RT.4 Complex Fractions

When working with algebraic expressions, sometimes we come across needing to simplify expressions like these:

$$\frac{\frac{x^2-9}{x+1}}{\frac{x+3}{x^2-1}}, \quad \frac{1+\frac{1}{x}}{1-\frac{1}{y}}, \quad \frac{\frac{1}{x+2}-\frac{1}{x+h+2}}{h}, \quad \frac{1}{\frac{1}{a}-\frac{1}{b}}$$

A complex fraction is a quotient of rational expressions (including sums of rational expressions) where at least one of these expressions contains a fraction itself. In this section, we will examine two methods of simplifying such fractions.

Simplifying Complex Fractions

Definition 4.1 ► A **complex fraction** is a **quotient** of rational expressions (including their sums) that result in a fraction with more than two levels. For example, $\frac{\frac{1}{2}}{3}$ has three levels while $\frac{\frac{1}{2x}}{\frac{3}{4x}}$ has four levels. Such fractions can be **simplified to a single fraction** with only two levels. For example,

$$\frac{\frac{1}{2}}{3} = \frac{1}{2}\cdot\frac{1}{3} = \frac{1}{6}, \qquad or \qquad \frac{\frac{1}{2x}}{\frac{3}{4x^2}} = \frac{1}{2\cancel{x}}\cdot\frac{4x^{\cancel{2}\,1}}{3} = \frac{2x}{3}$$

There are two common methods of simplifying complex fractions.

Method I (***multiplying by the reciprocal of the denominator***)

Replace the main division in the complex fraction with a multiplication of the numerator fraction by the reciprocal of the denominator fraction. We then simplify the resulting fraction if possible. Both examples given in *Definition 4.1* were simplified using this strategy.

Method I is the most convenient to use when both the numerator and the denominator of a complex fraction consist of single fractions. However, if either the numerator or the denominator of a complex fraction contains addition or subtraction of fractions, it is usually easier to use the method shown below.

Method II (***multiplying by LCD***)

Multiply the numerator and denominator of a complex fraction by the least common denominator of all the fractions appearing in the numerator or in the denominator of the complex fraction. Then, simplify the resulting fraction if possible. For example, to simplify $\frac{y+\frac{1}{x}}{x+\frac{1}{y}}$, multiply the numerator $y+\frac{1}{x}$ and the denominator $x+\frac{1}{y}$ by the $LCD\left(\frac{1}{x},\frac{1}{y}\right) = xy$. So,

$$\frac{\left(y+\frac{1}{x}\right)}{\left(x+\frac{1}{y}\right)}\cdot\frac{xy}{xy} = \frac{xy^2+y}{x^2y+x} = \frac{y\cancel{(xy+1)}}{x\cancel{(xy+1)}} = \frac{\boldsymbol{y}}{\boldsymbol{x}}$$

Example 1 ► **Simplifying Complex Fractions**

Use a method of your choice to simplify each complex fraction.

a. $\dfrac{\frac{x^2-x-12}{x^2-2x-15}}{\frac{x^2+8x+12}{x^2-5x-14}}$

b. $\dfrac{a+b}{\frac{1}{a^3}+\frac{1}{b^3}}$

c. $\dfrac{x+\frac{1}{5}}{x-\frac{1}{3}}$

d. $\dfrac{\frac{6}{x^2-4}-\frac{5}{x+2}}{\frac{7}{x^2-4}-\frac{4}{x-2}}$

Solution ►

a. Since the expression $\dfrac{\frac{x^2-x-12}{x^2-2x-15}}{\frac{x^2+8x+12}{x^2-5x-14}}$ contains a single fraction in both the numerator and denominator, we will simplify it using method I, as below.

$$\dfrac{\frac{x^2-2x-8}{x^2-2x-15}}{\frac{x^2+8x+12}{x^2-4x-21}} = \frac{(x-4)\cancel{(x+2)}}{(x-5)\cancel{(x+3)}} \cdot \frac{(x-7)\cancel{(x+3)}}{(x+6)\cancel{(x+2)}} = \boldsymbol{\frac{(x-4)(x-7)}{(x-5)(x+6)}}$$

factor and multiply by the reciprocal

b. $\dfrac{a+b}{\frac{1}{a^3}+\frac{1}{b^3}}$ can be simplified in the following two ways:

Method I

$$\frac{a+b}{\frac{1}{a^3}+\frac{1}{b^3}} = \frac{a+b}{\frac{b^3+a^3}{a^3b^3}} = \frac{(a+b)a^3b^3}{a^3+b^3}$$

$$= \frac{\cancel{(a+b)}a^3b^3}{\cancel{(a+b)}(a^2-ab+b^2)} = \boldsymbol{\frac{a^3b^3}{a^2-ab+b^2}}$$

Method II

$$\frac{a+b}{\frac{1}{a^3}+\frac{1}{b^3}} \cdot \frac{a^3b^3}{a^3b^3} = \frac{(a+b)a^3b^3}{b^3+a^3}$$

$$= \frac{\cancel{(a+b)}a^3b^3}{\cancel{(a+b)}(a^2-ab+b^2)} = \boldsymbol{\frac{a^3b^3}{a^2-ab+b^2}}$$

Caution: In Method II, the factor that we multiply the complex fraction by **must be equal to 1**. This means that **the numerator and denominator of this factor must be exactly the same**.

c. To simplify $\dfrac{x+\frac{1}{5}}{x-\frac{1}{3}}$, we will use method II. Multiplying the numerator and denominator by the $LCD\left(\frac{1}{5},\frac{1}{3}\right)=15$, we obtain

$$\frac{x+\frac{1}{5}}{x-\frac{1}{3}} \cdot \frac{15}{15} = \boldsymbol{\frac{15x+3}{15x-5}}$$

d. Again, to simplify $\frac{\frac{6}{x^2-4}-\frac{5}{x+2}}{\frac{7}{x^2-4}-\frac{4}{x-2}}$, we will use method II. Notice that the lowest common multiple of the denominators in blue is $(x+2)(x-2)$. So, after multiplying the numerator and denominator of the whole expression by the LCD, we obtain

$$\frac{\frac{6}{x^2-4}-\frac{5}{x+2}}{\frac{7}{x^2-4}-\frac{4}{x-2}}\cdot\frac{(x+2)(x-2)}{(x+2)(x-2)}=\frac{6-5(x-2)}{7-4(x+2)}=\frac{6-5x+10}{7-4x-8}$$

$$=\frac{-5x+16}{-4x-1}=\boldsymbol{\frac{5x-16}{4x+1}}$$

Example 2 ► **Simplifying Rational Expressions with Negative Exponents**

Simplify each expression. Leave the answer with only positive exponents.

a. $\frac{x^{-2}-y^{-1}}{y-x}$ **b.** $\frac{a^{-3}}{a^{-1}-b^{-1}}$

Solution ► **a.** If we write the expression with no negative exponents, it becomes a complex fraction, which can be simplified as in *Example 1*. So,

$$\frac{x^{-2}-y^{-1}}{y-x}=\frac{\frac{1}{x}-\frac{1}{y}}{y-x}\cdot\frac{xy}{xy}=\frac{y-x}{xy(y-x)}=\boldsymbol{\frac{1}{xy}}$$

b. As above, first, we rewrite the expression with only positive exponents and then simplify as any other complex fraction.

Remember! This factor must be = 1

$$\frac{a^{-3}}{a^{-1}-b^{-1}}=\frac{\frac{1}{a^3}}{\frac{1}{a}-\frac{1}{b}}\cdot\frac{a^3b}{a^3b}=\frac{b}{a^2b-a^3}=\boldsymbol{\frac{b}{a^2(b-a)}}$$

Example 3 ► **Simplifying the Difference Quotient for a Rational Function**

Find and simplify the expression $\frac{f(a+h)-f(a)}{h}$ for the function $f(x)=\frac{1}{x+1}$.

Solution ► Since $f(a+h)=\frac{1}{a+h+1}$ and $f(a)=\frac{1}{a+1}$, then

$$\frac{f(a+h)-f(a)}{h}=\frac{\frac{1}{a+h+1}-\frac{1}{a+1}}{h}$$

To simplify this expression, we can multiply the numerator and denominator by the lowest common denominator, which is $(a+h+1)(a+1)$. Thus,

This bracket is essential!

$$\frac{\frac{1}{a+h+1}-\frac{1}{a+1}}{h}\cdot\frac{(a+h+1)(a+1)}{(a+h+1)(a+1)}=\frac{a+1-(a+h+1)}{h(a+h+1)(a+1)}$$

$$=\frac{\cancel{a}+\cancel{1}-\cancel{a}-h-\cancel{1}}{h(a+h+1)(a+1)}=\frac{-\cancel{h}}{\cancel{h}(a+h+1)(a+1)}=\frac{\mathbf{-1}}{\mathbf{(a+h+1)(a+1)}}$$

keep the denominator in a factored form

RT.4 Exercises

Vocabulary Check *Complete each blank with the most appropriate term from the given list:* ***complex, LCD, multiply, reciprocal, same, single.***

1. A quotient of two rational expressions that results in a fraction with more than two levels is called a ________________ algebraic fraction.

2. To simplify a complex fraction means to find an equivalent ___________ fraction $\frac{P}{Q}$, where P and Q are polynomials with no essential common factors.

3. To simplify a complex rational expression using the _____________ method, first write both the numerator and the denominator as single fractions in simplified form.

4. To simplify a complex fraction using the _______ method, first find the LCD of all rational expressions within the complex fraction. Then, _______________ the numerator and denominator by the __________ LCD expression.

Concept Check *Simplify each complex fraction.*

5. $\frac{2-\frac{1}{3}}{3+\frac{7}{3}}$

6. $\frac{5-\frac{3}{4}}{4+\frac{1}{2}}$

7. $\frac{\frac{3}{8}-5}{\frac{2}{3}+6}$

8. $\frac{\frac{2}{3}+\frac{4}{5}}{\frac{3}{4}-\frac{1}{2}}$

Simplify each complex rational expression.

9. $\frac{\frac{x^3}{y}}{\frac{x^2}{y^3}}$

10. $\frac{\frac{n-5}{6n}}{\frac{n-5}{8n^2}}$

11. $\frac{1-\frac{1}{a}}{4+\frac{1}{a}}$

12. $\frac{\frac{2}{n}+3}{\frac{5}{n}-6}$

13. $\frac{\frac{9-3x}{4x+12}}{\frac{x-3}{6x-24}}$

14. $\frac{\frac{9}{y}}{\frac{15}{y}-6}$

15. $\frac{\frac{4}{x}-\frac{2}{y}}{\frac{4}{x}+\frac{2}{y}}$

16. $\frac{\frac{3}{a}+\frac{4}{b}}{\frac{4}{a}-\frac{3}{b}}$

17. $\frac{a-\frac{3a}{b}}{b-\frac{b}{a}}$ **18.** $\frac{\frac{1}{x}-\frac{1}{y}}{\frac{x^2-y^2}{xy}}$ **19.** $\frac{\frac{4}{y}-\frac{y}{x^2}}{\frac{1}{x}-\frac{2}{y}}$ **20.** $\frac{\frac{5}{p}-\frac{1}{q}}{\frac{1}{5q^2}-\frac{5}{p^2}}$

21. $\frac{\frac{n-12}{n}+n}{n+4}$ **22.** $\frac{2t-1}{\frac{3t-2}{t}+2t}$ **23.** $\frac{\frac{1}{a-h}-\frac{1}{a}}{h}$ **24.** $\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}$

25. $\frac{4+\frac{12}{2x-3}}{5+\frac{15}{2x-3}}$ **26.** $\frac{1+\frac{3}{x+2}}{1+\frac{6}{x-1}}$ **27.** $\frac{\frac{1}{b^2}-\frac{1}{a^2}}{\frac{1}{b}-\frac{1}{a}}$ **28.** $\frac{\frac{1}{x^2}-\frac{1}{y^2}}{\frac{1}{x}+\frac{1}{y}}$

29. $\frac{\frac{x+3}{x}-\frac{4}{x-1}}{\frac{x}{x-1}+\frac{1}{x}}$ **30.** $\frac{\frac{3}{x^2+6x+9}+\frac{3}{x+3}}{\frac{6}{x^2-9}+\frac{6}{3-x}}$ **31.** $\frac{\frac{1}{a^2}-\frac{1}{b^2}}{\frac{1}{a^3}+\frac{1}{b^3}}$ **32.** $\frac{\frac{4p^2-12p+9}{2p^2+7p-15}}{\frac{2p^2-15p+18}{p^2-p-30}}$

Discussion Point

33. Are the expressions $\frac{x^{-1}+y^{-1}}{x^{-2}+y^{-2}}$ and $\frac{x^2+y^2}{x+y}$ equivalent? Explain why or why not.

Simplify each expression. Leave your answer with only positive exponents.

34. $\frac{1}{a^{-2}-b^{-2}}$ **35.** $\frac{x^{-1}+x^{-2}}{3x^{-1}}$ **36.** $\frac{x^{-2}}{y^{-3}-x^{-3}}$ **37.** $\frac{1-(2n+1)^{-1}}{1+(2n+1)^{-1}}$

Analytic Skills *Find and simplify the difference quotient* $\frac{f(a+h)-f(a)}{h}$ *for the given function.*

38. $f(x)=\frac{5}{x}$ **39.** $f(x)=\frac{2}{x^2}$ **40.** $f(x)=\frac{1}{1-x}$ **41.** $f(x)=-\frac{1}{x-2}$

Analytic Skills *Simplify each continued fraction.*

42. $a-\frac{a}{1-\frac{a}{1-a}}$ **43.** $3-\frac{2}{1-\frac{2}{3-\frac{2}{x}}}$ **44.** $a+\frac{a}{2+\frac{1}{1-\frac{2}{a}}}$