RT.2 Rational Expressions and Functions; Multiplication and Division of Rational Expressions

In arithmetic, a rational number is a quotient of two integers with denominator different than zero. In algebra, a *rational expression*, offten called an *algebraic fraction*, is a quotient of two polynomials, also with denominator different than zero. In this section, we will examine rational expressions and functions, paying attention to their domains. Then, we will simplify, multiply, and divide rational expressions, employing the factoring skills developed in *Chapter P*.

Rational Expressions and Functions

Here are some examples of rational expressions:

$$-\frac{x^2}{2xy}, \qquad x^{-1}, \qquad \frac{x^2-4}{x-2}, \qquad \frac{8x^2+6x-5}{4x^2+5x}, \qquad \frac{x-3}{3-x}, \qquad x^2-25, \qquad 3x(x-1)^{-2}$$

Definition 2.1 ► A **rational expression** (**algebraic fraction**) is a quotient $\frac{P(x)}{Q(x)}$ of two polynomials $P(x)$ and $Q(x)$, where $Q(x) \neq 0$. Since division by zero is not permitted, a rational expression is defined only for the x-values that make the denominator of the expression different than zero. The set of such x-values is referred to as the **domain** of the expression.

Note 1: Negative exponents indicate hidden fractions and therefore represent rational expressions. For instance, $x^{-1} = \frac{1}{x}$.

Note 2: A single polynomial can also be seen as a rational expression because it can be considered as a fraction with a denominator of 1.
For instance, $x^2 - 25 = \frac{x^2-25}{1}$.

Definition 2.2 ► A **rational function** is a function defined by a rational expression,

$$\boldsymbol{f(x) = \frac{P(x)}{Q(x)}}.$$

The **domain** of such function consists of all real numbers except for the x-values that make the denominator $Q(x)$ equal to 0. So, the domain $\boldsymbol{D} = \mathbb{R} \setminus \{\boldsymbol{x} | \boldsymbol{Q(x) = 0}\}$

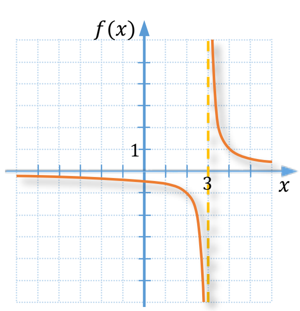

Figure 1

For example, the domain of the rational function $f(x) = \frac{1}{x-3}$ is the set of all real numbers except for 3 because 3 would make the denominator equal to 0. So, we write $D = \mathbb{R} \setminus \{3\}$. Sometimes, to make it clear that we refer to function f, we might denote the domain of f by D_f, rather than just D.

Figure 1 shows a graph of the function $f(x) = \frac{1}{x-3}$. Notice that the graph does not cross the dashed vertical line whose equation is $x = 3$. This is because $f(3)$ is not defined. A closer look at the graphs of rational functions will be given in *Section RT.5*.

Example 1 ► **Evaluating Rational Expressions or Functions**

Evaluate the given expression or function for $x = -1, 0, 1$. If the value cannot be calculated, write ***undefined***.

a. $3x(x-1)^{-2}$ **b.** $f(x) = \frac{x}{x^2+x}$

Solution ► **a.** If $x = -1$, then $3x(x-1)^{-2} = 3(-1)(-1-1)^{-2} = -3(-2)^{-2} = \frac{-3}{(-2)^2} = -\frac{\mathbf{3}}{\mathbf{4}}$.

If $x = 0$, then $3x(x-1)^{-2} = 3(0)(0-1)^{-2} = \mathbf{0}$.

If $x = 1$, then $3x(x-1)^{-2} = 3(1)(1-1)^{-2} = 3 \cdot 0^{-2} = \boldsymbol{undefined}$, as division by zero is not permitted.

Note: Since the expression $3x(x-1)^{-2}$ cannot be evaluated at $x = 1$, the number 1 does not belong to its domain.

b. $f(-1) = \frac{-1}{(-1)^2+(-1)} = \frac{-1}{1-1} = \boldsymbol{undefined}$.

$f(0) = \frac{0}{(0)^2+(0)} = \frac{0}{0} = \boldsymbol{undefined}$.

$f(1) = \frac{1}{(1)^2+(1)} = \frac{\mathbf{1}}{\mathbf{2}}$.

Observation: Function $f(x) = \frac{x}{x^2+x}$ is undefined at $x = 0$ and $x = -1$. This is because the denominator $x^2 + x = x(x+1)$ becomes zero when the x-value is 0 or -1.

Example 2 ► **Finding Domains of Rational Expressions or Functions**

Find the domain of each expression or function.

a. $\frac{4}{2x+5}$ **b.** $\frac{x-2}{x^2-2x}$

c. $f(x) = \frac{x^2-4}{x^2+4}$ **d.** $g(x) = \frac{2x-1}{x^2-4x-5}$

Solution ► **a.** The domain of $\frac{4}{2x+5}$ consists of all real numbers except for those that would make the denominator $2x + 5$ equal to zero. To find these numbers, we solve the equation

$$2x + 5 = 0$$
$$2x = -5$$
$$x = -\frac{\mathbf{5}}{\mathbf{2}}$$

So, the domain of $\frac{4}{2x+5}$ is the set of all real numbers except for $-\frac{5}{2}$. This can be recorded in set notation as $\mathbb{R} \setminus \left\{-\frac{5}{2}\right\}$, or in set-builder notation as $\left\{\boldsymbol{x} \middle| \boldsymbol{x} \neq -\frac{5}{2}\right\}$, or in interval notation as $\left(-\infty, -\frac{5}{2}\right) \cup \left(-\frac{5}{2}, \infty\right)$.

DOMAIN

b. To find the domain of $\frac{x-2}{x^2-2x}$, we want to exclude from the set of real numbers all the x-values that would make the denominator $x^2 - 2x$ equal to zero. After solving the equation

$$x^2 - 2x = 0$$

via factoring

$$x(x-2) = 0$$

and zero-product property

$$x = \mathbf{0} \quad \text{or} \quad x = \mathbf{2},$$

we conclude that the domain is the set of all real numbers except for 0 and 2, which can be recorded as $\mathbb{R} \setminus \{\mathbf{0}, \mathbf{2}\}$. This is because the x-values of 0 or 2 make the denominator of the expression $\frac{x-2}{x^2-2x}$ equal to zero.

c. To find the domain of the function $f(x) = \frac{x^2-4}{x^2+4}$, we first look for all the x-values that make the denominator $x^2 + 4$ equal to zero. However, $x^2 + 4$, as a sum of squares, is never equal to 0. So, the domain of function f is the set of all real numbers $\mathbb{R}$.

d. To find the domain of the function $g(x) = \frac{2x-1}{x^2-4x-5}$, we first solve the equation $x^2 - 4x - 5 = 0$ to find which x-values make the denominator equal to zero. After factoring, we obtain

$$(x-5)(x+1) = 0$$

which results in $\quad x = 5 \;$ and $\; x = -1$

Thus, the domain of g equals to $\boldsymbol{D_g} = \mathbb{R} \setminus \{-\mathbf{1}, \mathbf{5}\}$.

Equivalent Expressions

Definition 2.3 ► Two expressions are **equivalent** in the **common domain** iff (if and only if) they produce the same values for every input from the domain.

Consider the expression $\frac{x-2}{x^2-2x}$ from *Example 2b*. Notice that this expression can be simplified to $\frac{\cancel{x-2}}{x\cancel{(x-2)}} = \frac{1}{x}$ by reducing common factors in the numerator and the denominator. However, the domain of the simplified fraction, $\frac{1}{x}$, is the set $\mathbb{R} \setminus \{0\}$, which is different than the domain of the original fraction, $\mathbb{R} \setminus \{0,2\}$. Notice that for $x = 2$, the expression $\frac{x-2}{x^2-2x}$ is undefined while the value of the expression $\frac{1}{x}$ is $\frac{1}{2}$. So, the two expressions are not equivalent in the set of real numbers. However, if the domain of $\frac{1}{x}$ is

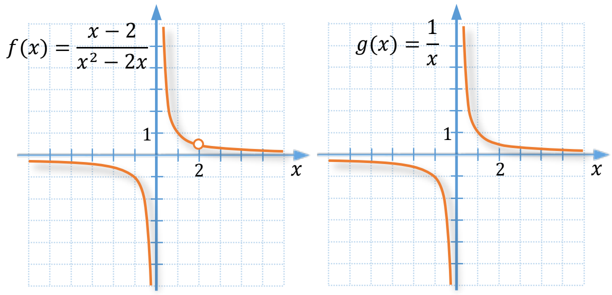

Figure 2

resticted to the set $\mathbb{R} \setminus \{0,2\}$, then the two expressions produce the same values and as such, they are equivalent. We say that the two expressions are **equivalent** in the **common domain**.

The above situation can be illustrated by graphing the related functions, $f(x) = \frac{x-2}{x^2-2x}$ and $g(x) = \frac{1}{x}$, as in *Figure 2*. The graphs of both functions are exactly the same except for the hole in the graph of f at the point $\left(2, \frac{1}{2}\right)$.

So, from now on, when writing statements like $\frac{x-2}{x^2-2x} = \frac{1}{x}$, we keep in mind that they apply only to real numbers which make both denominators different than zero. Thus, by saying in short that two **expressions are equivalent**, we really mean that they are **equivalent in the common domain**.

Note: The **domain** of $f(x) = \frac{x-2}{x^2-2x} = \frac{\cancel{x-2}}{x(\cancel{x-2})} = \frac{1}{x}$ is still $\mathbb{R} \setminus \{\mathbf{0}, \mathbf{2}\}$, even though the $(x-2)$ term was simplified.

The process of simplifying expressions involve creating equivalent expressions. In the case of rational expressions, equivalent expressions can be obtained by multiplying or dividing the numerator and denominator of the expression by the same nonzero polynomial. For example,

$$\frac{\boldsymbol{-x-3}}{\boldsymbol{-5x}} = \frac{(-x-3)\cdot(-1)}{(-5x)\cdot(-1)} = \frac{\boldsymbol{x+3}}{\boldsymbol{5x}}$$

$$\frac{\boldsymbol{x-3}}{\boldsymbol{3-x}} = \frac{\cancel{(x-3)}}{-1\cancel{(x-3)}} = \frac{1}{-1} = \boldsymbol{-1}$$

To simplify a rational expression:

- **Factor** the numerator and denominator **completely**.
- **Eliminate all common factors** by following the property of multiplicative identity. *Do <u>not eliminate common terms</u> - they must be factors!*

Example 3 ► **Simplifying Rational Expressions**

Simplify each expression.

a. $\frac{7a^2b^2}{21a^3b-14a^3b^2}$ **b.** $\frac{x^2-9}{x^2-6x+9}$ **c.** $\frac{20x-15x^2}{15x^3-5x^2-20x}$

Solution ► **a.** First, we factor the denominator and then reduce the common factors. So,

$$\frac{7a^2b^2}{21a^3b-14a^3b^2} = \frac{\cancel{7}\cancel{a^2}\cancel{b^2}^{\,1}}{\cancel{7}\cancel{a^3}^{\,1}\cancel{b}(3a-2ab)} = \frac{\boldsymbol{b}}{\boldsymbol{a(3-2b)}}$$

b. As before, we factor and then reduce. So,

$$\frac{x^2-9}{x^2-6x+9}=\frac{\cancel{(x-3)}(x+3)}{(x-3)^{\cancel{2}\,1}}=\boldsymbol{\frac{x+3}{x-3}}$$

Neither x nor 3 can be reduced, as they are NOT factors !

c. Factoring and reducing the numerator and denominator gives us

$$\frac{20x-15x^2}{15x^3-5x^2-20x}=\frac{\cancel{5x}(4-3x)}{\cancel{5x}(3x^2-x-4)}=\frac{4-3x}{(3x-4)(x+1)}$$

Since $\frac{4-3x}{3x-4}=\frac{-(3x-4)}{3x-4}=-1$, the above expression can be reduced further to

$$\frac{\cancel{4-3x}^{\,-1}}{\cancel{(3x-4)}(x+1)}=\boldsymbol{\frac{-1}{x+1}}$$

Notice: An opposite expression in the numerator and denominator can be reduced to -1. For example, since $a-b$ is opposite to $b-a$, then

$$\boldsymbol{\frac{a-b}{b-a}=-1}, \text{ as long as } a\neq b.$$

Caution: Note that $a-b$ is NOT opposite to $a+b$!

Multiplication and Division of Rational Expressions

Recall that to multiply common fractions, we multiply their numerators and denominators, and then simplify the resulting fraction. Multiplication of algebraic fractions is performed in a similar way.

To multiply rational expressions:

- **factor** each numerator and denominator **completely**,
- **reduce all common factors** in any of the numerators and denominators,
- **multiply** the remaining expressions by writing the product of their numerators over the product of their denominators.

  For instance,

$$\frac{3x}{x^2+5x}\cdot\frac{3x+15}{6x}=\frac{3\cancel{x}}{\cancel{x}\cancel{(x+5)}}\cdot\frac{\cancel{3}\cancel{(x+5)}}{\cancel{6}_{\,2}x}=\frac{3}{2x}$$

Example 4 ► **Multiplying Algebraic Fractions**

Multiply and simplify. Assume non-zero denominators.

a. $\frac{2x^2y^3}{3xy^2}\cdot\frac{(2x^3y)^2}{2(xy)^3}$ **b.** $\frac{x^3-y^3}{x+y}\cdot\frac{3x+3y}{x^2-y^2}$

Solution ► **a.** To multiply the two algebraic fractions, we use appropriate rules of powers to simplify each fraction, and then reduce all the remaining common factors. So,

$$\frac{2x^{\cancel{2}\,1}\cancel{y^3}^{\,1}}{3\cancel{x}\cancel{y^2}}\cdot\frac{(2x^3y)^2}{2(xy)^3}=\frac{2xy}{3}\cdot\frac{\cancel{4}^{2}x^{\cancel{6}\,3}\cancel{y^2}}{\cancel{2}x^3y^{\cancel{3}\,1}}=\frac{2x\cancel{y}\cdot 2x^3}{3\cdot\cancel{y}}=\frac{\mathbf{4x^4}}{\mathbf{3}}=\frac{\mathbf{4}}{\mathbf{3}}\boldsymbol{x^4}$$

equivalent answers

Recall: $x^3-y^3=(x-y)(x^2+xy+y^2)$

$x^2-y^2=(x+y)(x-y)$

b. After factoring and simplifying, we have

$$\frac{x^3-y^3}{x+y}\cdot\frac{3x+3y}{x^2-y^2}=\frac{\cancel{(x-y)}(x^2+xy+y^2)}{x+y}\cdot\frac{3\cancel{(x+y)}}{\cancel{(x-y)}\cancel{(x+y)}}=\frac{\mathbf{3(x^2+xy+y^2)}}{\boldsymbol{x+y}}$$

To divide rational expressions, **multiply** the first, the *dividend*, **by the reciprocal** of the second, the *divisor*.

For instance,

$$\frac{5x-10}{3x}\div\frac{3x-6}{2x^2}=\frac{5x-10}{3x}\cdot\frac{2x^2}{3x-6}=\frac{5\cancel{(x-2)}}{3\cancel{x}}\cdot\frac{2x^{\cancel{2}\,1}}{3\cancel{(x-2)}}=\frac{10x}{9}$$

multiply by the **reciprocal**

follow multiplication rules

Example 5 ► Dividing Algebraic Fractions

Perform operations and simplify. Assume non-zero denominators.

a. $\frac{2x^2+2x}{x-1}\div(x+1)$ **b.** $\frac{x^2-25}{x^2+5x+4}\div\frac{x^2-10x+25}{2x^2+8x}\cdot\frac{x^2+x}{4x^2}$

Solution ► **a.** To divide by $(x+1)$ we multiply by the reciprocal $\frac{1}{(x+1)}$. So,

$$\frac{2x^2+2x}{x-1}\div(x+1)=\frac{2x\cancel{(x+1)}}{x-1}\cdot\frac{1}{\cancel{(x+1)}}=\frac{2x}{x-1}$$

b. The order of operations indicates to perform the division first. To do this, we convert the division into multiplication by the reciprocal of the middle expression. Therefore,

$$\frac{x^2-25}{x^2+5x+4}\div\frac{x^2-10x+25}{2x^2+8x}\cdot\frac{x^2+x}{4x^2}$$

$$=\frac{(x-5)(x+5)}{(x+4)\cancel{(x+1)}}\cdot\frac{2x^2+8x}{x^2-10x+25}\cdot\frac{\cancel{x}\cancel{(x+1)}}{4x^{\cancel{2}\,1}}$$

$$=\frac{\cancel{(x-5)}(x+5)}{\cancel{(x+4)}}\cdot\frac{2\cancel{x}\cancel{(x+4)}}{(x-5)^{\cancel{2}\,1}}\cdot\frac{1}{\cancel{4}_{2}x}=\frac{\boldsymbol{(x+5)}}{\mathbf{2(x-5)}}$$

Reduction of common factors can be done gradually, especially if there is many common factors to divide out.

RT.2 Exercises

Vocabulary Check *Complete each blank with the most appropriate term or phrase from the given list:* ***rational, zero, common, factor, reciprocal****.*

1. If $P(x)$ and $Q(x)$ are polynomials, then $f(x) = \frac{P(x)}{Q(x)}$ is a __________ function provided that $Q(x)$ is not the zero polynomial.

2. The domain of a rational function consists of all real numbers except for the x-values that will make the denominator equal to________.

3. Two rational expressions are equivalent if they produce the same values for any inputs from their _____________ domain.

4. To simplify, multiply, or divide rational expressions, we first _____________ each numerator and denominator.

5. To divide by a rational expression, we multiply by its ______________.

Concept Check *True or false.*

6. $f(x) = \frac{4}{\sqrt{x}-4}$ is a rational function.

7. The domain of $f(x) = \frac{x-2}{4}$ is the set of all real numbers.

8. $\frac{x-3}{4-x}$ is equivalent to $-\frac{x-3}{x-4}$.

9. $\frac{n^2+1}{n^2-1}$ is equivalent to $\frac{n+1}{n-1}$.

Concept Check *Given the rational function f, find $f(-1)$, $f(0)$, and $f(2)$.*

10. $f(x) = \frac{x}{x-2}$

11. $f(x) = \frac{5x}{3x-x^2}$

12. $f(x) = \frac{x-2}{x^2+x-6}$

Concept Check *For each rational function, find all numbers that are not in the domain. Then give the domain, using both set notation and interval notation.*

13. $f(x) = \frac{x}{x+2}$

14. $g(x) = \frac{x}{x-6}$

15. $h(x) = \frac{2x-1}{3x+7}$

16. $f(x) = \frac{3x+2}{5x-4}$

17. $g(x) = \frac{x+2}{x^2-4}$

18. $h(x) = \frac{x-2}{x^2+4}$

19. $f(x) = \frac{5}{3x-x^2}$

20. $g(x) = \frac{x^2+x-6}{x^2+12x+35}$

21. $h(x) = \frac{7}{|4x-3|}$

Discussion Point

22. Is there a rational function f such that 2 is not in its domain and $f(0) = 5$? If yes, how many such functions are there? If not, explain why not.

Concept Check

23. *Which rational expressions are equivalent and what is the simplest expression that they are equivalent to?*

a. $\frac{2x+3}{2x-3}$ **b.** $\frac{2x-3}{3-2x}$ **c.** $\frac{2x+3}{3+2x}$ **d.** $\frac{2x+3}{-2x-3}$ **e.** $\frac{3-2x}{2x-3}$

Concept Check

24. *Which rational expressions can be simplified?*

a. $\frac{x^2+2}{x^2}$ **b.** $\frac{x^2+2}{2}$ **c.** $\frac{x^2-x}{x^2}$ **d.** $\frac{x^2-y^2}{y^2}$ **e.** $\frac{x}{x^2-x}$

Simplify each expression, if possible.

25. $\frac{24a^3b}{3ab^3}$ **26.** $\frac{-18x^2y^3}{8x^3y}$ **27.** $\frac{7-x}{x-7}$ **28.** $\frac{x+2}{x-2}$

29. $\frac{a-5}{-5+a}$ **30.** $\frac{(3-y)(x+1)}{(y-3)(x-1)}$ **31.** $\frac{12x-15}{21}$ **32.** $\frac{18a-2}{22}$

33. $\frac{4y-12}{4y+12}$ **34.** $\frac{7x+14}{7x-14}$ **35.** $\frac{6m+18}{7m+21}$ **36.** $\frac{3z^2+z}{18z+6}$

37. $\frac{m^2-25}{20-4m}$ **38.** $\frac{9n^2-3}{4-12n^2}$ **39.** $\frac{t^2-25}{t^2-10t+25}$ **40.** $\frac{p^2-36}{p^2+12t+36}$

41. $\frac{x^2-9x+8}{x^2+3x-4}$ **42.** $\frac{p^2+8p-9}{p^2-5p+4}$ **43.** $\frac{x^3-y^3}{x^2-y^2}$ **44.** $\frac{b^2-a^2}{a^3-b^3}$

Perform operations and simplify. Assume non-zero denominators.

45. $\frac{18a^4}{5b^2} \cdot \frac{25b^4}{9a^3}$ **46.** $\frac{28}{xy} \div \frac{63x^3}{2y^2}$ **47.** $\frac{12x}{49(xy^2)^3} \cdot \frac{(7xy)^2}{8}$

48. $\frac{x+1}{2x-3} \cdot \frac{2x-3}{2x}$ **49.** $\frac{10a}{6a-12} \cdot \frac{20a-40}{30a^3}$ **50.** $\frac{a^2-1}{4a} \cdot \frac{2}{1-a}$

51. $\frac{y^2-25}{4y} \cdot \frac{2}{5-y}$ **52.** $(8x-16) \div \frac{3x-6}{10}$ **53.** $(y^2-4) \div \frac{2-y}{8y}$

54. $\frac{3n-9}{n^2-9} \cdot (n^3+27)$ **55.** $\frac{x^2-16}{x^2} \cdot \frac{x^2-4x}{x^2-x-12}$ **56.** $\frac{y^2+10y+25}{y^2-9} \cdot \frac{y^2-3y}{y+5}$

57. $\frac{b-3}{b^2-4b+3} \div \frac{b^2-b}{b-1}$ **58.** $\frac{x^2-6x+9}{x^2+3x} \div \frac{x^2-9}{x}$ **59.** $\frac{x^2-2x}{3x^2-5x-2} \cdot \frac{9x^2-4}{9x^2-12x+4}$

60. $\frac{t^2-49}{t^2+4t-21} \cdot \frac{t^2+8t+15}{t^2-2t-35}$ **61.** $\frac{a^3-b^3}{a^2-b^2} \div \frac{2a-2b}{2a+2b}$ **62.** $\frac{64x^3+1}{4x^2-100} \cdot \frac{4x+20}{64x^2-16x+4}$

63. $\frac{x^3y-64y}{x^3y+64y} \div \frac{x^2y^2-16y^2}{x^2y^2-4xy^2+16y^2}$

64. $\frac{p^3-27q^3}{p^2+pq-12q^2} \cdot \frac{p^2-2pq-24q^2}{p^2-5pq-6q^2}$

65. $\frac{4x^2-9y^2}{8x^3-27y^3} \cdot \frac{4x^2+6xy+9y^2}{4x^2+12xy+9y^2}$

66. $\frac{2x^2+x-1}{6x^2+x-2} \div \frac{2x^2+5x+3}{6x^2+13x+6}$

67. $\frac{6x^2-13x+6}{14x^2-25x+6} \div \frac{14-21x}{49x^2+7x-6}$

68. $\frac{4y^2-12y+36}{27-3y^2} \div (y^3+27)$

69. $\frac{3y}{x^2} \div \frac{y^2}{x} \div \frac{y}{5x}$

70. $\frac{x+1}{y-2} \div \frac{2x+2}{y-2} \div \frac{x}{y}$

71. $\frac{a^2-4b^2}{a+2b} \div (a+2b) \cdot \frac{2b}{a-2b}$

72. $\frac{9x^2}{x^2-16y^2} \div \frac{1}{x^2+4xy} \cdot \frac{x-4y}{3x}$

73. $\frac{x^2-25}{x-4} \div \frac{x^2-2x-15}{x^2-10x+24} \cdot \frac{x+3}{x^2+10x+25}$

74. $\frac{y-3}{y^2-8y+16} \cdot \frac{y^2-16}{y+4} \div \frac{y^2+3y-18}{y^2+11y+30}$

Given $f(x)$ and $g(x)$, find $f(x) \cdot g(x)$ and $f(x) \div g(x)$.

75. $f(x) = \frac{x-4}{x^2+x}$ and $g(x) = \frac{2x}{x+1}$

76. $f(x) = \frac{x^3-3x^2}{x+5}$ and $g(x) = \frac{4x^2}{x-3}$

77. $f(x) = \frac{x^2-7x+12}{x+3}$ and $g(x) = \frac{9-x^2}{x-4}$

78. $f(x) = \frac{x+6}{4-x^2}$ and $g(x) = \frac{2-x}{x^2+8x+12}$