RT.5	**Rational Equations and Graphs**

In previous sections of this chapter, we worked with rational expressions. If two rational expressions are equated, a *rational equation* arises. Such equations often appear when solving application problems that involve rates of work or amounts of time considered in motion problems. In this section, we will discuss how to solve rational equations, with close attention to their domains. We will also take a look at the graphs of reciprocal functions, their properties and transformations.

Rational Equations

Definition 5.1 ► A **rational equation** is an equation involving only rational expressions and containing at least one fractional expression.

Here are some examples of rational equations:

$$\frac{x}{2} - \frac{12}{x} = -1, \qquad \frac{x^2}{x-5} = \frac{25}{x-5}, \qquad \frac{2x}{x-3} - \frac{6}{x} = \frac{18}{x^2-3x}$$

Attention! A rational equation contains an *equals* sign, while a rational expression does not. An equation can be solved for a given variable, while an expression can only be simplified or evaluated. For example, $\frac{x}{2} - \frac{12}{x}$ is an **expression** to simplify, while $\frac{x}{2} = \frac{12}{x}$ is an **equation** to solve.

When working with algebraic structures, it is essential to identify whether they are equations or expressions before applying appropriate strategies.

By *Definition 5.1*, rational equations contain one or more denominators. Since division by zero is not allowed, we need to pay special attention to the variable values that would make any of these denominators equal to zero. Such values would have to be excluded from the set of possible solutions. For example, neither 0 nor 3 can be solutions to the equation

$$\frac{2x}{x-3} - \frac{6}{x} = \frac{18}{x^2-3x},$$

as it is impossible to evaluate either of its sides for $x = 0$ or 3. So, when solving a rational equation, it is important to find its domain first.

Definition 5.2 ► The **domain** of the variable(s) of a rational equation (in short, the **domain of a rational equation**) is the **intersection of the domains** of all rational expressions within the equation.

As stated in *Definition 2.1*, the domain of each single algebraic fraction is the set of all real numbers except for the **zeros of the denominator** (the variable values that would make the denominator equal to zero). Therefore, the **domain of a rational equation is** the set of **all real numbers except for the zeros of all the denominators** appearing in this equation.

Example 1 ► **Determining Domains of Rational Equations**

Find the domain of the variable in each of the given equations.

a. $\frac{x}{2} - \frac{12}{x} = -1$ **b.** $\frac{2x}{x-2} = \frac{-3}{x} + \frac{4}{x-2}$

c. $\frac{2}{y^2-2y-8} - \frac{4}{y^2+6y+8} = \frac{2}{y^2-16}$

Solution ► **a.** The equation $\frac{x}{2} - \frac{12}{x} = -1$ contains two denominators, 2 and x. 2 is never equal to zero and x becomes zero when $x = 0$. Thus, the domain of this equation is $\mathbb{R} \setminus \{0\}$.

b. The equation $\frac{2x}{x-2} = \frac{-3}{x} + \frac{4}{x-2}$ contains two types of denominators, $x - 2$ and x. The $x - 2$ becomes zero when $x = 2$, and x becomes zero when $x = 0$. Thus, the domain of this equation is $\mathbb{R} \setminus \{0, 2\}$.

c. The equation $\frac{2}{y^2-2y-8} - \frac{4}{y^2+6y+8} = \frac{2}{y^2-16}$ contains three different denominators. To find the zeros of these denominators, we solve the following equations by factoring:

$$y^2 - 2y - 8 = 0 \qquad y^2 + 6y + 8 = 0 \qquad y^2 - 16 = 0$$

$$(y - 4)(y + 2) = 0 \qquad (y + 4)(y + 2) = 0 \qquad (y - 4)(y + 4) = 0$$

$$y = 4 \text{ or } y = -2 \qquad y = -4 \text{ or } y = -2 \qquad y = 4 \text{ or } y = -4$$

So, -4, -2, and 4 must be excluded from the domain of this equation. Therefore, the domain $D = \mathbb{R} \setminus \{-4, -2, 4\}$.

To solve a rational equation, it is convenient to clear all the fractions first and then solve the resulting polynomial equation. This can be achieved by multiplying all the terms of the equation by the least common denominator.

Caution! Only **equations**, not expressions, can be changed equivalently by **multiplying** both of their sides by the **LCD**.
Multiplying **expressions** by any number other than 1 creates expressions that are **NOT equivalent** to the original ones. So, avoid multiplying rational expressions by the LCD.

Example 2 ► **Solving Rational Equations**

Solve each equation.

a. $\frac{x}{2} - \frac{12}{x} = -1$ **b.** $\frac{2x}{x-2} = \frac{-3}{x} + \frac{4}{x-2}$

c. $\frac{2}{y^2-2y-8} - \frac{4}{y^2+6y+8} = \frac{2}{y^2-16}$ **d.** $\frac{x-1}{x-3} = \frac{2}{x-3}$

Solution ▶ **a.** The domain of the equation $\frac{x}{2} - \frac{12}{x} = -1$ is the set $\mathbb{R} \setminus \{0\}$, as discussed in *Example 1a*. The $LCM(2, x) = 2x$, so we calculate

$$\frac{x}{2} - \frac{12}{x} = -1 \qquad\qquad / \cdot 2x$$

$$2x \cdot \frac{x}{2} - 2x \cdot \frac{12}{x} = -1 \cdot 2x \qquad\boxed{\text{multiply each term by the } \textbf{LCD}}$$

$$x^2 - 24 = -2x$$

$$x^2 + 2x - 24 = 0$$

$$(x + 6)(x - 4) = 0 \qquad\boxed{\text{factor to find the } \textbf{possible roots}}$$

$$x = -6 \ \text{ or } \ x = 4$$

Since both of these numbers belong to the domain, the solution set of the original equation is $\{\mathbf{-6, 4}\}$.

b. The domain of the equation $\frac{2x}{x-2} = \frac{-3}{x} + \frac{4}{x-2}$ is the set $\mathbb{R} \setminus \{0, 2\}$, as discussed in *Example 1b*. The $LCM(x - 2, x) = x(x - 2)$, so we calculate

$$\frac{2x}{x-2} = \frac{-3}{x} + \frac{4}{x-2} \qquad\qquad / \cdot x(x-2)$$

$$x(x-2) \cdot \frac{2x}{x-2} = \frac{-3}{x} \cdot x(x-2) + \frac{4}{x-2} \cdot x(x-2)$$

$$2x^2 = -3(x-2) + 4x$$

$$2x^2 = -3x + 6 + 4x \qquad\boxed{\begin{array}{c}\text{expand the bracket,}\\\text{collect like terms, and}\\\text{bring the terms over to one side}\end{array}}$$

$$2x^2 - x + 6 = 0$$

$$(2x + 3)(x - 2) = 0 \qquad\boxed{\text{factor to find the } \textbf{possible roots}}$$

$$x = -\frac{3}{2} \ \text{ or } \ x = 2$$

Since 2 is excluded from the domain, there is only one solution to the original equation, $x = -\frac{3}{2}$.

c. The domain of the equation $\frac{2}{y^2-2y-8} - \frac{4}{y^2+6y+8} = \frac{2}{y^2-16}$ is the set $\mathbb{R} \setminus \{-4, -2, 4\}$, as discussed in *Example 1c*. To find the LCD, it is useful to factor the denominators first. Since
$y^2 - 2y - 8 = (y - 4)(y + 2)$,
$y^2 + 6y + 8 = (y + 4)(y + 2)$, and
$y^2 - 16 = (y - 4)(y + 4)$, then the LCD needed to clear the fractions in the original equation is $(y - 4)(y + 4)(y + 2)$. So, we calculate

$$\frac{2}{(y-4)(y+2)} - \frac{4}{(y+4)(y+2)} = \frac{2}{(y-4)(y+4)} \qquad / \cdot (y-4)(y+4)(y+2)$$

$$(y-4)(y+4)(y+2) \cdot \frac{2}{(y-4)(y+2)} - (y-4)(y+4)(y+2) \cdot \frac{4}{(y+4)(y+2)}$$

$$= \frac{2}{(y-4)(y+4)} \cdot (y-4)(y+4)(y+2)$$

$$2(y+4) - 4(y-4) = 2(y+2)$$

$$2y + 8 - 4y + 16 = 2y + 4$$

$$20 = 4y$$

$$y = 5$$

Since 5 is in the domain, this is the true solution.

d. First, we notice that the domain of the equation $\frac{x-1}{x-3} = \frac{2}{x-3}$ is the set $\mathbb{R} \setminus \{3\}$. To solve this equation, we can multiply it by the $LCD = x - 3$, as in the previous examples, or we can apply the method of cross-multiplication, as the equation is a proportion. Here, we show both methods.

Use the method of your choice – either one is fine.

Multiplication by LCD:

$$\frac{x-1}{x-3} = \frac{2}{x-3} \qquad / \cdot (x-3)$$

$$x - 1 = 2$$

this multiplication is permitted as $x - 3 \neq 0$

$$x = 3$$

Cross-multiplication:

$$\frac{x-1}{x-3} \diagdown \frac{2}{x-3}$$

$$(x-1)(x-3) = 2(x-3) \qquad / \div (x-3)$$

$$x - 1 = 2$$

this division is permitted as $x - 3 \neq 0$

$$x = 3$$

Since 3 is excluded from the domain, there is **no solution** to the original equation.

Summary of Solving Rational Equations in One Variable

1. **Determine the domain** of the variable.

2. **Clear** all the **fractions by multiplying** both sides of the equation **by the LCD** of these fractions.

3. **Find possible solutions** by solving the resulting equation.

4. **Check** the possible solutions **against the domain**. The solution set consists of only these possible solutions that belong to the domain.

Graphs of Basic Rational Functions

So far, we discussed operations on rational expressions and solving rational equations. Now, we will look at rational functions, such as

$$f(x) = \frac{1}{x}, \qquad g(x) = \frac{-2}{x+3}, \qquad or \qquad h(x) = \frac{x-3}{x-2}.$$

Definition 5.3 ► A **rational function** is any function that can be written in the form

$$f(x) = \frac{P(x)}{Q(x)},$$

where P and Q are polynomials and Q is not a zero polynomial.

The **domain D_f** of such function f includes all x-values for which $Q(x) \neq 0$.

Example 3 ► **Finding the Domain of a Rational Function**

Find the domain of each function.

    a. $\quad g(x) = \frac{-2}{x+3}$ **b.** $\quad h(x) = \frac{x-3}{x-2}$

Solution ► **a.**   Since $x + 3 = 0$ for $x = -3$, the domain of g is the set of all real numbers except for -3. So, the domain $D_g = \mathbb{R} \setminus \{-3\}$.

           b.   Since $x - 2 = 0$ for $x = 2$, the domain of h is the set of all real numbers except for 2. So, the domain $D_h = \mathbb{R} \setminus \{2\}$.

Note: The subindex f in the notation D_f indicates that the domain is of function f.

To graph a rational function, we usually start by making a table of values. Because the graphs of rational functions are typically nonlinear, it is a good idea to plot at least 3 points on each side of each x-value where the function is undefined. For example, to graph the

x	$f(x)$
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$
0	*undefined*
$-\frac{1}{2}$	-2
-1	-1
-2	$-\frac{1}{2}$

basic rational function, $f(x) = \frac{1}{x}$, called the **reciprocal function**, we evaluate f for a few points to the right of zero and to the left of zero. This is because f is undefined at $x = 0$, which means that the graph of f does not cross the y-axis. After plotting the obtained points, we connect them within each group, to the right of zero and to the left of zero, creating two disjoint curves. To see the shape of each curve clearly, we might need to evaluate f at some additional points.
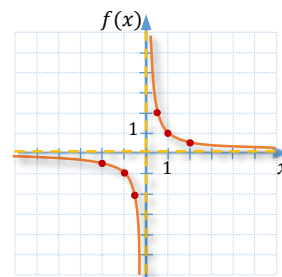
The **domain** of the reciprocal function $f(x) = \frac{1}{x}$ is $\mathbb{R} \setminus \{0\}$, as the denominator x must be different than zero. Projecting the graph of this function onto the y-axis helps us determine the **range**, which is also $\mathbb{R} \setminus \{0\}$.

There is another interesting feature of the graph of the reciprocal function $f(x) = \frac{1}{x}$. Observe that the graph approaches two lines, $y = 0$, the x-axis, and $x = 0$, the y-axis. These lines are called **asymptotes**. They effect the shape of the graph, but they themselves do not belong to the graph. To indicate the fact that asymptotes do not belong to the graph, we use a dashed line when graphing them.

$f(x)$

Vertical Asymptote

In general, if the y-values of a rational function approach ∞ or $-\infty$ as the x-values approach a real number a, the vertical line $x = a$ is a vertical asymptote of the graph. This can be recorded with the use of arrows, as follows:

read: approaches

$x = a$ is a **vertical asymptote** $\Leftrightarrow y \to \infty$ (or $-\infty$) when $x \to a$.

Also, if the y-values approach a real number b as x-values approach ∞ or $-\infty$, the horizontal line $y = b$ is a horizontal asymptote of the graph. Again, using arrows, we can record this statement as:

$f(x)$

Horizontal Asymptote

$y = a$ is a **horizontal asymptote** $\Leftrightarrow y \to b$ when $x \to \infty$ (or $-\infty$).

Example 4 ▶ **Graphing and Analysing the Graphs of Basic Rational Functions**

For each function, state its domain and the equation of the vertical asymptote, graph it, and then state its range and the equation of the horizontal asymptote.

a. $g(x) = \frac{-2}{x+3}$

b. $h(x) = \frac{x-3}{x-2}$

Solution ▶ a. The domain of function $g(x) = \frac{-2}{x+3}$ is $D_g = \mathbb{R} \setminus \{-3\}$, as discussed in *Example 3a*. Since -3 is excluded from the domain, we expect the vertical asymptote to be at $x = -3$.

x	$g(x)$
$-\frac{5}{2}$	-4
-2	-2
-1	-1
1	$-\frac{1}{2}$
-3	undefined
$-\frac{7}{2}$	4
-4	2
-5	1
-6	$\frac{2}{3}$

To graph function g, we evaluate it at some points to the right and to the left of -3. The reader is encouraged to check the values given in the table. Then, we draw the vertical asymptote $x = -3$ and plot and join the obtained points on each side of this asymptote. The graph suggests that the horizontal asymptote is the x-axis. Indeed, the value of zero cannot be attained by the function $g(x) = \frac{-2}{x+3}$, as in order for a fraction to become zero, its numerator would have to be zero. So, the range of function g is $\mathbb{R} \setminus \{0\}$ and $y = 0$ is the equation of the horizontal asymptote.

b. The domain of function $h(x) = \frac{x-3}{x-2}$ is $D_h = \mathbb{R} \setminus \{2\}$, as discussed in *Example 3b*. Since 2 is excluded from the domain, we expect the vertical asymptote to be at $x = 2$.

x	$h(x)$
-1	$\frac{4}{3}$
0	$\frac{3}{2}$
1	2
$\frac{3}{2}$	3
2	*undefined*
$\frac{5}{2}$	-1
3	0
4	$\frac{1}{2}$
6	$\frac{3}{4}$

As before, to graph function h, we evaluate it at some points to the right and to the left of 2. Then, we draw the vertical asymptote $x = 2$ and plot and join the obtained points on each side of this asymptote. The graph suggests that the horizontal asymptote is the line $y = 1$. Thus, the range of function h is $\mathbb{R} \setminus \{1\}$.
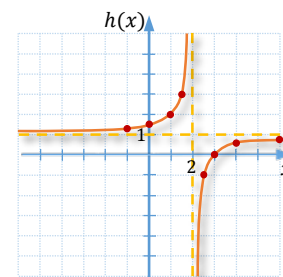
Notice that $\frac{x-3}{x-2} = \frac{x-2-1}{x-2} = \frac{x-2}{x-2} - \frac{1}{x-2} = 1 - \frac{1}{x-2}$. Since $\frac{1}{x-2}$ is never equal to zero than $1 - \frac{1}{x-2}$ is never equal to 1. This confirms the range and the horizontal asymptote stated above.

Example 5 ▶ **Connecting the Algebraic and Graphical Solutions of Rational Equations**

Given that $f(x) = \frac{x+2}{x-1}$, find all the x-values for which $f(x) = 2$. Illustrate the situation with a graph.

Solution ▶ To find all the x-values for which $f(x) = 2$, we replace $f(x)$ in the equation $f(x) = \frac{x+2}{x-1}$ with 2 and solve the resulting equation. So, we have

$$2 = \frac{x+2}{x-1} \qquad\qquad / \cdot (x-1)$$

$$2x - 2 = x + 2 \qquad\qquad / -x, \ +2$$

$$x = 4$$

Thus, $f(x) = 2$ for $x = 4$.
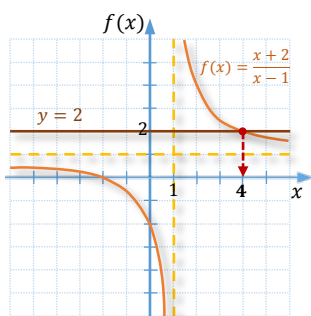
The geometrical connection can be observed by graphing the function $f(x) = \frac{x+2}{x-1} = \frac{x-1+3}{x-1} = 1 + \frac{3}{x-1}$ and the line $y = 2$ on the same grid, as illustrated by the accompanying graph. The x-coordinate of the intersection of the two graphs is the solution to the equation $2 = \frac{x+2}{x-1}$. This also means that $f(4) = \frac{4+2}{4-1} = 2$. So, we can say that $f(4) = 2$.

Example 6 ▶ **Graphing the Reciprocal of a Linear Function**

Suppose $f(x) = 2x - 3$.

a. Determine the reciprocal function $g(x) = \frac{1}{f(x)}$ and its domain D_g.

b. Determine the equation of the vertical asymptote of the reciprocal function g.

c. Graph the function f and its reciprocal function g on the same grid. Then, describe the relations between the two graphs.

Solution ▶ **a.** The reciprocal of $f(x) = 2x - 3$ is the function $g(x) = \frac{1}{2x-3}$. Since $2x - 3 = 0$ for $x = \frac{3}{2}$, then the domain $D_g = \mathbb{R} \setminus \left\{\frac{3}{2}\right\}$.

b. A vertical asymptote of a rational function in simplified form is a vertical line passing through any of the x-values that are excluded from the domain of such a function. So, the equation of the vertical asymptote of function $g(x) = \frac{1}{2x-3}$ is $x = \frac{3}{2}$.

c. To graph functions f and g, we can use a table of values as below.

x	$f(x)$	$g(x)$
$-\frac{1}{2}$	-4	$-\frac{1}{4}$
$\frac{1}{2}$	-2	$-\frac{1}{2}$
1	-1	-1
$\frac{5}{4}$	$\frac{1}{2}$	2
$\frac{3}{2}$	0	undefined
$\frac{7}{4}$	$-\frac{1}{2}$	-2
2	1	1
$\frac{5}{2}$	2	$\frac{1}{2}$
$\frac{7}{2}$	4	$\frac{1}{4}$

$g(x) = \frac{1}{2x-3}$

$f(x) = 2x - 3$
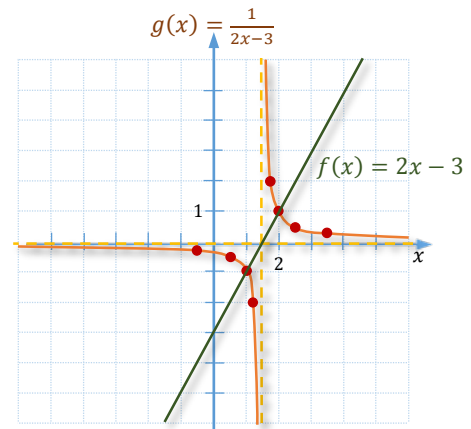
Notice that the vertical asymptote of the reciprocal function comes through the zero of the linear function. Also, the values of both functions are positive to the right of $\frac{3}{2}$ and negative to the left of $\frac{3}{2}$. In addition, $f(2) = g(2) = 1$ and $f(1) = g(1) = -1$. This is because the reciprocal of 1 is 1 and the reciprocal of -1 is -1. For the rest of the values, observe that the values of the linear function that are very close to zero become very large in the reciprocal function and conversely, the values of the linear function that are very far from zero become very close to zero in the reciprocal function. This suggests the horizontal asymptote at zero.

Example 7 ▶ **Using Properties of a Rational Function in an Application Problem**

When curves are designed for train tracks, the outer rail is usually elevated so that a locomotive and its cars can safely take the curve at a higher speed than if the tracks were at the same level. Suppose that a circular curve with a radius of r feet is being designed for a train traveling 60 miles per hour. The function $f(r) = \frac{2540}{r}$ calculates the proper elevation $y = f(r)$, in inches, for the outer rail.

elevation

a. Evaluate $f(300)$ and interpret the result.
b. Suppose that the outer rail for a curve is elevated 6 inches. What should the radius of the curve be?
c. Observe the accompanying graph of the function f and discuss how the elevation of the outer rail changes as the radius r increases.
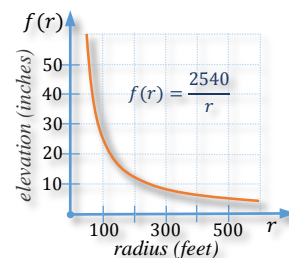
Solution ▶

a. $f(300) = \frac{2540}{300} \approx 8.5$. Thus, the outer rail on a curve with radius 300 ft should be elevated about **8.5 inches** for a train to safely travel through it at 60 miles per hour.

b. Since the elevation $y = f(r) = 6$ inches, to find the corresponding value of r, we need to solve the equation

$$6 = \frac{2540}{r}.$$

After multiplying this equation by r and dividing it by 6, we obtain

$$r = \frac{2540}{6} \approx 423$$

So, the radius of the curve should be about **423 feet**.

c. As the radius increases, the outer rail needs less elevation.

RT.5 Exercises

Vocabulary Check   *Complete each blank with one of the suggested words, or with the most appropriate term from the given list:* ***asymptote, cross-product, domain, equations, LCD, numerator, rational.***

1. A _____ equation involves fractional expressions.

2. To solve a rational equation, first find the _____ of all the rational expressions.

3. Multiplication by any quantity other than zero can be used to create equivalent _____.

4. When simplifying rational expressions, we _____ multiply by LCD.
   $\underset{can\,/\,cannot}{}$

5. The _____ of a rational function is the set of all real numbers except for the variable values that would make the denominator of this function equal to zero.

6. An _____ is a line that a given graph approaches in a long run.

7. A fractional expression is equal to zero when its _____ is equal to zero.

8. To solve a rational proportion we can use either the LCD method or the _____ method.

_____
Rational Equations and Graphs

Concept Check   State the **domain** for each equation. There is no need to solve it.

9. $\dfrac{x+5}{4} - \dfrac{x+3}{3} = \dfrac{x}{6}$

10. $\dfrac{5}{6a} - \dfrac{a}{4} = \dfrac{8}{2a}$

11. $\dfrac{3}{x+4} = \dfrac{2}{x-9}$

12. $\dfrac{4}{3x-5} + \dfrac{2}{x} = \dfrac{9}{4x+7}$

13. $\dfrac{4}{y^2-25} - \dfrac{1}{y+5} = \dfrac{2}{y-7}$

14. $\dfrac{x}{2x-6} - \dfrac{3}{x^2-6x+9} = \dfrac{x-2}{3x-9}$

Solve each equation.

15. $\dfrac{3}{8} + \dfrac{1}{3} = \dfrac{x}{12}$

16. $\dfrac{1}{4} - \dfrac{5}{6} = \dfrac{1}{y}$

17. $x + \dfrac{8}{x} = -9$

18. $\dfrac{4}{3a} - \dfrac{3}{a} = \dfrac{10}{3}$

19. $\dfrac{r}{8} + \dfrac{r-4}{12} = \dfrac{r}{24}$

20. $\dfrac{n-2}{2} - \dfrac{n}{6} = \dfrac{4n}{9}$

21. $\dfrac{5}{r+20} = \dfrac{3}{r}$

22. $\dfrac{5}{a+4} = \dfrac{3}{a-2}$

23. $\dfrac{y+2}{y} = \dfrac{5}{3}$

24. $\dfrac{x-4}{x+6} = \dfrac{2x+3}{2x-1}$

25. $\dfrac{x}{x-1} - \dfrac{x^2}{x-1} = 5$

26. $3 - \dfrac{12}{x^2} = \dfrac{5}{x}$

27. $\dfrac{1}{3} - \dfrac{x-1}{x} = \dfrac{x}{3}$

28. $\dfrac{1}{x} + \dfrac{2}{x+10} = \dfrac{x}{x+10}$

29. $\dfrac{1}{y-1} + \dfrac{5}{12} = \dfrac{-2}{3y-3}$

30. $\dfrac{7}{6x+3} - \dfrac{1}{3} = \dfrac{2}{2x+1}$

31. $\dfrac{8}{3k+9} - \dfrac{8}{15} = \dfrac{2}{5k+15}$

32. $\dfrac{6}{m-4} + \dfrac{5}{m} = \dfrac{2}{m^2-4m}$

33. $\dfrac{3}{y-2} + \dfrac{2y}{4-y^2} = \dfrac{5}{y+2}$

34. $\dfrac{x}{x-2} + \dfrac{x}{x^2-4} = \dfrac{x+3}{x+2}$

35. $\dfrac{1}{2x+10} = \dfrac{8}{x^2-25} - \dfrac{2}{x-5}$

36. $\dfrac{5}{y+3} = \dfrac{1}{4y^2-36} + \dfrac{2}{y-3}$

37. $\dfrac{6}{x^2-4x+3} - \dfrac{1}{x-3} = \dfrac{1}{4x-4}$

38. $\dfrac{7}{x-2} - \dfrac{8}{x+5} = \dfrac{1}{2x^2+6x-20}$

39. $\dfrac{5}{x-4} - \dfrac{3}{x-1} = \dfrac{x^2-1}{x^2-5x+4}$

40. $\dfrac{y}{y+1} + \dfrac{3y+5}{y^2+4y+3} = \dfrac{2}{y+3}$

41. $\dfrac{3x}{x+2} + \dfrac{72}{x^3+8} = \dfrac{24}{x^2-2x+4}$

42. $\dfrac{4}{x+3} + \dfrac{7}{x^2-3x+9} = \dfrac{108}{x^3+27}$

43. $\dfrac{x}{2x-9} - 3x = \dfrac{10}{9-2x}$

44. $\dfrac{-2}{x^2+2x-3} - \dfrac{5}{3-3x} = \dfrac{4}{3x+9}$

For the given rational function f, find all values of x for which $f(x)$ has the indicated value.

45. $f(x) = 2x - \dfrac{15}{x}$; $f(x) = 1$

46. $f(x) = \dfrac{x-5}{x+1}$; $f(x) = \dfrac{3}{5}$

47. $g(x) = \dfrac{-3x}{x+3} + x$; $g(x) = 4$

48. $g(x) = \dfrac{4}{x} + \dfrac{1}{x-2}$; $g(x) = 3$

*Graph each rational function. State its **domain**, **range** and the equations of the **vertical** and **horizontal** asymptotes.*

49. $f(x) = \frac{2}{x}$

50. $g(x) = -\frac{1}{x}$

51. $h(x) = \frac{2}{x-3}$

52. $f(x) = \frac{-1}{x+1}$

53. $g(x) = \frac{x-1}{x+2}$

54. $h(x) = \frac{x+2}{x-3}$

Analytic Skills *For each function f, find its reciprocal function $g(x) = \frac{1}{f(x)}$ and graph both functions on the same grid. Then, state the equations of the vertical and horizontal asymptotes of function g.*

55. $f(x) = \frac{1}{2}x + 1$

56. $f(x) = -x + 2$

57. $f(x) = -2x - 3$

Analytic Skills *Solve each equation.*

58. $\dfrac{x}{1 + \frac{1}{x+1}} = x - 3$

59. $\dfrac{2 - \frac{1}{x}}{4 - \frac{1}{x^2}} = 1$

Solve each problem.

60. The average number of vehicles waiting in line to enter a parking area is modeled by the function defined by

$$w(x) = \frac{x^2}{2(1-x)},$$

where x is a quantity between 0 and 1 known as the ***traffic intensity***.

a. For each traffic intensity, find the average number of vehicles waiting. *Round the answer to the nearest one.*
   i. 0.1   **ii.** 0.8   **iii.** 0.9

b. What happens to the number of vehicles waiting in line as traffic intensity increases?

61. The percent of deaths caused by smoking, called the ***incidence rate***, is modeled by the rational function

$$p(x) = \frac{x - 1}{x},$$

where x is the number of times a smoker is more likely to die of lung cancer than a non-smoker is. For example, $x = 10$ means that a smoker is 10 times more likely than a non-smoker to die from lung cancer.

a. Find $p(x)$ if x is 10.
b. For what values of x is $p(x) = 80\%$? (*Hint:* Change 80% to a decimal.)
c. Can the incidence rate equal 0? Explain.
d. Can the incidence rate equal 1? Explain.