RT.3 Addition and Subtraction of Rational Expressions

Many real-world applications involve adding or subtracting algebraic fractions. Similarly as in the case of common fractions, to add or subtract algebraic fractions, we first need to change them equivalently to fractions with the same denominator. Thus, we begin by discussing the techniques of finding the least common denominator.

Least Common Denominator

The **least common denominator (LCD)** for fractions with given denominators is the same as the **least common multiple (LCM)** of these denominators. The methods of finding the LCD for fractions with numerical denominators were reviewed in *section R3*. For example,

$$LCD(4,6,8) = 24,$$

because 24 is a multiple of 4, 6, and 8, and there is no smaller natural number that would be divisible by all three numbers, 4, 6, and 8.

Suppose the denominators of three algebraic fractions are $4(x^2 - y^2)$, $-6(x + y)^2$, and $8x$. The numerical factor of the least common multiple is 24. The variable part of the LCM is built by taking the product of all the different variable factors from each expression, with each factor raised to the **greatest** exponent that occurs in any of the expressions. In our example, since $4(x^2 - y^2) = 4(x + y)(x - y)$, then

$$LCD(\ 4(x+y)(x-y),\ \ -6(x+y)^2,\ \ 8x\ ) = \mathbf{24x(x+y)^2(x-y)}$$

Notice that we do not worry about the negative sign of the middle expression. This is because a negative sign can always be written in front of a fraction or in the numerator rather than in the denominator. For example,

$$\frac{1}{-6(x+y)^2} = -\frac{1}{6(x+y)^2} = \frac{-1}{6(x+y)^2}$$

In summary, to find the LCD for algebraic fractions, follow the steps:

- **Factor** each denominator **completely**.
- Build the LCD for the denominators by including the following as factors:
  - **LCD of all numerical coefficients**,
  - all of the **different factors** from each denominator, with each factor **raised to the greatest exponent** that occurs in any of the denominators.
    Note: Disregard any factor of -1.

Example 1 ► **Determining the LCM for the Given Expressions**

Find the LCM for the given expressions.

a. $12x^3y$ and $15xy^2(x - 1)$ **b.** $x^2 - 2x - 8$ and $x^2 + 3x + 2$

c. $y^2 - x^2,\ 2x^2 - 2xy,$ and $x^2 + 2xy + y^2$

Solution

a. Notice that both expressions, $12x^3y$ and $15xy^2(x-1)$, are already in factored form. The $LCM(12{,}15) = 60$, as

divide by 3

$$\begin{array}{c|cc} 3 & 12 & 15 \\ \hline \cdot & 4 & \cdot 5 \end{array} = \mathbf{60}$$

no more common factors, so we multiply the numbers in the letter L

The highest power of x is 3, the highest power of y is 2, and $(x-1)$ appears in the first power. Therefore,

$$LCM\left(12x^3y,\ \ 15xy^2(x-1)\right) = \mathbf{60x^3y^2(x-1)}$$

b. To find the LCM of x^2-2x-8 and x^2+3x+2, we factor each expression first:

$$x^2-2x-8 = (x-4)(x+2)$$
$$x^2+3x+2 = (x+1)(x+2)$$

There are three different factors in these expressions, $(x-4)$, $(x+2)$, and $(x+1)$. All of these factors appear in the first power, so

$$LCM(\,x^2-2x-8,\ \ x^2+3x+2\,) = \mathbf{(x-4)(x+2)(x+1)}$$

notice that $(x+2)$ is taken only ones!

c. As before, to find the LCM of y^2-x^2, $2x^2-2xy$, and $x^2+2xy+y^2$, we factor each expression first:

$$y^2-x^2 = (y+x)(y-x) = -(x+y)(x-y)$$
$$2x^2-2xy = 2x(x-y)$$
$$x^2+2xy+y^2 = (x+y)^2$$

as $\boldsymbol{y-x=-(x-y)}$ and $\boldsymbol{y+x=x+y}$

Since the factor of -1 can be disregarded when finding the LCM, the opposite factors can be treated as the same by factoring the -1 out of one of the expressions. So, there are four different factors to consider, 2, x, $(x+y)$, and $(x-y)$. The highest power of $(x+y)$ is 2 and the other factors appear in the first power. Therefore,

$$LCM(\,y^2-x^2,\ \ 2x^2-2xy,\ \ x^2+2xy+y^2\,) = \mathbf{2x(x-y)(x+y)^2}$$

Addition and Subtraction of Rational Expressions

Observe addition and subtraction of common fractions, as review in *section R3*.

work out the numerator

$$\frac{1}{2}+\frac{2}{3}-\frac{5}{6} = \frac{1\cdot 3+2\cdot 2-5}{6} = \frac{3+4-5}{6} = \frac{2}{6} = \mathbf{\frac{1}{3}}$$

convert fractions to the lowest common denominator

simplify, if possible

To add or subtract algebraic fractions, follow the steps:

Step 1: **Factor** the denominators of all algebraic fractions **completely**.
Step 2: **Find the LCD** of all the denominators.
Step 3: **Convert each algebraic fraction to the lowest common denominator** found in *Step 2* and write the sum (or difference) as a single fraction.
Step 4: **Simplify** the numerator and the whole fraction, if possible.

Example 2 ► **Adding and Subtracting Rational Expressions**

Perform the operations and simplify if possible.

a. $\frac{a}{5}-\frac{3b}{2a}$

b. $\frac{x}{x-y}+\frac{y}{y-x}$

c. $\frac{3x^2+3xy}{x^2-y^2}-\frac{2-3x}{x-y}$

d. $\frac{y+1}{y^2-7y+6}+\frac{y+2}{y^2-5y-6}$

e. $\frac{2x}{x^2-4}+\frac{5}{2-x}-\frac{1}{2+x}$

f. $(2x-1)^{-2}+(2x-1)^{-1}$

Solution ►

Multiplying the numerator and denominator of a fraction by the same factor is equivalent to multiplying the whole fraction by 1, which does not change the value of the fraction.

a. Since $LCM(5,2a)=10a$, we would like to rewrite expressions, $\frac{a}{5}$ and $\frac{3b}{2a}$, so that they have a denominator of 10a. This can be done by multiplying the numerator and denominator of each expression by the factors of 10a that are missing in each denominator. So, we obtain

$$\frac{a}{5}-\frac{3b}{2a}=\frac{a}{5}\cdot\frac{2a}{2a}-\frac{3b}{2a}\cdot\frac{5}{5}=\frac{\mathbf{2a^2-15b}}{\mathbf{10a}}$$

b. Notice that the two denominators, $x-y$ and $y-x$, are opposite expressions. If we write $y-x$ as $-(x-y)$, then

$$\frac{x}{x-y}+\frac{y}{y-x}=\frac{x}{x-y}+\frac{y}{-(x-y)}=\frac{x}{x-y}-\frac{y}{x-y}=\frac{x-y}{x-y}=\mathbf{1}$$

combine the signs

c. To find the LCD, we begin by factoring $x^2-y^2=(x-y)(x+y)$. Since this expression includes the second denominator as a factor, the LCD of the two fractions is $(x-y)(x+y)$. So, we calculate

keep the bracket after a "–" sign

$$\frac{3x^2+3xy}{x^2-y^2}-\frac{2-3x}{x-y}=\frac{(3x^2+3xy)\cdot 1+(2+3x)\cdot(x+y)}{(x-y)(x+y)}=$$

$$\frac{3x^2+3xy-(2x+2y+3x^2+3xy)}{(x-y)(x+y)}=\frac{3x^2+3xy-2x-2y-3x^2-3xy}{(x-y)(x+y)}=$$

$$\frac{-2x-2y}{(x-y)(x+y)}=\frac{-2\cancel{(x+y)}}{(x-y)\cancel{(x+y)}}=\frac{\mathbf{-2}}{\mathbf{(x-y)}}$$

d. To find the LCD, we first factor each denominator. Since

$$y^2 - 7y + 6 = (y-6)(y-1) \text{ and } y^2 - 5y - 6 = (y-6)(y+1),$$

then $LCD = (y-6)(y-1)(y+1)$ and we calculate

multiply by the missing bracket

$$\frac{y+1}{y^2-7y+6} + \frac{y-1}{y^2-5y-6} = \frac{y+1}{(y-6)(y-1)} + \frac{y-1}{(y-6)(y+1)} =$$

$$\frac{(y+1)\cdot(y+1)+(y-1)\cdot(y-1)}{(y-6)(y-1)(y+1)} = \frac{y^2+2y+1+(y^2-1)}{(y-6)(y-1)(y+1)} =$$

$$\frac{2y^2+2y}{(y-6)(y-1)(y+1)} = \frac{2y\cancel{(y+1)}}{(y-6)(y-1)\cancel{(y+1)}} = \frac{\mathbf{2y}}{\mathbf{(y-6)(y-1)}}$$

e. As in the previous examples, we first factor the denominators, including factoring out a negative from any opposite expression. So,

$$\frac{2x}{x^2-4} + \frac{5}{2-x} - \frac{1}{2+x} = \frac{2x}{(x-2)(x+2)} \overset{+}{-} \frac{5}{(x-2)} - \frac{1}{x+2} =$$

$LCD = (x-2)(x+2)$

$$\frac{2x - 5(x+2) - 1(x-2)}{(x-2)(x+2)} = \frac{2x - 5x - 10 - x + 2}{(x-2)(x+2)} =$$

$$\frac{-4x-8}{(x-2)(x+2)} = \frac{-4\cancel{(x+2)}}{(x-2)\cancel{(x+2)}} = \frac{\mathbf{-4}}{\mathbf{(x-2)}}$$

e. Recall that a negative exponent really represents a hidden fraction. So, we may choose to rewrite the negative powers as fractions, and then add them using techniques as shown in previous examples.

$$3(2x-1)^{-2} + (2x-1)^{-1} = \frac{1}{(2x-1)^2} + \frac{1}{2x-1} = \frac{1+1\cdot(2x-1)}{(2x-1)^2} =$$

$$\frac{3+2x-1}{(2x-1)^2} = \frac{2x+2}{(2x-1)^2} = \frac{\mathbf{2(x+1)}}{\mathbf{(2x-1)^2}}$$

nothing to simplify this time

Note: Since addition (or subtraction) of rational expressions results in a rational expression, from now on the term "rational expression" will include sums of rational expressions as well.

Example 3 ► **Adding Rational Expressions in Application Problems**

An airplane flies m miles with a windspeed of w mph. On the return flight, the airplane flies against the same wind. The expression $\frac{m}{s+w} + \frac{m}{s-w}$, where s is the speed of the airplane in still air, represents the total time in hours that it takes to make the round-trip flight. Write a single rational expression representing the total time.

Solution To find a single rational expression representing the total time, we perform the addition using $(s+w)(s-w)$ as the lowest common denominator. So,

$$\frac{m}{s+w}+\frac{m}{s-w}=\frac{m(s-w)+m(s+w)}{(s+w)(s-w)}=\frac{ms-mw+ms+mw}{(s+w)(s-w)}=\boldsymbol{\frac{2ms}{s^2-w^2}}$$

Example 4 **Adding and Subtracting Rational Functions**

Given $f(x)=\frac{1}{x^2+10x+24}$ and $g(x)=\frac{2}{x^2+4x}$, find

a. $(f+g)(x)$ **b.** $(f-g)(x)$.

Solution

a. $$(\boldsymbol{f}+\boldsymbol{g})(\boldsymbol{x})=f(x)+g(x)=\frac{1}{x^2+10x+24}+\frac{2}{x^2+4x}$$

$$=\frac{1}{(x+6)(x+4)}+\frac{2}{x(x+4)}=\frac{1\cdot x+2(x+6)}{x(x+6)(x+4)}=\frac{x+2x+12}{x(x+6)(x+4)}$$

$$=\frac{3x+12}{x(x+6)(x+4)}=\frac{3(x+4)}{x(x+6)(x+4)}=\boldsymbol{\frac{3}{x(x+6)}}$$

b. $$(\boldsymbol{f}-\boldsymbol{g})(\boldsymbol{x})=f(x)-g(x)=\frac{1}{x^2+10x+24}-\frac{2}{x^2+4x}$$

$$=\frac{1}{(x+6)(x+4)}-\frac{2}{x(x+4)}=\frac{1\cdot x-2(x+6)}{x(x+6)(x+4)}=\frac{x-2x-12}{x(x+6)(x+4)}$$

$$=\boldsymbol{\frac{-x-12}{x(x+6)(x+4)}}$$

RT.3 Exercises

Vocabulary Check *Complete each blank with the most appropriate term from the given list:* ***denominator, different, factor, highest, rational.***

1. To add (or subtract) rational expressions with the same denominator, add (or subtract) the numerators and keep the same __________________.

2. To find the LCD of rational expressions, first ______________ each denominator completely.

3. To add (or subtract) rational expressions with _______________ denominators, first find the LCD for all the involved expressions.

4. The LCD of rational expressions is the product of the different factors in the denominators, where the power of each factor is the __________ number of times that it occurs in any single denominator.

5. A polynomial expression raised to a negative exponent represents a ______________ expression.

Concept Check

6. **a.** What is the LCM for 6 and 9? **b.** What is the LCD for $\frac{1}{6}$ and $\frac{1}{9}$?

7. **a.** What is the LCM for $x^2 - 25$ and $x + 5$? **b.** What is the LCD for $\frac{1}{x^2-25}$ and $\frac{1}{x+5}$?

Concept Check *Find the LCD and then perform the indicated operations. Simplify the resulting fraction.*

8. $\frac{5}{12} + \frac{13}{18}$

9. $\frac{11}{30} - \frac{19}{75}$

10. $\frac{3}{4} + \frac{7}{30} - \frac{1}{16}$

11. $\frac{5}{8} - \frac{7}{12} + \frac{11}{40}$

Concept Check *Find the least common multiple (LCM) for each group of expressions.*

12. $24a^3b^4,\ 18a^5b^2$

13. $6x^2y^2,\ 9x^3y,\ 15y^3$

14. $x^2 - 4,\ x^2 + 2x$

15. $10x^2,\ 25(x^2 - x)$

16. $(x-1)^2,\ 1 - x$

17. $y^2 - 25,\ 5 - y$

18. $x^2 - y^2,\ xy + y^2$

19. $5a - 15,\ a^2 - 6a + 9$

20. $x^2 + 2x + 1,\ x^2 - 4x - 1$

21. $n^2 - 7n + 10,\ n^2 - 8n + 15$

22. $2x^2 - 5x - 3,\ 2x^2 - x - 1,\ x^2 - 6x + 9$

23. $1 - 2x,\ 2x + 1,\ 4x^2 - 1$

24. $x^5 - 4x^4 + 4x^3,\ 12 - 3x^2,\ 2x + 4$

Concept Check *True or false? If true, explain why. If false, correct it.*

25. $\frac{1}{2x} + \frac{1}{3x} = \frac{1}{5x}$

26. $\frac{1}{x-3} + \frac{1}{3-x} = 0$

27. $\frac{1}{x} + \frac{1}{y} = \frac{1}{x+y}$

28. $\frac{3}{4} + \frac{x}{5} = \frac{3+x}{20}$

Perform the indicated operations and simplify if possible.

29. $\frac{x-2y}{x+y} + \frac{3y}{x+y}$

30. $\frac{a+3}{a+1} - \frac{a-5}{a+1}$

31. $\frac{4a+3}{a-3} - 1$

32. $\frac{n+1}{n-2} + 2$

33. $\frac{x^2}{x-y} + \frac{y^2}{y-x}$

34. $\frac{4a-2}{a^2-49} + \frac{5+3a}{49-a^2}$

35. $\frac{2y-3}{y^2-1} - \frac{4-y}{1-y^2}$

36. $\frac{a^3}{a-b} + \frac{b^3}{b-a}$

37. $\frac{1}{x+h} - \frac{1}{h}$

38. $\frac{x-2}{x+3} + \frac{x+2}{x-4}$

39. $\frac{x-1}{3x+1} + \frac{2}{x-3}$

40. $\frac{4xy}{x^2-y^2} + \frac{x-y}{x+y}$

41. $\frac{x-1}{3x+15} - \frac{x+3}{5x+25}$

42. $\frac{y-2}{4y+8} - \frac{y+6}{5y+10}$

43. $\frac{4x}{x-1} - \frac{2}{x+1} - \frac{4}{x^2-1}$

44. $\frac{-2}{y+2} + \frac{5}{y-2} + \frac{y+3}{y^2-4}$

45. $\frac{y}{y^2-y-20} + \frac{2}{y+4}$

46. $\frac{5x}{x^2-6x+8} - \frac{3x}{x^2-x-12}$

47. $\frac{9x+2}{3x^2-2x-8} + \frac{7}{3x^2+x-4}$

48. $\frac{3y+2}{2y^2-y-10} + \frac{8}{2y^2-7y+5}$

49. $\frac{6}{y^2+6y+9} + \frac{5}{y^2-9}$

50. $\frac{3x-1}{x^2+2x-3} - \frac{x+4}{x^2-9}$

51. $\frac{1}{x+1} - \frac{x}{x-2} + \frac{x^2+2}{x^2-x-2}$

52. $\frac{2}{y+3} - \frac{y}{y-1} + \frac{y^2+2}{y^2+2y-3}$

53. $\frac{4x}{x^2-1}+\frac{3x}{1-x}-\frac{4}{x-1}$

54. $\frac{5y}{1-2y}-\frac{2y}{2y+1}+\frac{3}{4y^2-1}$

55. $\frac{x+5}{x-3}-\frac{x+2}{x+1}-\frac{6x+10}{x^2-2x-3}$

Discussion Point

56. Consider the following calculation

$$\frac{x}{x-2}-\frac{4x-1}{x^2-4}=\frac{x(x+2)-4x-1}{x^2-4}=\frac{x^2+2x-4x-1}{x^2-4}=\frac{x^2-2x-1}{x^2-4}$$

Is this correct? If *yes*, check if the result can be simplified. If *no*, correct it.

Perform the indicated operations and simplify if possible.

57. $2x^{-3}+(3x)^{-1}$

58. $(x^2-9)^{-1}+2(x-3)^{-1}$

59. $\left(\frac{x+1}{3}\right)^{-1}-\left(\frac{x-4}{2}\right)^{-1}$

60. $\left(\frac{a-3}{a^2}-\frac{a-3}{9}\right)\div\frac{a^2-9}{3a}$

61. $\frac{x^2-4x+4}{2x+1}\cdot\frac{2x^2+x}{x^3-4x}-\frac{3x-2}{x+1}$

62. $\frac{2}{x-3}-\frac{x}{x^2-x-6}\cdot\frac{x^2-2x-3}{x^2-x}$

Given $f(x)$ and $g(x)$, find $(f+g)(x)$ and $(f-g)(x)$. Leave the answer in simplified single fraction form.

63. $f(x)=\frac{x}{x+2},\ \ g(x)=\frac{4}{x-3}$

64. $f(x)=\frac{x}{x^2-4},\ \ g(x)=\frac{1}{x^2+4x+4}$

65. $f(x)=\frac{3x}{x^2+2x-3},\ \ g(x)=\frac{1}{x^2-2x+1}$

66. $f(x)=x+\frac{1}{x-1},\ \ g(x)=\frac{1}{x+1}$

Solve each problem.

67. Two friends work part-time at a store. The first person works every sixth day and the second person works every tenth day. If they are both working today, how many days pass before they both work on the same day again?

68. Suppose a cylindrical water tank is being drained. The change in the water level can be found using the expression $\frac{V_1}{\pi r^2}-\frac{V_2}{\pi r^2}$, where V_1 and V_2 represent the original and new volume, respectively, and r is the radius of the tank. Write the change in water level as a single algebraic fraction.

69. To determine the percent growth in sales from the previous year, the owner of a company uses the expression $100\left(\frac{S_1}{S_0}-1\right)$, where S_1 represents the current year's sales and S_0 represents last year's sales. Write this expression as a single algebraic fraction.

70. A boat travels d mi against the current whose rate is c mph. On the return trip, the boat travels with the same current. The expression $\frac{d}{r-c}+\frac{d}{r+c}$, where r is the speed of the boat in calm water, represents the total amount of time in hours it takes for the entire boating trip. Represent this amount of time as a single algebraic fraction.