 7 \cdot 3\sqrt{5} + 4\sqrt{5} - 2\sqrt{3} \\ &= 21\sqrt{5} + 4\sqrt{5} - 2\sqrt{3} = 25\sqrt{5} - 2\sqrt{3} \end{aligned}$$

- d. As in the previous example, we simplify each radical expression before attempting to combine them.

$$\begin{aligned} 3^3\sqrt{y^5} - 5y^3\sqrt{y^2} + \sqrt[5]{32y^7} &= 3y^3\sqrt{y^2} - 5y^3\sqrt{y^2} + 2y^5\sqrt{y^2} \\ &= (3y - 5y)^3\sqrt{y^2} + 2y^5\sqrt{y^2} = -2y^3\sqrt{y^2} + 2y^5\sqrt{y^2} \end{aligned}$$

Remember to write the index with each radical.

Note: The last two radical expressions cannot be combined because of different indexes.

- e. To perform the addition $\sqrt{\frac{x}{16}} + 2\sqrt{\frac{x^3}{9}}$, we may simplify each radical expression first. Then, we add the expressions by bringing them to the least common denominator and finally, factor the common radical, as shown below.

$$\sqrt{\frac{x}{16}} + 2\sqrt{\frac{x^3}{9}} = \frac{\sqrt{x}}{\sqrt{16}} + 2\frac{\sqrt{x^3}}{\sqrt{9}} = \frac{\sqrt{x}}{4} + 2\frac{\sqrt{x^3}}{3} = \frac{3\sqrt{x} + 2 \cdot 4 \cdot x\sqrt{x}}{12} = \left(\frac{3+8x}{12}\right)\sqrt{x}$$

- f. In an attempt to simplify radicals in the expression $\sqrt{25x^2 - 25} - \sqrt{9x^2 - 9}$, we factor each radicand first. So, we obtain

$$\begin{aligned}\sqrt{25x^2 - 25} - \sqrt{9x^2 - 9} &= \sqrt{25(x^2 - 1)} - \sqrt{9(x^2 - 1)} = 5\sqrt{x^2 - 1} - 3\sqrt{x^2 - 1} \\ &= 2\sqrt{x^2 - 1}\end{aligned}$$

Caution! The root of a sum does not equal the sum of the roots. For example,

$$\sqrt{5} = \sqrt{1 + 4} \neq \sqrt{1} + \sqrt{4} = 1 + 2 = 3$$

So, radicals such as $\sqrt{25x^2 - 25}$ or $\sqrt{9x^2 - 9}$ can be simplified only via factoring a perfect square out of their radicals while $\sqrt{x^2 - 1}$ cannot be simplified any further.

Multiplication of Radical Expressions with More than One Term

Similarly as in the case of multiplication of polynomials, multiplication of radical expressions where at least one factor consists of more than one term is performed by applying the distributive property.

Example 2 ▶ Multiplying Radical Expressions with More than One Term

Multiply and then simplify each product. Assume that all variables represent positive real numbers.

- a. $5\sqrt{2}(3\sqrt{2x} - \sqrt{6})$ b. $\sqrt[3]{x}(\sqrt[3]{3x^2} - \sqrt[3]{81x^2})$
 c. $(2\sqrt{3} + \sqrt{2})(\sqrt{3} - 3\sqrt{2})$ d. $(x\sqrt{x} - \sqrt{y})(x\sqrt{x} + \sqrt{y})$
 e. $(3\sqrt{2} + 2\sqrt[3]{x})(3\sqrt{2} - 2\sqrt[3]{x})$ f. $(\sqrt{5y} + y\sqrt{y})^2$

Solution ▶

a.
$$5\sqrt{2}(3\sqrt{2x} - \sqrt{6}) = 15\sqrt{4x} - 5\sqrt{2 \cdot 2 \cdot 3} = 15 \cdot 2\sqrt{x} - 5 \cdot 2\sqrt{3} = 30\sqrt{x} - 10\sqrt{3}$$

$5\sqrt{2} \cdot 3\sqrt{2x} = 5 \cdot 3\sqrt{2 \cdot 2x}$

These are unlike terms. So, they cannot be combined.

b.
$$\sqrt[3]{x}(\sqrt[3]{3x^2} - \sqrt[3]{81x^2}) = \sqrt[3]{3x^2 \cdot x} - \sqrt[3]{81x^2 \cdot x} = x\sqrt[3]{3} - 3x\sqrt[3]{3} = -2x\sqrt[3]{3}$$

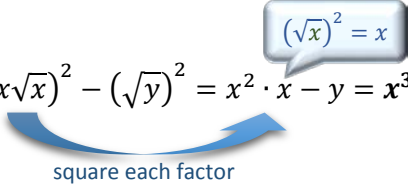
distribution simplification combining like terms

- c. To multiply two binomial expressions involving radicals we may use the **FOIL** method. Recall that the acronym **FOIL** refers to multiplying the **F**irst, **O**uter, **I**nner, and **L**ast terms of the binomials.

$$\begin{aligned}
 (2\sqrt{3} + \sqrt{2})(\sqrt{3} - 3\sqrt{2}) &= \overset{\mathbf{F}}{2} \cdot \overset{\mathbf{O}}{3} - \overset{\mathbf{I}}{6}\sqrt{\overset{\mathbf{L}}{3} \cdot 2} + \sqrt{2} \cdot \overset{\mathbf{I}}{3} - \overset{\mathbf{L}}{3} \cdot 2 = \cancel{6} - 6\sqrt{6} + \sqrt{6} - \cancel{6} \\
 &= -5\sqrt{6}
 \end{aligned}$$

- d. To multiply two conjugate binomial expressions we follow the difference of squares formula, $(a - b)(a + b) = a^2 - b^2$. So, we obtain

$$\begin{aligned}
 (x\sqrt{x} - \sqrt{y})(x\sqrt{x} + \sqrt{y}) &= (x\sqrt{x})^2 - (\sqrt{y})^2 = x^2 \cdot x - y = x^3 - y
 \end{aligned}$$



 square each factor

- e. Similarly as in the previous example, we follow the difference of squares formula.

$$(3\sqrt{2} + 2\sqrt[3]{x})(3\sqrt{2} - 2\sqrt[3]{x}) = (3\sqrt{2})^2 - (2\sqrt[3]{x})^2 = 9 \cdot 2 - 4\sqrt[3]{x^2} = 18 - 4\sqrt[3]{x^2}$$

- f. To multiply two identical binomial expressions we follow the perfect square formula, $(a + b)(a + b) = a^2 + 2ab + b^2$. So, we obtain

$$\begin{aligned}
 (\sqrt{5y} + y\sqrt{y})^2 &= (\sqrt{5y})^2 + 2(\sqrt{5y})(y\sqrt{y}) + (y\sqrt{y})^2 = 5y + 2y\sqrt{5y^2} + y^2y \\
 &= 5y + 2\sqrt{5y^2} + y^3
 \end{aligned}$$

Rationalization of Denominators

As mentioned in *section RD3*, a process of simplifying radicals involves rationalization of any emerging denominators. Similarly, a radical expression is not in its simplest form unless all its denominators are rational. This agreement originated before the days of calculators when computation was a tedious process performed by hand. Nevertheless, even in present time, the agreement of keeping denominators rational does not lose its validity, as we often work with variable radical expressions. For example, the expressions $\frac{2}{\sqrt{2}}$ and $\sqrt{2}$ are equivalent, as

$$\frac{2}{\sqrt{2}} = \frac{2}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}}{2} = \sqrt{2}$$

Similarly, $\frac{x}{\sqrt{x}}$ is equivalent to \sqrt{x} , as

$$\frac{x}{\sqrt{x}} = \frac{x}{\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}} = \frac{x\sqrt{x}}{x} = \sqrt{x}$$

While one can argue that evaluating $\frac{2}{\sqrt{2}}$ is as easy as evaluating $\sqrt{2}$ when using a calculator, the expression \sqrt{x} is definitely easier to use than $\frac{x}{\sqrt{x}}$ in any further algebraic manipulations.

Definition 4.2 ► The process of removing radicals from a denominator so that the denominator contains only rational numbers is called **rationalization** of the denominator.

Rationalization of denominators is carried out by multiplying the given fraction by a factor of 1, as shown in the next two examples.

Example 3 ▶ Rationalizing Monomial Denominators

Simplify, if possible. Leave the answer with a rational denominator. Assume that all variables represent positive real numbers.

a. $\frac{-1}{3\sqrt{5}}$

b. $\frac{5}{\sqrt[3]{32x}}$

c. $\sqrt[4]{\frac{81x^5}{y}}$

Solution ▶

- a. Notice that $\sqrt{5}$ can be converted to a rational number by multiplying it by another $\sqrt{5}$. Since the denominator of a fraction cannot be changed without changing the numerator in the same way, we multiply both, the numerator and denominator of $\frac{-1}{3\sqrt{5}}$ by $\sqrt{5}$. So, we obtain

$$\frac{-1}{3\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{-\sqrt{5}}{3 \cdot 5} = -\frac{\sqrt{5}}{15}$$

- b. First, we may want to simplify the radical in the denominator. So, we have

$$\frac{5}{\sqrt[3]{32x}} = \frac{5}{\sqrt[3]{8 \cdot 4x}} = \frac{5}{2\sqrt[3]{4x}}$$

Then, notice that since $\sqrt[3]{4x} = \sqrt[3]{2^2x}$, it is enough to multiply it by $\sqrt[3]{2x^2}$ to nihilate the radical. This is because $\sqrt[3]{2^2x} \cdot \sqrt[3]{2x^2} = \sqrt[3]{2^3x^3} = 2x$. So, we proceed

$$\frac{5}{\sqrt[3]{32x}} = \frac{5}{2\sqrt[3]{4x}} \cdot \frac{\sqrt[3]{2x^2}}{\sqrt[3]{2x^2}} = \frac{5\sqrt[3]{2x^2}}{2 \cdot 2x} = \frac{5\sqrt[3]{2x^2}}{4x}$$

Caution: A common mistake in the rationalization of $\sqrt[3]{4x}$ is the attempt to multiply it by a copy of $\sqrt[3]{4x}$. However, $\sqrt[3]{4x} \cdot \sqrt[3]{4x} = \sqrt[3]{16x^2} = 2\sqrt[3]{3x^2}$ is still not rational. This is because we work with a cubic root, not a square root. So, to rationalize $\sqrt[3]{4x}$ we must look for ‘filling’ the radicand to a perfect cube. This is achieved by multiplying $4x$ by $2x^2$ to get $8x^3$.

- c. To simplify $\sqrt[4]{\frac{81x^5}{y}}$, first, we apply the quotient rule for radicals, then simplify the radical in the numerator, and finally, rationalize the denominator. So, we have

$$\sqrt[4]{\frac{81x^5}{y}} = \frac{\sqrt[4]{81x^5}}{\sqrt[4]{y}} = \frac{3x\sqrt[4]{x}}{\sqrt[4]{y}} \cdot \frac{\sqrt[4]{y^3}}{\sqrt[4]{y^3}} = \frac{3x\sqrt[4]{xy^3}}{y}$$

To rationalize a binomial containing square roots, such as $2 - \sqrt{x}$ or $\sqrt{2} - \sqrt{3}$, we need to find a way to square each term separately. This can be achieved through multiplying by a conjugate binomial, in order to benefit from the difference of squares formula. In particular, we can rationalize denominators in expressions below as follows:

$$\frac{1}{2 - \sqrt{x}} = \frac{1}{2 - \sqrt{x}} \cdot \frac{(2 + \sqrt{x})}{(2 + \sqrt{x})} = \frac{2 + \sqrt{x}}{4 - x}$$

Apply the difference of squares formula:
 $(a - b)(a + b) = a^2 - b^2$

or

$$\frac{\sqrt{2}}{\sqrt{2} + \sqrt{3}} = \frac{\sqrt{2}}{(\sqrt{2} + \sqrt{3})} \cdot \frac{(\sqrt{2} - \sqrt{3})}{(\sqrt{2} - \sqrt{3})} = \frac{2 - \sqrt{6}}{2 - 3} = \frac{2 - \sqrt{6}}{-1} = \sqrt{6} - 2$$

Example 4 ▶ Rationalizing Binomial Denominators

Rationalize each denominator and simplify, if possible. Assume that all variables represent positive real numbers.

a. $\frac{1 - \sqrt{3}}{1 + \sqrt{3}}$

b. $\frac{\sqrt{xy}}{2\sqrt{x} - \sqrt{y}}$

Solution ▶

a. $\frac{1 - \sqrt{3}}{1 + \sqrt{3}} \cdot \frac{(1 - \sqrt{3})}{(1 - \sqrt{3})} = \frac{1 - 2\sqrt{3} + 3}{1 - 3} = \frac{4 - 2\sqrt{3}}{-2} \stackrel{\text{factor}}{=} \frac{-2(-2 + \sqrt{3})}{-2} = \sqrt{3} - 2$

b. $\frac{\sqrt{xy}}{2\sqrt{x} - \sqrt{y}} \cdot \frac{(2\sqrt{x} + \sqrt{y})}{(2\sqrt{x} + \sqrt{y})} = \frac{2x\sqrt{y} + y\sqrt{x}}{4x - y}$

Some of the challenges in algebraic manipulations involve simplifying quotients with radical expressions, such as $\frac{4 - 2\sqrt{3}}{-2}$, which appeared in the solution to *Example 4a*. The key concept that allows us to simplify such expressions is **factoring**, as only common factors can be reduced.

Example 5 ▶ Writing Quotients with Radicals in Lowest Terms

Write each quotient in lowest terms.

a. $\frac{15 - 6\sqrt{5}}{6}$

b. $\frac{3x + \sqrt{8x^2}}{9x}$

Solution ▶

a. To reduce this quotient to the lowest terms we may factor the numerator first,

$$\frac{15 - 6\sqrt{5}}{6} = \frac{\cancel{3}(5 - 2\sqrt{5})}{\cancel{6}_2} = \frac{5 - 2\sqrt{5}}{2}$$

or alternatively, rewrite the quotient into two fractions and then simplify,

$$\frac{15 - 6\sqrt{5}}{6} = \frac{15}{6} - \frac{6\sqrt{5}}{6} = \frac{5}{2} - \sqrt{5}.$$

Caution: Here are the common errors to avoid:

$$\frac{15-6\sqrt{5}}{6} = 15 - \sqrt{5} \quad \text{- only common factors can be reduced!}$$

$$\frac{15-6\sqrt{5}}{6} = \frac{9\sqrt{5}}{6} = \frac{3\sqrt{5}}{2} \quad \text{- subtraction is performed after multiplication!}$$

- b. To reduce this quotient to the lowest terms, we simplify the radical and factor the numerator first. So,

$$\frac{3x + \sqrt{8x^2}}{6x} = \frac{3x + 2x\sqrt{2}}{6x} = \frac{\cancel{x}(3 + 2\sqrt{2})}{6\cancel{x}} = \frac{3 + 2\sqrt{2}}{6}$$

This expression cannot be simplified any further.

RD.4 Exercises

Vocabulary Check Complete each blank with the most appropriate term or phrase from the given list: *coefficients, denominator, distributive, factoring, like, rationalize, simplified.*

- Radicals that have the same index and the same radicand are called _____ radicals.
- Some unlike radicals can become like radicals after they are _____.
- Like radicals can be added by combining their _____ and then multiplying this sum by the common radical.
- To multiply radical expressions containing more than one term, we use the _____ property.
- To _____ the denominator means to rewrite an expression that has a radical in its denominator as an equivalent expression that does not have one.
- To rationalize a denominator with two terms involving square roots, multiply both the numerator and the denominator by the conjugate of the _____.
- The key strategy for simplifying quotients such as $\frac{a+b\sqrt{c}}{d}$ is _____ the numerator, if possible.

Concept Check

8. A student claims that $24 - 4\sqrt{x} = 20\sqrt{x}$ because for $x = 1$ both sides of the equation equal to 20. Is this a valid justification? Explain.
9. For most values of a and b , $\sqrt{a} + \sqrt{b} \neq \sqrt{a+b}$. Are there any real numbers a and b such that $\sqrt{a} + \sqrt{b} = \sqrt{a+b}$? Justify your answer.

Perform operations and simplify, if possible. Assume that all variables represent positive real numbers.

- | | | |
|---|---|---|
| 10. $2\sqrt{3} + 5\sqrt{3}$ | 11. $6\sqrt[3]{x} - 4\sqrt[3]{x}$ | 12. $9y\sqrt{3x} + 4y\sqrt{3x}$ |
| 13. $12a\sqrt{5b} - 4a\sqrt{5b}$ | 14. $5\sqrt{32} - 3\sqrt{8} + 2\sqrt{3}$ | 15. $-2\sqrt{48} + 4\sqrt{75} - \sqrt{5}$ |
| 16. $\sqrt[3]{16} + 3\sqrt[3]{54}$ | 17. $\sqrt[4]{32} - 3\sqrt[4]{2}$ | 18. $\sqrt{5a} + 2\sqrt{45a^3}$ |
| 19. $\sqrt[3]{24x} - \sqrt[3]{3x^4}$ | 20. $4\sqrt{x^3} - 2\sqrt{9x}$ | 21. $7\sqrt{27x^3} + \sqrt{3x}$ |
| 22. $6\sqrt{18x} - \sqrt{32x} + 2\sqrt{50x}$ | 23. $2\sqrt{128a} - \sqrt{98a} + 2\sqrt{72a}$ | |
| 24. $\sqrt[3]{6x^4} + \sqrt[3]{48x} - \sqrt[3]{6x}$ | 25. $9\sqrt{27y^2} - 14\sqrt{108y^2} + 2\sqrt{48y^2}$ | |
| 26. $3\sqrt{98n^2} - 5\sqrt{32n^2} - 3\sqrt{18n^2}$ | 27. $-4y\sqrt{xy^3} + 7x\sqrt{x^3y}$ | |
| 28. $6a\sqrt{ab^5} - 9b\sqrt{a^3b}$ | 29. $\sqrt[3]{-125p^9} + p\sqrt[3]{-8p^6}$ | |
| 30. $3\sqrt[4]{x^5y} + 2x\sqrt[4]{xy}$ | 31. $\sqrt{125a^5} - 2\sqrt[3]{125a^4}$ | 32. $x\sqrt[3]{16x} + \sqrt{2} - \sqrt[3]{2x^4}$ |
| 33. $\sqrt{9a-9} + \sqrt{a-1}$ | 34. $\sqrt{4x+12} - \sqrt{x+3}$ | 35. $\sqrt{x^3-x^2} - \sqrt{4x-4}$ |
| 36. $\sqrt{25x-25} - \sqrt{x^3-x^2}$ | 37. $\frac{4\sqrt{3}}{3} - \frac{2\sqrt{3}}{9}$ | 38. $\frac{\sqrt{27}}{2} - \frac{3\sqrt{3}}{4}$ |
| 39. $\sqrt{\frac{49}{x^4}} + \sqrt{\frac{81}{x^8}}$ | 40. $2a\sqrt[4]{\frac{a}{16}} - 5a\sqrt[4]{\frac{a}{81}}$ | 41. $-4\sqrt[3]{\frac{4}{y^9}} + 3\sqrt[3]{\frac{9}{y^{12}}}$ |

Discussion Point

42. A student simplifies an expression *incorrectly*:

$$\begin{aligned} \sqrt{8} + \sqrt[3]{16} &\stackrel{?}{=} \sqrt{4 \cdot 2} + \sqrt[3]{8 \cdot 2} \\ &\stackrel{?}{=} \sqrt{4} \cdot \sqrt{2} + \sqrt[3]{8} \cdot \sqrt[3]{2} \\ &\stackrel{?}{=} 2\sqrt{2} + 2\sqrt[3]{2} \\ &\stackrel{?}{=} 4\sqrt{4} \\ &\stackrel{?}{=} 8 \end{aligned}$$

Explain any errors that the student made. What would you do differently?

Concept Check

43. Match each expression from **Column I** with the equivalent expression in **Column II**. Assume that A and B represent positive real numbers.

Column I

A. $(A + \sqrt{B})(A - \sqrt{B})$

B. $(\sqrt{A} + B)(\sqrt{A} - B)$

C. $(\sqrt{A} + \sqrt{B})(\sqrt{A} - \sqrt{B})$

D. $(\sqrt{A} + \sqrt{B})^2$

E. $(\sqrt{A} - \sqrt{B})^2$

F. $(\sqrt{A} + B)^2$

Column II

a. $A - B$

b. $A + 2B\sqrt{A} + B^2$

c. $A - B^2$

d. $A - 2\sqrt{AB} + B$

e. $A^2 - B$

f. $A + 2\sqrt{AB} + B$

Multiply, and then simplify each product. Assume that all variables represent positive real numbers.

44. $\sqrt{5}(3 - 2\sqrt{5})$

45. $\sqrt{3}(3\sqrt{3} - \sqrt{2})$

46. $\sqrt{2}(5\sqrt{2} - \sqrt{10})$

47. $\sqrt{3}(-4\sqrt{3} + \sqrt{6})$

48. $\sqrt[3]{2}(\sqrt[3]{4} - 2\sqrt[3]{32})$

49. $\sqrt[3]{3}(\sqrt[3]{9} + 2\sqrt[3]{21})$

50. $(\sqrt{3} - \sqrt{2})(\sqrt{3} + \sqrt{2})$

51. $(\sqrt{5} + \sqrt{7})(\sqrt{5} - \sqrt{7})$

52. $(2\sqrt{3} + 5)(2\sqrt{3} - 5)$

53. $(6 + 3\sqrt{2})(6 - 3\sqrt{2})$

54. $(5 - \sqrt{5})^2$

55. $(\sqrt{2} + 3)^2$

56. $(\sqrt{a} + 5\sqrt{b})(\sqrt{a} - 5\sqrt{b})$

57. $(2\sqrt{x} - 3\sqrt{y})(2\sqrt{x} + 3\sqrt{y})$

58. $(\sqrt{3} + \sqrt{6})^2$

59. $(\sqrt{5} - \sqrt{10})^2$

60. $(2\sqrt{5} + 3\sqrt{2})^2$

61. $(2\sqrt{3} - 5\sqrt{2})^2$

62. $(4\sqrt{3} - 5)(\sqrt{3} - 2)$

63. $(4\sqrt{5} + 3\sqrt{3})(3\sqrt{5} - 2\sqrt{3})$

64. $(\sqrt[3]{2y} - 5)(\sqrt[3]{2y} + 1)$

65. $(\sqrt{x+5} - 3)(\sqrt{x+5} + 3)$

66. $(\sqrt{x+1} - \sqrt{x})(\sqrt{x+1} + \sqrt{x})$

67. $(\sqrt{x+2} + \sqrt{x-2})^2$

Concept Check

Given $f(x)$ and $g(x)$, find $(f + g)(x)$ and $(fg)(x)$.

68. $f(x) = 5x\sqrt{20x}$ and $g(x) = 3\sqrt{5x^3}$

69. $f(x) = 2x^4\sqrt{64x}$ and $g(x) = -3^4\sqrt{4x^5}$

Rationalize each denominator and simplify, if possible. Assume that all variables represent positive real numbers.

70. $\frac{\sqrt{5}}{2\sqrt{2}}$

71. $\frac{3}{5\sqrt{3}}$

72. $\frac{12}{\sqrt{6}}$

73. $-\frac{15}{\sqrt{24}}$

74. $-\frac{10}{\sqrt{20}}$

75. $\sqrt{\frac{3x}{20}}$

76. $\sqrt{\frac{5y}{32}}$

77. $\frac{\sqrt[3]{7a}}{\sqrt[3]{3b}}$

78. $\frac{\sqrt[3]{2y^4}}{\sqrt[3]{6x^4}}$

79. $\frac{\sqrt[3]{3n^4}}{\sqrt[3]{5m^2}}$

80. $\frac{pq}{\sqrt[4]{p^3q}}$

81. $\frac{2x}{\sqrt[5]{18x^8}}$

82. $\frac{17}{6+\sqrt{2}}$

83. $\frac{4}{3-\sqrt{5}}$

84. $\frac{2\sqrt{3}}{\sqrt{3}-\sqrt{2}}$

85. $\frac{6\sqrt{3}}{3\sqrt{2}-\sqrt{3}}$

86. $\frac{3}{3\sqrt{5}+2\sqrt{3}}$

87. $\frac{\sqrt{2}+\sqrt{3}}{\sqrt{3}+5\sqrt{2}}$

88. $\frac{m-4}{\sqrt{m}+2}$

89. $\frac{4}{\sqrt{x}-2\sqrt{y}}$

90. $\frac{\sqrt{3}+2\sqrt{x}}{\sqrt{3}-2\sqrt{x}}$

91. $\frac{\sqrt{x}-2}{3\sqrt{x}+\sqrt{y}}$

92. $\frac{2\sqrt{a}}{\sqrt{a}-\sqrt{b}}$

93. $\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$

Write each quotient in lowest terms. Assume that all variables represent positive real numbers.

94. $\frac{10-20\sqrt{5}}{10}$

95. $\frac{12+6\sqrt{3}}{6}$

96. $\frac{12-9\sqrt{72}}{18}$

97. $\frac{2x+\sqrt{8x^2}}{2x}$

98. $\frac{6p-\sqrt{24p^3}}{3p}$

99. $\frac{9x+\sqrt{18}}{15}$

Discussion Point

100. In a certain problem in trigonometry, a student obtained the answer $\frac{\sqrt{3}+1}{1-\sqrt{3}}$. The textbook answer to this problem was $-2 - \sqrt{3}$. Was the student's answer equivalent to the textbook answer?

Analytic Skills Solve each problem.

101. The Great Pyramid at Giza has a square base with an area of 52,900 m². What is the perimeter of its base?
102. The areas of two types of square floor tiles sold at a home improvement store are shown. How much longer is the side of the larger tile? Express the answer as a radical in simplest form and as a decimal to the nearest tenth.

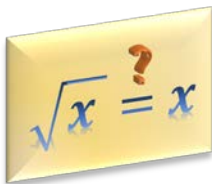


Area =
72 cm²

Area =
128 cm²

RD.5

Radical Equations



In this section, we discuss techniques for solving radical equations. These are equations containing at least one radical expression with a variable, such as $\sqrt{3x-2} = x$, or a variable expression raised to a fractional exponent, such as $(2x)^{\frac{1}{3}} + 1 = 5$.

At the end of this section, we revisit working with formulas involving radicals as well as application problems that can be solved with the use of radical equations.

Radical Equations

Definition 5.1 ▶ A **radical equation** is an equation in which a variable appears in one or more radicands. This includes radicands ‘hidden’ under fractional exponents.

For example, since $(x-1)^{\frac{1}{2}} = \sqrt{x-1}$, then the base $x-1$ is, in fact, the ‘hidden’ radicand.

Some examples of radical equations are

$$x = \sqrt{2x}, \quad \sqrt{x} + \sqrt{x-2} = 5, \quad (x-4)^{\frac{3}{2}} = 8, \quad \sqrt[3]{3+x} = 5$$

Note that $x = \sqrt{2}$ is not a radical equation since there is no variable under the radical sign.

The process of solving radical equations involves clearing radicals by raising both sides of an equation to an appropriate power. This method is based on the following property of equality.

Power Rule:

For any **odd** natural number n , the equation $a = b$ is equivalent to the equation $a^n = b^n$.

For any **even** natural number n , if an equation $a = b$ is true, then $a^n = b^n$ is true.

When rephrased, the power rule for odd powers states that the solution sets to both equations, $a = b$ and $a^n = b^n$, are exactly the same.

However, the power rule for even powers states that the solutions to the original equation $a = b$ are among the solutions to the ‘power’ equation $a^n = b^n$.

Unfortunately, the reverse implication does not hold for even numbers n . We cannot conclude that $a = b$ from the fact that $a^n = b^n$ is true. For instance, $3^2 = (-3)^2$ is true but $3 \neq -3$. This means that not all solutions of the equation $a^n = b^n$ are in fact true solutions to the original equation $a = b$. Solutions that do not satisfy the original equation are called **extraneous solutions** or **extraneous roots**. Such solutions must be rejected.

For example, to solve $\sqrt{2-x} = x$, we may square both sides of the equation to obtain the quadratic equation

$$2 - x = x^2.$$

Then, we solve it via factoring and the zero-product property:

$$x^2 + x - 2 = 0$$

$$(x + 2)(x - 1) = 0$$

So, the possible solutions are $x = -2$ and $x = 1$.

Notice that $x = 1$ satisfies the original equation, as $\sqrt{2 - 1} = 1$ is true. However, $x = -2$ does not satisfy the original equation as its left side equals to $\sqrt{2 - (-2)} = \sqrt{4} = 2$, while the right side equals to -2 . Thus, $x = -2$ is the extraneous root and as such, it does not belong to the solution set of the original equation. So, the solution set of the original equation is $\{1\}$.

Caution: When the power rule for **even powers** is used to solve an equation, **every solution** of the ‘power’ equation **must be checked in the original equation**.

Example 1 ▶ Solving Equations with One Radical

Solve each equation.

a. $\sqrt{3x + 4} = 4$

b. $\sqrt{2x - 5} + 4 = 0$

c. $2\sqrt{x + 1} = x - 7$

d. $\sqrt[3]{x - 8} + 2 = 0$

Solution ▶

- a. Since the radical in $\sqrt{3x + 4} = 4$ is isolated on one side of the equation, squaring both sides of the equation allows for clearing (reversing) the square root. Then, by solving the resulting polynomial equation, one can find the possible solution(s) to the original equation.

$(\sqrt{a})^2 = (a^{\frac{1}{2}})^2 = a$

$(\sqrt{3x + 4})^2 = (4)^2$
 $3x + 4 = 16$
 $3x = 12$
 $x = 4$

To check if 4 is a true solution, it is enough to check whether or not $x = 4$ satisfies the original equation.

$$\begin{aligned} \sqrt{3 \cdot 4 + 4} &\stackrel{?}{=} 4 \\ \sqrt{16} &\stackrel{?}{=} 4 \\ 4 &= 4 \quad \checkmark \dots \text{true} \end{aligned}$$

Since $x = 4$ satisfies the original equation, the solution set is $\{4\}$.

- b. To solve $\sqrt{2x - 5} + 4 = 0$, it is useful to isolate the radical on one side of the equation. So, consider the equation

$$\sqrt{2x - 5} = -4$$

Notice that the left side of the above equation is nonnegative for any x -value while the right side is constantly negative. Thus, such an equation cannot be satisfied by any x -value. Therefore, this equation has **no solution**.

- c. Squaring both sides of the equation gives us

$$(2\sqrt{x+1})^2 = (x-7)^2$$

the bracket is essential here

$$4(x+1) = x^2 - 14x + 49$$

$$4x + 4 = x^2 - 14x + 49$$

$$x^2 - 18x + 45 = 0$$

$$(x-3)(x-15) = 0$$

apply the perfect square formula
 $(a-b)^2 = a^2 - 2ab + b^2$

So, the possible solutions are $x = 3$ or $x = 15$. We check each of them by substituting them into the original equation.

If $x = 3$, then

$$2\sqrt{3+1} \stackrel{?}{=} 3-7$$

$$2\sqrt{4} \stackrel{?}{=} -4$$

$$4 \neq -4 \quad \times \dots \text{false}$$

So $x = 3$ is the extraneous root.

If $x = 15$, then

$$2\sqrt{15+1} \stackrel{?}{=} 15-7$$

$$2\sqrt{16} \stackrel{?}{=} 8$$

$$8 = 8 \quad \checkmark \dots \text{true}$$

Since only 15 satisfies the original equation, the solution set is $\{15\}$.

- d. To solve $\sqrt[3]{x-8} + 2 = 0$, we first isolate the radical by subtracting 2 from both sides of the equation.

$$\sqrt[3]{x-8} = -2$$

Then, to clear the cube root, we raise both sides of the equation to the third power.

$$(\sqrt[3]{x-8})^3 = (-2)^3$$

So, we obtain

$$x-8 = -8$$

$$x = 0$$

Since we applied the power rule for odd powers, the obtained solution is the true solution. So the solution set is $\{0\}$.

Observation: When using the power rule for odd powers checking the obtained solutions against the original equation is not necessary. This is because there is no risk of obtaining extraneous roots when applying the power rule for odd powers.

To solve radical equations with more than one radical term, we might need to apply the power rule repeatedly until all radicals are cleared. In an efficient solution, each application

of the power rule should cause clearing of at least one radical term. For that reason, it is a good idea to isolate a single radical term on one side of the equation before each application of the power rule. For example, to solve the equation

$$\sqrt{x-3} + \sqrt{x+5} = 4,$$

we isolate one of the radicals before squaring both sides of the equation. So, we have

$$(\sqrt{x-3})^2 = (4 - \sqrt{x+5})^2$$

$$x - 3 = \underbrace{16}_{a^2} - \underbrace{8\sqrt{x+5}}_{2ab} + \underbrace{x+5}_{b^2}$$

Remember that the perfect square formula consists of **three terms**.

Then, we isolate the remaining radical term and simplify, if possible. This gives us

$$8\sqrt{x+5} = 24 \quad / \div 8$$

$$\sqrt{x+5} = 3 \quad / ()^2$$

Squaring both sides of the last equation gives us

$$x + 5 = 9 \quad / -5$$

$$x = 4$$

The reader is encouraged to check that $x = 4$ is the true solution to the original equation.

A general strategy for solving radical equations, including those with two radical terms, is as follows.

Summary of Solving a Radical Equation

1. **Isolate one of the radical terms.** Make sure that one radical term is alone on one side of the equation.
2. **Apply an appropriate power rule.** Raise each side of the equation to a power that is the same as the index of the isolated radical.
3. **Solve the resulting equation.** If it still contains a radical, repeat steps 1 and 2.
4. **Check** all proposed solutions in the original equation.
5. **State the solution set** to the original equation.

Example 2 ▶ Solving Equations Containing Two Radical Terms

Solve each equation.

a. $\sqrt{3x+1} - \sqrt{x+4} = 1$

b. $\sqrt[3]{4x-5} = 2\sqrt{x+1}$

Solution

- a.** We start solving the equation $\sqrt{3x+1} - \sqrt{x+4} = 1$ by isolating one radical on one side of the equation. This can be done by adding $\sqrt{x+4}$ to both sides of the equation. So, we have

$$\sqrt{3x+1} = 1 + \sqrt{x+4}$$

which after squaring give us

$$(\sqrt{3x+1})^2 = (1 + \sqrt{x+4})^2$$

$$3x + 1 = 1 + 2\sqrt{x+4} + x + 4$$

$$2x - 4 = 2\sqrt{x+4} \quad / \div 2$$

$$x - 2 = \sqrt{x+4}$$

To clear the remaining radical, we square both sides of the above equation again.

$$(x - 2)^2 = (\sqrt{x+4})^2$$

$$x^2 - 4x + 4 = x + 4$$

$$x^2 - 5x = 0$$

The resulting polynomial equation can be solved by factoring and applying the zero-product property. Thus,

$$x(x - 5) = 0.$$

So, the possible roots are $x = 0$ or $x = 5$.

We check each of them by substituting to the original equation.

If $x = 0$, then

$$\sqrt{3 \cdot 0 + 1} - \sqrt{0 + 4} \stackrel{?}{=} 1$$

$$\sqrt{1} - \sqrt{4} \stackrel{?}{=} 1$$

$$1 - 2 \stackrel{?}{=} 1$$

$$-1 \neq 1 \quad \times \dots \text{false}$$

If $x = 5$, then

$$\sqrt{3 \cdot 5 + 1} - \sqrt{5 + 4} \stackrel{?}{=} 1$$

$$\sqrt{16} - \sqrt{9} \stackrel{?}{=} 1$$

$$4 - 3 \stackrel{?}{=} 1$$

$$1 = 1 \quad \checkmark \dots \text{true}$$

Since $x = 0$ is the **extraneous** root, it does not belong to the solution set.

Only 5 satisfies the original equation. So, the solution set is $\{5\}$.

- b.** To solve the equation $\sqrt[3]{4x-5} = 2\sqrt[3]{x+1}$, we would like to clear the cubic roots. This can be done by cubing both of its sides, as shown below.

$$(\sqrt[3]{4x-5})^3 = (2\sqrt[3]{x+1})^3$$

$$4x - 5 = 2^3(x + 1)$$

the bracket is essential here

$$4x - 5 = 8x + 8$$

$$/ -4x, -8$$

$$-13 = 4x$$

$$/ \div (-13)$$

$$x = -\frac{4}{13}$$

Since we applied the power rule for cubes, the obtained root is the true solution of the original equation.

Formulas Containing Radicals



Many formulas involve radicals. For example, the period T , in seconds, of a pendulum of length L , in feet, is given by the formula

$$T = 2\pi \sqrt{\frac{L}{32}}$$

Sometimes, we might need to solve a radical formula for a specified variable. In addition to all the strategies for solving formulas for a variable, discussed in *sections L2, F4, and RT6*, we may need to apply the power rule to clear the radical(s) in the formula.

Example 3 ▶ Solving Radical Formulas for a Specified Variable

Solve each formula for the indicated variable.

a. $N = \frac{1}{2\pi} \sqrt{\frac{a}{r}}$ for a

b. $r = \sqrt[3]{\frac{A}{P}} - 1$ for P

Solution ▶

- a. Since a appears in the radicand, to solve $N = \frac{1}{2\pi} \sqrt{\frac{a}{r}}$ for a , we may want to clear the radical by squaring both sides of the equation. So, we have

$$\begin{aligned} N^2 &= \left(\frac{1}{2\pi} \sqrt{\frac{a}{r}} \right)^2 \\ N^2 &= \frac{1}{(2\pi)^2} \cdot \frac{a}{r} && / \cdot 4\pi^2 r \\ 4\pi^2 N^2 r &= a \end{aligned}$$

Note: We could also first multiply by 2π and then square both sides of the equation.

- b. First, observe the position of P in the equation $r = \sqrt[3]{\frac{A}{P}} - 1$. It appears in the denominator of the radical. Therefore, to solve for P , we may plan to isolate the cube root first, cube both sides of the equation to clear the radical, and finally bring P to the numerator. So, we have

$$r = \sqrt[3]{\frac{A}{P}} - 1 \quad / +1$$

$$(r + 1)^3 = \left(\sqrt[3]{\frac{A}{P}} \right)^3$$

$$(r + 1)^3 = \frac{A}{P} \quad / \cdot P, \div (r + 1)^3$$

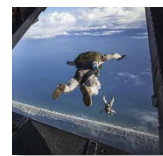
$$P = \frac{A}{(r + 1)^3}$$

Radicals in Applications

Many application problems in sciences, engineering, or finances translate into radical equations.

Example 4 ▶ Finding the Velocity of a Skydiver

When skydivers initially fall from an airplane, their velocity v in kilometers per hour after free falling d meters can be approximated by $v = 15.9\sqrt{d}$. Approximately how far, in meters, do skydivers need to fall to attain 100 kph?



Solution ▶ We may substitute $v = 100$ into the equation $v = 15.9\sqrt{d}$ and solve it for d . Thus,

$$100 = 15.9\sqrt{d} \quad / \div 15.9$$

$$6.3 \approx \sqrt{d} \quad / \text{square both sides}$$

$$40 \approx d$$

So, skydivers fall at 100 kph approximately after 40 meters of free falling.

RD.5 Exercises

Vocabulary Check Complete each blank with the most appropriate term or phrase from the given list: *even, extraneous, original, power, radical, solution*.

1. A _____ equation contains at least one radical with a variable.

2. Radical equations are solved by applying an appropriate _____ rule to both sides of the equation.
3. The roots obtained in the process of solving a radical equation that do not satisfy the original equation are called _____ roots. They do not belong to the _____ set of the original equation.
4. If a power rule for _____ powers is applied, every possible solution must be checked in the _____ equation.

Concept Check True or false.

5. $\sqrt{2}x = x^2 - \sqrt{5}$ is a radical equation.
6. When raising each side of a radical equation to a power, the resulting equation is equivalent to the original equation.
7. $\sqrt{3x + 9} = x$ cannot have negative solutions.
8. -9 is a solution to the equation $\sqrt{x} = -3$.

Solve each equation.

- | | | | |
|-----------------------------|--------------------------------|-------------------------------|------------------------------|
| 9. $\sqrt{7x - 3} = 6$ | 10. $\sqrt{5y + 2} = 7$ | 11. $\sqrt{6x} + 1 = 3$ | 12. $\sqrt{2k} - 4 = 6$ |
| 13. $\sqrt{x + 2} = -6$ | 14. $\sqrt{y - 3} = -2$ | 15. $\sqrt[3]{x} = -3$ | 16. $\sqrt[3]{a} = -1$ |
| 17. $\sqrt[4]{y - 3} = 2$ | 18. $\sqrt[4]{n + 1} = 3$ | 19. $5 = \frac{1}{\sqrt{a}}$ | 20. $\frac{1}{\sqrt{y}} = 3$ |
| 21. $\sqrt{3r + 1} - 4 = 0$ | 22. $\sqrt{5x - 4} - 9 = 0$ | 23. $4 - \sqrt{y - 2} = 0$ | |
| 24. $9 - \sqrt{4a + 1} = 0$ | 25. $x - 7 = \sqrt{x - 5}$ | 26. $x + 2 = \sqrt{2x + 7}$ | |
| 27. $2\sqrt{x + 1} - 1 = x$ | 28. $3\sqrt{x - 1} - 1 = x$ | 29. $y - 4 = \sqrt{4 - y}$ | |
| 30. $x + 3 = \sqrt{9 - x}$ | 31. $x = \sqrt{x^2 + 4x - 20}$ | 32. $x = \sqrt{x^2 + 3x + 9}$ | |

Concept Check

33. When solving the equation $\sqrt{3x + 4} = 8 - x$, a student wrote the following for the first step.

$$3x + 4 = 64 + x^2$$

Is this correct? Justify your answer.

34. When solving the equation $\sqrt{5x + 6} - \sqrt{x + 3} = 3$, a student wrote the following for the first step.

$$(5x + 6) + (x + 3) = 9$$

Is this correct? Justify your answer.

Solve each equation.

35. $\sqrt{5x+1} = \sqrt{2x+7}$

36. $\sqrt{5y-3} = \sqrt{2y+3}$

37. $\sqrt[3]{p+5} = \sqrt[3]{2p-4}$

38. $\sqrt[3]{x^2+5x+1} = \sqrt[3]{x^2+4x}$

39. $2\sqrt{x-3} = \sqrt{7x+15}$

40. $\sqrt{6x-11} = 3\sqrt{x-7}$

41. $3\sqrt{2t+3} - \sqrt{t+10} = 0$

42. $2\sqrt{y-1} - \sqrt{3y-1} = 0$

43. $\sqrt{x-9} + \sqrt{x} = 1$

44. $\sqrt{y-5} + \sqrt{y} = 5$

45. $\sqrt{3n} + \sqrt{n-2} = 4$

46. $\sqrt{x+5} - 2 = \sqrt{x-1}$

47. $\sqrt{14-n} = \sqrt{n+3} + 3$

48. $\sqrt{p+15} - \sqrt{2p+7} = 1$

49. $\sqrt{4a+1} - \sqrt{a-2} = 3$

50. $4 - \sqrt{a+6} = \sqrt{a-2}$

51. $\sqrt{x-5} + 1 = -\sqrt{x+3}$

52. $\sqrt{3x-5} + \sqrt{2x+3} + 1 = 0$

53. $\sqrt{2m-3} + 2 - \sqrt{m+7} = 0$

54. $\sqrt{x+2} + \sqrt{3x+4} = 2$

55. $\sqrt{6x+7} - \sqrt{3x+3} = 1$

56. $\sqrt{4x+7} - 4 = \sqrt{4x-1}$

57. $\sqrt{5y+4} - 3 = \sqrt{2y-2}$

58. $\sqrt{2\sqrt{x+11}} = \sqrt{4x+2}$

59. $\sqrt{1 + \sqrt{24 + 10x}} = \sqrt{3x+5}$

60. $(2x-9)^{\frac{1}{2}} = 2 + (x-8)^{\frac{1}{2}}$

61. $(3k+7)^{\frac{1}{2}} = 1 + (k+2)^{\frac{1}{2}}$

62. $(x+1)^{\frac{1}{2}} - (x-6)^{\frac{1}{2}} = 1$

63. $\sqrt{(x^2-9)^{\frac{1}{2}}} = 2$

64. $\sqrt{\sqrt{x}+4} = \sqrt{x}-2$

65. $\sqrt{a^2+30a} = a + \sqrt{5a}$

Discussion Point

66. Can the expression $\sqrt{7+4\sqrt{3}} - \sqrt{7-4\sqrt{3}}$ be evaluated without the use of a calculator?

Solve each formula for the indicated variable.

67. $Z = \sqrt{\frac{L}{C}}$ for L

68. $V = \sqrt{\frac{2K}{m}}$ for K

69. $V = \sqrt{\frac{2K}{m}}$ for m

70. $r = \sqrt{\frac{Mm}{F}}$ for M

71. $r = \sqrt{\frac{Mm}{F}}$ for F

72. $Z = \sqrt{L^2 + R^2}$ for R

73. $F = \frac{1}{2\pi\sqrt{LC}}$ for C

74. $N = \frac{1}{2\pi}\sqrt{\frac{a}{r}}$ for a

75. $N = \frac{1}{2\pi}\sqrt{\frac{a}{r}}$ for r

Analytic Skills Solve each problem.

76. According to Einstein's theory of relativity, time passes more quickly for bodies that travel very close to the speed of light. The aging rate compared to the time spent on earth is given by the formula $r = \frac{\sqrt{c^2 - v^2}}{c^2}$, where c is the speed of light, and v is the speed of the traveling body. For example, the aging rate of 0.5 means that one year for the person travelling at the speed v corresponds to two years spent on earth.

- Find the aging rate for a person traveling at 90% of the speed of light.
- Find the elapsed time on earth for one year of travelling time at 90% of the speed of light.



77. Before determining the dosage of a drug for a patient, doctors will sometimes calculate the patient's **Body Surface Area** (or BSA). One way to determine a person's BSA, in square meters, is to use the formula



$BSA = \sqrt{\frac{wh}{3600}}$, where w is the weight, in pounds, and h is the height, in centimeters, of the patient. Jason weighs 160 pounds and has a BSA of about $2\sqrt{2}$ m². How tall is he? Round the answer to the nearest centimeter.

78. The distance, d , to the horizon for an object h miles above the Earth's surface is given by the equation $d = \sqrt{8000h + h^2}$. How many miles above the Earth's surface is a satellite if the distance to the horizon is 900 miles?
79. To calculate the minimum speed S , in miles per hour, that a car was traveling before skidding to a stop, traffic accident investigators use the formula $S = \sqrt{30fL}$, where f is the drag factor of the road surface and L is the length of a skid mark, in feet. Calculate the length of the skid marks for a car traveling at a speed of 30 mph that skids to a stop on a road surface with a drag factor of 0.5.

