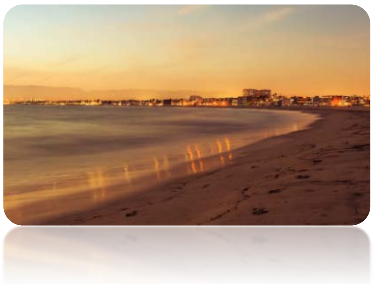
Rational Expressions and Functions



In the previous two chapters we discussed algebraic expressions, equations, and functions related to polynomials. In this chapter, we will examine a broader category of algebraic expressions, *rational expressions*, also referred to as *algebraic fractions*. Similarly as in arithmetic, where a rational number is a quotient of two integers with a denominator that is different than zero, a rational expression is a quotient of two polynomials, also with a denominator that is different than zero.

We start by introducing the related topic of integral exponents, including scientific notation. Then, we discuss operations on algebraic fractions, solving rational equations, and properties and graphs of rational functions with an emphasis on such features as domain, range, and asymptotes. At the end of this chapter, we show examples of applied problems, including work problems, that require solving rational equations.

RT.1 Integral Exponents and Scientific Notation

Integral Exponents

In *section P.2,* we discussed the following power rules, using whole numbers for the exponents.

product rule	$\boldsymbol{a^m \cdot a^n = a^{m+n}}$	$\boldsymbol{(ab)^n = a^n b^n}$
quotient rule	$\boldsymbol{\frac{a^m}{a^n} = a^{m-n}}$	$\boldsymbol{\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}}$
power rule	$\boldsymbol{(a^m)^n = a^{mn}}$	$\boldsymbol{a^0 = 1}$ for $\boldsymbol{a \neq 0}$ $\boldsymbol{0^0}$ is undefined

Observe that these rules gives us the following result.

$$\boldsymbol{a^{-1}} = a^{n-(n+1)} \underset{\text{quotient rule}}{=} \frac{a^n}{a^{n+1}} \underset{\text{product rule}}{=} \frac{\cancel{a^n}}{\cancel{a^n}\cdot a} = \frac{\boldsymbol{1}}{\boldsymbol{a}}$$

Consequantly, $\boldsymbol{a^{-n}} \underset{\text{power rule}}{=} (a^n)^{-1} = \frac{\boldsymbol{1}}{\boldsymbol{a^n}}$.

Since $\boxed{\boldsymbol{a^{-n} = \frac{1}{a^n}}}$, then the expression a^n is meaningful for any integral exponent n and a nonzero real base a. So, the above rules of exponents can be extended to include integral exponents.

In practice, to work out the negative sign of an exponent, take the **reciprocal of the base**, or equivalently, "**change the level**" of the power. For example,

$$3^{-2} = \left(\frac{1}{3}\right)^2 = \frac{1^2}{3^2} = \frac{1}{9} \quad \text{and} \quad \frac{2^{-3}}{3^{-1}} = \frac{3^1}{2^3} = \frac{3}{8}.$$

Attention! Exponent refers to the immediate number, letter, or expression in a bracket. For example,

$$\boldsymbol{x^{-2} = \frac{1}{x^2}}, \quad (-\boldsymbol{x})^{-2} = \frac{1}{(-x)^2} = \boldsymbol{\frac{1}{x^2}}, \quad but \quad \boldsymbol{-x^{-2} = -\frac{1}{x^2}}.$$

Example 1 ► **Evaluating Expressions with Integral Exponents**

Evaluate each expression.

a. $3^{-1} + 2^{-1}$ **b.** $\frac{5^{-2}}{2^{-5}}$

c. $\frac{-2^2}{2^{-7}}$ **d.** $\frac{-2^{-2}}{3 \cdot 2^{-3}}$

Solution ►

a. $3^{-1} + 2^{-1} = \frac{1}{3} + \frac{1}{2} = \frac{2}{6} + \frac{3}{6} = \boldsymbol{\frac{5}{6}}$

Caution! $3^{-1} + 2^{-1} \neq (3+2)^{-1}$, because the value of $3^{-1} + 2^{-1}$ is $\frac{5}{6}$, as shown in the example, while the value of $(3+2)^{-1}$ is $\frac{1}{5}$.

b. $\frac{5^{-2}}{2^{-5}} = \frac{2^5}{5^2} = \boldsymbol{\frac{32}{25}}$

Note: To work out the negative exponent, move the power from the numerator to the denominator or vice versa.

c. $\frac{-2^2}{2^{-7}} = -2^2 \cdot 2^7 = \boldsymbol{-2^9}$

Attention! The role of a negative sign in front of a base number or in front of an exponent is different. To work out the negative in 2^{-7}, we either take the reciprocal of the base, or we change the position of the power to a different level in the fraction. So, $2^{-7} = \left(\frac{1}{2}\right)^7$ or $2^{-7} = \frac{1}{2^7}$. However, the negative sign in -2^2 just means that the number is negative. So, $-2^2 = -4$. ***Caution!*** $-2^2 \neq \frac{1}{4}$

d. $\frac{-2^{-2}}{3 \cdot 2^{-3}} = \frac{-2^{\cancel{3}^{\,1}}}{3 \cdot 2^{\cancel{2}}} = \boldsymbol{-\frac{2}{3}}$

Note: Exponential expressions can be simplified in many ways. For example, to simplify $\frac{2^{-2}}{2^{-3}}$, we can work out the negative exponents first by moving the powers to a different level, $\frac{2^3}{2^2}$, and then reduce the common factors as shown in the example; or we can employ the quotient rule of powers to obtain

$$\frac{2^{-2}}{2^{-3}} = 2^{-2-(-3)} = 2^{-2+3} = 2^1 = 2.$$

Example 2 ► **Simplifying Exponential Expressions Involving Negative Exponents**

Simplify the given expression. Leave the answer with only positive exponents.

a. $4x^{-5}$ **b.** $(x+y)^{-1}$

c. $x^{-1}+y^{-1}$ **d.** $(-2^3x^{-2})^{-2}$

e. $\frac{x^{-4}y^2}{x^2y^{-5}}$ **f.** $\left(\frac{-4m^5n^3}{24mn^{-6}}\right)^{-2}$

Solution ►

a. $4x^{-5}=\frac{4}{x^5}$

exponent -5 refers to x only!

b. $(x+y)^{-1}=\frac{1}{x+y}$

these expressions are NOT equivalent!

c. $x^{-1}+y^{-1}=\frac{1}{x}+\frac{1}{y}$

d. $(-2^3x^{-2})^{-2}=\left(\frac{-2^3}{x^2}\right)^{-2}=\left(\frac{x^2}{-2^3}\right)^{2}=\frac{(x^2)^2}{(-1)^2(2^3)^2}=\frac{x^4}{2^6}$

power rule – multiply exponents

work out the negative exponents inside the bracket

work out the negative exponents outside the bracket

a "−" sign can be treated as a factor of -1

e. $\frac{x^{-4}y^2}{x^2y^{-5}}=\frac{y^2y^5}{x^2x^4}=\frac{y^7}{x^6}$

product rule – add exponents

f. $\left(\frac{-\cancelto{}{4}\cancelto{4}{m^5}n^3}{\cancelto{6}{24}\cancel{m}n^{-6}}\right)^{-2}=\left(\frac{-m^4n^3n^6}{6}\right)^{-2}=\left(\frac{(-1)m^4n^9}{6}\right)^{-2}=\left(\frac{6}{(-1)m^4n^9}\right)^{2}=\frac{36}{m^8n^{18}}$

$(-1)^2=1$

Scientific Notation

Integral exponents allow us to record numbers with a very large or very small absolute value in a shorter, more convenient form.

For example, the average distance from the Sun to the Saturn is 1,430,000,000 km, which can be recorded as $1.43\cdot 10{,}000{,}000$ or more concisely as $1.43\cdot 10^9$.

Similarly, the mass of an electron is 0.0000000000000000000000000009 grams, which can be recorded as $9\cdot 0.0000000000000000000000000001$, or more concisely as $9\cdot 10^{-28}$.

This more concise representation of numbers is called **scientific notation** and it is frequently used in sciences and engineering.

Definition 1.1 ► A real number x is written in **scientific notation** iff $x=a\cdot 10^n$,
where the coefficient a is such that $|a|\in[1,10)$, and the exponent n is an integer.

Example 3 ► **Converting Numbers to Scientific Notation**

Convert each number to scientific notation.

a. $520{,}000$ **b.** -0.000102 **c.** $12.5 \cdot 10^3$

Solution ►

an integer has its **decimal** dot **after the last digit**

a. To represent $520{,}000$ in scientific notation, we place a decimal point after the first nonzero digit,

$$5.20000$$

and then count the number of decimal places needed for the decimal point to move to its original position, which by default was after the last digit. In our example the number of places we need to move the decimal place is 5. This means that 5.2 needs to be multiplied by 10^5 in order to represent the value of 520,000. So, $\mathbf{520{,}000 = 5.2 \cdot 10^5}$.

Note: To comply with the scientific notation format, we always place the decimal point after the first nonzero digit of the given number. This will guarantee that the coefficient $\boldsymbol{a}$ satisfies the condition $\mathbf{1 \le |a| < 10}$.

b. As in the previous example, to represent -0.000102 in scientific notation, we place a decimal point after the first nonzero digit,

$$-0.0001.02$$

and then count the number of decimal places needed for the decimal point to move to its original position. In this example, we move the decimal 4 places to the left. So the number 1.02 needs to be divided by 10^4, or equivalently, multiplied by 10^{-4} in order to represent the value of -0.000102. So, $\mathbf{-0.000102 = -1.02 \cdot 10^{-4}}$.

Observation: Notice that moving the decimal to the **right** corresponds to using a **positive** exponent, as in *Example 3a*, while moving the decimal to the **left** corresponds to using a **negative** exponent, as in *Example 3b*.

c. Notice that $12.5 \cdot 10^3$ is not in scientific notation as the coefficient 12.5 is <u>not</u> smaller than 10. To convert $12.5 \cdot 10^3$ to scientific notation, first, convert 12.5 to scientific notation and then multiply the powers of 10. So,

$$12.5 \cdot 10^3 = 1.25 \cdot 10 \cdot 10^3 = \mathbf{1.25 \cdot 10^4}$$

multiply powers by adding exponents

Example 4 ► Converting from Scientific to Decimal Notation

Convert each number to decimal notation.

a. $-6.57 \cdot 10^6$ **b.** $4.6 \cdot 10^{-7}$

Solution ►

a. The exponent 6 indicates that the decimal point needs to be moved 6 places to the right. So,

$$-6.57 \cdot 10^6 = -6\,.\,5\,7\,\_\,\_\,\_\,\_\,. = \mathbf{-6,570,000}$$

fill the empty places by zeros

b. The exponent -7 indicates that the decimal point needs to be moved 7 places to the left. So,

$$4.6 \cdot 10^{-7} = 0.\_\,\_\,\_\,\_\,\_\,\_\,4\,.\,6 = \mathbf{0.00000046}$$

fill the empty places by zeros

Example 5 ► Using Scientific Notation in Computations

Evaluate. Leave the answer in scientific notation.

a. $6.5 \cdot 10^7 \cdot 3 \cdot 10^5$ **b.** $\frac{3.6 \cdot 10^3}{9 \cdot 10^{14}}$

Solution ►

a. Since the product of the coefficients $6.5 \cdot 3 = 19.5$ is larger than 10, we convert it to scientific notation and then multiply the remaining powers of 10. So,

$$6.5 \cdot 10^7 \cdot 3 \cdot 10^5 = 19.5 \cdot 10^7 \cdot 10^5 = 1.95 \cdot 10 \cdot 10^{12} = \mathbf{1.95 \cdot 10^{13}}$$

b. Similarly as in the previous example, since the quotient $\frac{3.6}{9} = 0.4$ is smaller than 1, we convert it to scientific notation and then work out the remaining powers of 10. So,

$$\frac{3.6 \cdot 10^3}{9 \cdot 10^{14}} = 0.4 \cdot 10^{-11} = 4 \cdot 10^{-1} \cdot 10^{-11} = \mathbf{4 \cdot 10^{-12}}$$

divide powers by subtracting exponents

Example 6 ► Using Scientific Notation to Solve Problems

The average distance from Earth to the sun is $1.5 \cdot 10^8$ km. How long would it take a rocket, traveling at $4.7 \cdot 10^3$ km/h, to reach the sun?

Solution ► To find time T needed for the rocket traveling at the rate $R = 4.7 \cdot 10^3$ km/h to reach the sun that is at the distance $D = 1.5 \cdot 10^8$ km from Earth, first, we solve the motion formula $R \cdot T = D$ for T. Since $T = \frac{D}{R}$, we calculate,

$$T = \frac{1.5 \cdot 10^8}{4.7 \cdot 10^3} \cong 0.32 \cdot 10^5 = 3.2 \cdot 10^4$$

So, it will take approximately $\mathbf{3.2 \cdot 10^4}$ **hours** for the rocket to reach the sun.

RT.1 Exercises

Vocabulary Check *Complete each blank with the most appropriate term or phrase from the given list:* ***numerator, opposite, reciprocal, scientific.***

1. To raise a base to a negative exponent, raise the _______________ of the base to the opposite exponent.

2. A power in a numerator of a fraction can be written equivalently as a power with the _______________ exponent in the denominator of this fraction. Similarly, a power in a denominator of a fraction can be written equivalently as a power with the opposite exponent in the _______________ of this fraction.

3. A number with a very large or very small absolute value is often written in the _______________ notation.

Concept Check *True or false.*

4. $\left(\frac{3}{4}\right)^{-2} = \left(\frac{4}{3}\right)^2$

5. $10^{-4} = 0.00001$

6. $(0.25)^{-1} = 4$

7. $-4^5 = \frac{1}{4^5}$

8. $(-2)^{-10} = 4^{-5}$

9. $2 \cdot 2 \cdot 2^{-1} = \frac{1}{8}$

10. $3x^{-2} = \frac{1}{3x^2}$

11. $-2^{-2} = -\frac{1}{4}$

12. $\frac{5^{10}}{5^{-12}} = 5^{-2}$

13. The number $0.68 \cdot 10^{-5}$ is written in scientific notation.

14. $98.6 \cdot 10^7 = 9.86 \cdot 10^6$

Concept Check

15. *Match the expression in Row I with its equivalent expression in Row II. Choices may be used once, more than once, or not at all.*

a. 5^{-2} **b.** -5^{-2} **c.** $(-5)^{-2}$ **d.** $-(-5)^{-2}$ **e.** $-5 \cdot 5^{-2}$

A. 25 **B.** $\frac{1}{25}$ **C.** -25 **D.** $-\frac{1}{5}$ **E.** $-\frac{1}{25}$

Concept Check *Evaluate each expression.*

16. $4^{-6} \cdot 4^3$

17. $-9^3 \cdot 9^{-5}$

18. $\frac{2^{-3}}{2^6}$

19. $\frac{2^{-7}}{2^{-5}}$

20. $\frac{-3^{-4}}{5^{-3}}$ **21.** $-\left(\frac{3}{2}\right)^{-2}$ **22.** $2^{-2}+2^{-3}$ **23.** $(2^{-1}-3^{-1})^{-1}$

Concept Check *Simplify each expression, if possible. Leave the answer with only* ***positive exponents****. Assume that all variables represent nonzero real numbers. Keep large numerical coefficients as powers of prime numbers, if possible.*

24. $(-2x^{-3})(7x^{-8})$ **25.** $(5x^{-2}y^{3})(-4x^{-7}y^{-2})$ **26.** $(9x^{-4n})(-4x^{-8n})$

27. $(-3y^{-4a})(-5y^{-3a})$ **28.** $-4x^{-3}$ **29.** $\frac{x^{-4n}}{x^{6n}}$

30. $\frac{3n^5}{nm^{-2}}$ **31.** $\frac{14a^{-4}b^{-3}}{-8a^8b^{-5}}$ **32.** $\frac{-18x^{-3}y^3}{-12x^{-5}y^5}$

33. $(2^{-1}p^{-7}q)^{-4}$ **34.** $(-3a^2b^{-5})^{-3}$ **35.** $\left(\frac{5x^{-2}}{y^3}\right)^{-3}$

36. $\left(\frac{2x^3y^{-2}}{3y^{-3}}\right)^{-3}$ **37.** $\left(\frac{-4x^{-3}}{5x^{-1}y^4}\right)^{-4}$ **38.** $\left(\frac{125x^2y^{-3}}{5x^4y^{-2}}\right)^{-5}$

39. $\left(\frac{-200x^3y^{-5}}{8x^5y^{-7}}\right)^{-4}$ **40.** $[(-2x^{-4}y^{-2})^{-3}]^{-2}$ **41.** $\frac{12a^{-2}(a^{-3})^{-2}}{6a^7}$

42. $\frac{(-2k)^2m^{-5}}{(km)^{-3}}$ **43.** $\left(\frac{2p}{q^2}\right)^3\left(\frac{3p^4}{q^{-4}}\right)^{-1}$ **44.** $\left(\frac{-3x^4y^6}{15x^{-6}y^7}\right)^{-3}$

45. $\left(\frac{-4a^3b^2}{12a^6b^{-5}}\right)^{-3}$ **46.** $\left(\frac{-9^{-2}x^{-4}y}{3^{-3}x^{-3}y^2}\right)^{8}$ **47.** $(4^{-x})^{2y}$

48. $(5^a)^{-a}$ **49.** x^ax^{-a} **50.** $\frac{9n^{2-x}}{3n^{2-2x}}$

51. $\frac{12x^{a+1}}{-4x^{2-a}}$ **52.** $(x^{b-1})^3(x^{b-4})^{-2}$ **53.** $\frac{25x^{a+b}y^{b-a}}{-5x^{a-b}y^{b+a}}$

Convert each number to scientific notation.

54. 26,000,000,000 **55.** -0.000132 **56.** 0.0000000105 **57.** 705.6

Convert each number to decimal notation.

58. $6.7 \cdot 10^8$ **59.** $5.072 \cdot 10^{-5}$ **60.** $2 \cdot 10^{12}$ **61.** $9.05 \cdot 10^{-9}$

62. One gigabyte of computer memory equals 2^{30} bytes. Write the number of bytes in 1 gigabyte, using decimal notation. Then, using scientific notation, approximate this number by rounding the scientific notation coefficient to two decimals places.

Evaluate. State your answer in scientific notation.

63. $(6.5 \cdot 10^3)(5.2 \cdot 10^{-8})$ **64.** $(2.34 \cdot 10^{-5})(5.7 \cdot 10^{-6})$

65. $(3.26 \cdot 10^{-6})(5.2 \cdot 10^{-8})$ **66.** $\frac{4 \cdot 10^{-7}}{8 \cdot 10^{-3}}$

67. $\frac{7.5 \cdot 10^9}{2.5 \cdot 10^4}$

68. $\frac{4 \cdot 10^{-7}}{8 \cdot 10^{-3}}$

69. $\frac{0.05 \cdot 16000}{0.0004}$

70. $\frac{0.003 \cdot 40{,}000}{0.00012 \cdot 600}$

Analytic Skills *Solve each problem. State your answer in scientific notation.*

71. A *light-year* is the distance that light travels in one year. Find the number of kilometers in a light-year if light travels approximately $3 \cdot 10^5$ kilometers per second.

72. In 2017, the national debt in Canada was about $6.4 \cdot 10^{11}$ dollars. If Canadian population in 2017 was approximately $3.56 \cdot 10^7$, what was the share of this debt per person?

73. The brightest star in the night sky, Sirius, is about $4.704 \cdot 10^{13}$ miles from Earth. If one light-year is approximately $5.88 \cdot 10^{12}$ miles, how many light-years is it from Earth to Sirius?

74. The average discharge at the mouth of the Amazon River is 4,200,000 cubic feet per second. How much water is discharged from the Amazon River in one hour? in one year?

75. If current trends continue, world population P in billions may be modeled by the equation $P = 6(1.014)^x$, where x is in years and $x = 0$ corresponds to the year 2000. Estimate the world population in 2020 and 2025.

76. The mass of the sun is $1.989 \cdot 10^{30}$ kg and the mass of the earth is $5.976 \cdot 10^{24}$ kg. How many times larger in mass is the sun than the earth?

Discussion Point

77. Many calculators can only handle scientific notation for powers of 10 between -99 and 99. How can we compute $(4 \cdot 10^{220})^2$?