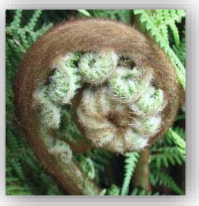
RT.6 Applications of Rational Equations



In previous sections of this chapter, we studied operations on rational expressions, simplifying complex fractions, and solving rational equations. These skills are needed when working with real-world problems that lead to a rational equation. The common types of such problems are motion or work problems. In this section, we first discuss how to solve a rational formula for a given variable, and then present several examples of application problems involving rational equations.

Formulas Containing Rational Expressions

Solving application problems often involves working with formulas. We might need to form a formula, evaluate it, or solve it for a desired variable. The basic strategies used to solve a formula for a variable were shown in *section L2* and *F4.* Recall the guidelines that we used to isolate the desired variable:

- **Reverse operations** to clear unwanted factors or addends;
  Example: To solve $\frac{A+B}{2}=C$ for A, we multiply by 2 and then subtract B.
- **Multiply by the LCD to keep** the desired variable **in the numerator**;
  Example: To solve $\frac{A}{1+r}=P$ for r, first, we multiply by $(1+r)$.
- **Take the reciprocal** of both sides of the equation **to keep** the desired variable **in the numerator** (this applies to proportions only);
  Example: To solve $\frac{1}{C}=\frac{A+B}{AB}$ for C, we can take the reciprocal of both sides to obtain $C=\frac{AB}{A+B}$.
- **Factor** to **keep** the desired variable **in one place**.
  Example: To solve $P+Prt=A$ for P, we first factor P out.

Below we show how to solve formulas containing rational expressions, using a combination of the above strategies.

Example 1 ► **Solving Rational Formulas for a Given Variable**

Solve each formula for the indicated variable.

a. $\frac{1}{f}=\frac{1}{p}+\frac{1}{q}$, for p **b.** $L=\frac{dR}{D-d}$, for D **c.** $L=\frac{dR}{D-d}$, for d

Solution ► **a.** ***Solution I:*** First, we isolate the term containing p, by 'moving' $\frac{1}{q}$ to the other side of the equation. So,

$$\frac{1}{f}=\frac{1}{p}+\frac{1}{q} \qquad / -\frac{1}{q}$$

$$\frac{1}{f}-\frac{1}{q}=\frac{1}{p}$$

$$\frac{1}{p}=\frac{q-f}{fq}$$

rewrite from the right to the left,

and perform the subtraction to leave this side as a single fraction

Then, to bring p to the numerator, we can take the reciprocal of both sides of the equation, obtaining

$$\boldsymbol{p = \frac{fq}{q-f}}$$

Caution! This method can be applied only to a proportion (an equation with a **single fraction on each side**).

Solution II: The same result can be achieved by multiplying the original equation by the $LCD = fpq$, as shown below

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{q} \qquad / \cdot fpq$$

$$pq = fq + fp \qquad / -fp$$

$$pq - fp = fq$$

$$p(q-f) = fq \qquad / \div (q-f)$$

$$\boldsymbol{p = \frac{fq}{q-f}}$$

factor p out

b. To solve $L = \frac{dR}{D-d}$ for D, we may start with multiplying the equation by the denominator to bring the variable D to the numerator. So,

$$L = \frac{dR}{D-d} \qquad / \cdot (D-d)$$

$$L(D-d) = d \qquad / \div L$$

$$D - d = \frac{dR}{L} \qquad / +d$$

$$\boldsymbol{D = \frac{dR}{L} + d = \frac{dR + dL}{L}}$$

This can be done in one step by interchanging L with $D-d$. The movement of the expressions resembles that of a teeter-totter.

Both forms are correct answers.

c. When solving $L = \frac{dR}{D-d}$ for d, we first observe that the variable d appears in both the numerator and denominator. Similarly as in the previous example, we bring the d from the denominator to the numerator by multiplying the formula by the denominator $D - d$. Thus,

$$L = \frac{dR}{D-d} \qquad / \cdot (D-d)$$

$$L(D-d) = dR.$$

Then, to keep the d in one place, we need to expand the bracket, collect terms with d, and finally factor the d out. So, we have

$$LD - Ld = dR \qquad /+Ld$$

$$LD = dR + Ld$$

$$LD = d(R + L) \qquad /\div (R + L)$$

$$\frac{LD}{R + L} = d$$

Obviously, the final formula can be written starting with d,

$$\boldsymbol{d = \frac{LD}{R + L}}.$$

Example 2 ► **Forming and Evaluating a Rational Formula**

Suppose a trip consists of two parts, each of length x.

a. Find a formula for determining the average speed v for the whole trip, given the speed v_1 for the first part of the trip and v_2 for the second part of the trip.

b. Find the average speed v for the whole trip, if the speed for the first part of the trip was 60 km/h and the speed for the second part of the trip was 90 km/h.

Solution ► **a.** The total distance, d, for the whole trip is $x + x = 2x$. The total time, t, for the whole trip is the sum of the times for the two parts of the trip, t_1 and t_2. From the relation $rate \cdot time = distance$, we have

$$t_1 = \frac{x}{v_1} \quad \text{and} \quad t_2 = \frac{x}{v_2}.$$

Therefore,

$$t = \frac{x}{v_1} + \frac{x}{v_2},$$

which after substituting to the formula for the average speed, $v = \frac{d}{t}$, gives us

$$v = \frac{2x}{\frac{x}{v_1} + \frac{x}{v_2}}.$$

Since the formula involves a complex fraction, it should be simplified. We can do this by multiplying the numerator and denominator by the $LCD = v_1 v_2$. So, we have

$$v = \frac{2x}{\frac{x}{v_1} + \frac{x}{v_2}} \cdot \frac{v_1 v_2}{v_1 v_2}$$

$$v = \frac{2x v_1 v_2}{\frac{x \cancel{v_1} v_2}{\cancel{v_1}} + \frac{x v_1 \cancel{v_2}}{\cancel{v_2}}}$$

$$v = \frac{2x v_1 v_2}{x v_2 + x v_1}$$ factor the x

$$v = \frac{2\cancel{x}v_1v_2}{\cancel{x}(v_2+v_1)}$$

$$\boldsymbol{v = \frac{2v_1v_2}{v_2+v_1}}$$

Note 1: The average speed in this formula does not depend on the distance travelled.

Note 2: The average speed for the total trip is not the average (arithmetic mean) of the speeds for each part of the trip. In fact, this formula represents the **harmonic mean** of the two speeds.

b. Since $v_1 = 60$ km/h and $v_2 = 90$km/h, using the formula developed in *Example 2a*, we calculate

$$v = \frac{2 \cdot 60 \cdot 90}{60+90} = \frac{10800}{150} = \mathbf{72\ km/h}$$

Observation: Notice that the average speed for the whole trip is lower than the average of the speeds for each part of the trip, which is $\frac{60+90}{2} = 75$ km/h.

Applied Problems

Many types of application problems were already introduced in *sections L3* and *E2*. Some of these types, for example motion problems, may involve solving rational equations. Below we show examples of proportion and motion problems as well as introduce another type of problems, work problems.

Proportion Problems

When forming a proportion,

$$\frac{category\ I\ before}{category\ II\ before} = \frac{category\ I\ after}{category\ II\ after},$$

it is essential that the same type of data are placed in the same row or the same column.

Recall: To solve a proportion

$$\frac{a}{b} = \frac{c}{d},$$

for example, for a, it is enough to multiply the equation by b. This gives us

$$\boldsymbol{a = \frac{bc}{d}}.$$

Similarly, to solve

$$\frac{a}{b} = \frac{c}{d}$$

for b, we can use the cross-multiplication method, which eventually (we encourage the reader to check this) leads us to

$$a = \frac{ad}{c}.$$

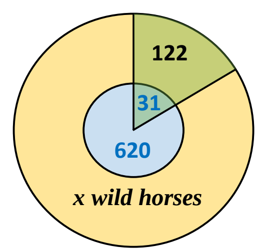
Notice that in both cases the desired variable equals the **product** of the blue variables lying **across** each other, **divided by the remaining** purple variable. This is often referred to as the 'cross multiply and divide' approach to solving a proportion.

In statistics, proportions are often used to estimate the population by analysing its sample in situations where the exact count of the population is too costly or not possible to obtain.

Example 3 ► **Estimating Numbers of Wild Animals**

To estimate the number of wild horses in Utah, a forest ranger catches 620 wild horses, tags them, and releases them. Later, 122 horses are caught and it is found that 31 of them are tagged. Assuming that the horses mix freely when they are released, estimate how many wild horses there are in Utah.

Solution ►



Suppose there are x wild horses in Utah. 620 of them were tagged, so the ratio of the tagged horses in the whole population of the wild horses in Utah is

$$\frac{620}{x}$$

The ratio of the tagged horses found in the sample of 122 horses caught in the later time is

$$\frac{31}{122}$$

So, we form the proportion:

$$\frac{620}{x} = \frac{31}{122}$$

tagged horses
all horses
population
sample

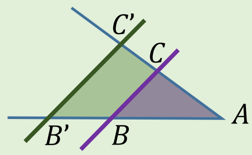
After solving for x, we have

$$x = \frac{620 \cdot 122}{31} = \mathbf{2440}$$

So, we can estimate that over 2400 wild horses live in Utah.

In geometry, proportions are the defining properties of similar figures. One frequently used theorem that involves proportions is the theorem about similar triangles, attributed to the Greek mathematician Thales.

Thales' Theorem ► Two triangles are **similar** iff the ratios of the corresponding sides are the same.



$$\triangle ABC \sim \triangle AB'C' \Leftrightarrow \frac{AB}{AB'} = \frac{AC}{AC'} = \frac{BC}{B'C'}$$

Example 4 ► **Using Similar Triangles in an Application Problem**

A cross-section of a small storage room is in the shape of a right triangle with a height of 2 meters and a base of 1.2 meters, as shown in *Figure 6.1*. What is the largest cubic box that can fit in this room? *Assume that the base of the box is positioned on the floor of the storage room.*

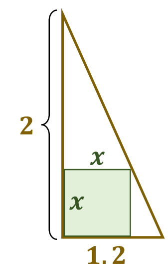


Figure 6.1a

Solution ► Suppose that the height of the box is x meters. Since the height of the storage room is 2 meters, the expression $2 - x$ represents the height of the wall above the box, as shown in *Figure 6.1b*.

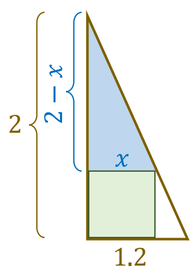


Figure 6.1b

Since the blue and brown triangles are similar, we can use the Thales' Theorem to form the proportion

$$\frac{2-x}{2} = \frac{x}{1.2}.$$

Employing cross-multiplication, we obtain

$$2.4 - 1.2x = 2x$$

$$2.4 = 3.2x$$

$$x = \frac{2.4}{3.2} = \mathbf{0.75}$$

So, the dimensions of the largest cubic box fitting in this storage room are 75 cm by 75 cm by 75 cm.

Motion Problems

Motion problems in which we compare times usually involve solving rational equations. This is because when solving the motion formula $rate\ \boldsymbol{R} \cdot time\ \boldsymbol{T} = distance\ \boldsymbol{D}$ for time, we create a fraction

$$time\ \boldsymbol{T} = \frac{distance\ \boldsymbol{D}}{rate\ \boldsymbol{R}}$$

Example 5 ► Solving a Motion Problem Where Times are the Same

The speed of one mountain biker is 3 km/h faster than the speed of another biker. The first biker travels 40 km in the same amount of time that it takes the second to travel 30 km. Find the speed of each biker.

Solution ► Let r represent the speed of the slower biker. Then $r+3$ represents the speed of the faster biker. The slower biker travels 30 km, while the faster biker travels 40 km. Now, we can complete the table

	R ·	T	= D
slower biker	r	$\frac{30}{r}$	30
faster biker	$r+3$	$\frac{40}{r+3}$	40

To complete the ***Time*** column, we divide the ***Distance*** by the ***Rate***.

Since the time of travel is the same for both bikers, we form and then solve the equation:

$$\frac{30}{r} = \frac{40}{r+3} \qquad /\div 10 \text{ and cross-multiply}$$

$$3(r+3) = 4r$$

$$3r + 9 = 4r \qquad /-3r$$

$$r = 9$$

Thus, the speed of the slower biker is $r = \mathbf{9}$ km/h and the speed of the faster biker is $r + 3 = \mathbf{12}$ km/h.

Example 6 ► Solving a Motion Problem Where the Total Time is Given

Kris and Pat are driving from Vancouver to Princeton, a distance of 297 km. Kris, whose average rate is 6 mph faster than Pat's, will drive the first 153 km of the trip, and then Pat will drive the rest of the way to their destination. If the total driving time is 3 hours, determine the average rate of each driver.

Solution ► Let r represent Pat's average rate. Then $r+6$ represents Kris' average rate. Kris travelled 153 km, while Pat travelled $297 - 153 = 144$ km. Now, we can complete the table:

	R ·	T	= D
Kris	$r+6$	$\frac{153}{r+6}$	153
Pat	r	$\frac{144}{r}$	144
total		3	297

Note: In motion problems we may add ***times*** or ***distances*** but we usually **do not add rates!**

The equation to solve comes from the ***Time*** column.

$$\frac{153}{r+6}+\frac{144}{r}=3 \qquad /\cdot r(r+6)$$

$$153r+144(r+6)=3r(r+6) \qquad /\div 3 \text{ and distribute; then collect like terms on one side}$$

$$51r+48r+288=r^2+6r$$

$$0=r^2-93r-288$$

$$(r-96)(r+3)=0 \qquad \text{factor}$$

$$r=96 \quad or \quad r=-3$$

Since a rate cannot be negative, we discard the solution $r=-3$. Therefore, Pat's average rate was $r=\mathbf{96}$ km/h and Kris' average rate was $r+6=\mathbf{102}$ km/h.

Work Problems

Notice the similarity to the formula $R\cdot T=D$ used in motion problems.

When solving work problems, refer to the formula

$$\boldsymbol{Rate} \textit{ of work} \cdot \boldsymbol{Time} = \textit{amount of } \boldsymbol{Job} \textit{ completed}$$

and organize data in a table like this:

	R ·	***T***	= ***J***
worker I			
worker II			
together			

Note: In work problems we usually *add rates* but **do not add times!**

Example 7 ► **Solving a Work Problem Involving Addition of Rates**

Alex can trim the shrubs at Beecher Community College in 6 hr. Bruce can do the same job in 4 hr. How long would it take them to complete the same trimming job if they work together?

Solution ► Let t be the time needed to trim the shrubs when Alex and Bruce work together. Since trimming the shrubs at Beecher Community College is considered to be the whole one job to complete, then the rate R in which this work is done equals

$$\boldsymbol{R}=\frac{\boldsymbol{Job}}{\boldsymbol{Time}}=\frac{\mathbf{1}}{\boldsymbol{Time}}.$$

To organize the information, we can complete the table below.

	R ·	T	= J
Alex	$\frac{1}{6}$	6	1
Bruce	$\frac{1}{4}$	4	1
together	$\frac{1}{t}$	t	1

The job column is often equal to **1**, although sometimes other values might need to be used.

To complete the ***Rate*** column, we divide the ***Job*** by the ***Time***.

Since the rate of work when both Alex and Bruce trim the shrubs is the sum of rates of individual workers, we form and solve the equation

$$\frac{1}{6}+\frac{1}{4}=\frac{1}{t} \qquad /\cdot 12t$$

$$2t+3t=12$$

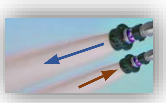
$$5t=12 \qquad /\div 5$$

$$t=\frac{12}{5}=2.4$$

So, if both Alex and Bruce work together, the amount of time needed to complete the job if 2.4 hours = **2 hours 24 minutes**.

Note: The time needed for both workers is **shorter** than either of the individual times.

Example 8 ► **Solving a Work Problem Involving Subtraction of Rates**



One pipe can fill a swimming pool in 10 hours, while another pipe can empty the pool in 15 hours. How long would it take to fill the pool if both pipes were left open?

Solution ► Suppose t is the time needed to fill the pool when both pipes are left open. If filling the pool is the whole one job to complete, then emptying the pool corresponds to -1 job. This is because when emptying the pool, we reverse the filling job.

To organize the information given in the problem, we complete the following table.

	R ·	T	= J
pipe I	$\frac{1}{10}$	10	1
pipe II	$-\frac{1}{15}$	15	-1
both pipes	$\frac{1}{t}$	t	1

The equation to solve comes from the ***Rate*** column.

$$\frac{1}{10} - \frac{1}{15} = \frac{1}{t} \qquad /\cdot 30t$$

$$3t - 2t = 30$$

$$t = 30$$

So, it will take **30 hours** to fill the pool when both pipes are left open.

RT.6 Exercises

Vocabulary Check *Complete each blank with the most appropriate term or phrase from the given list:* ***column, fraction, motion, numerator, proportions, row, similar, work.***

1. When solving a formula for a specified variable, we want to keep the variable in the _______________.

2. The strategy of taking reciprocal of each side of an equation is applicable only in solving _________________. This means that each side of such equation must be in the form of a single __________.

3. When forming a proportion, it is essential that the same type of data are placed in the same _________ or in the same ____________.

4. The Thales' Theorem states that for any two ____________ triangles the ratios of the corresponding sides are the same.

5. In ___________ problems, we usually add times or distances but not the rates.

6. In ___________ problems, we usually add rates but not times.

Concept Check

7. Using the formula $\frac{1}{a} = \frac{1}{b} + \frac{1}{c}$, find b if $a = 8$ and $c = 12$.

8. The gravitational force between two masses is given by the formula $F = \frac{GMm}{d^2}$.
Find M if $F = 10$, $G = 6.67 \cdot 10^{-11}$, $m = 1$, and $d = 3 \cdot 10^{-6}$. *Round your answer to two decimal places.*

Concept Check

9. What is the first step in solving the formula $rp - rq = p + q$ for r?

10. What is the first step in solving the formula $m = \frac{ab}{a-b}$ for a?

Solve each formula for the specified variable.

11. $m = \frac{F}{a}$ for a

12. $I = \frac{E}{R}$ for R

13. $\frac{W_1}{W_2} = \frac{d_1}{d_2}$ for d_1

14. $F = \frac{GMm}{d^2}$ for m

15. $s = \frac{(v_1+v_2)t}{2}$ for t

16. $s = \frac{(v_1+v_2)t}{2}$ for v_1

17. $\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2}$ for R

18. $\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2}$ for r_1

19. $\frac{1}{p} + \frac{1}{q} = \frac{1}{f}$ for q

20. $\frac{t}{a} + \frac{t}{b} = 1$ for a

21. $\frac{PV}{T} = \frac{pv}{t}$ for v

22. $\frac{PV}{T} = \frac{pv}{t}$ for T

23. $A = \frac{h(a+b)}{2}$ for b

24. $a = \frac{V-v}{t}$ for V

25. $R = \frac{gs}{g+s}$ for s

26. $I = \frac{2V}{V+2r}$ for V

27. $I = \frac{nE}{E+nr}$ for n

28. $\frac{E}{e} = \frac{R+r}{r}$ for e

29. $\frac{E}{e} = \frac{R+r}{r}$ for r

30. $S = \frac{H}{m(t_1-t_2)}$ for t_1

31. $V = \frac{\pi h^2(3R-h)}{3}$ for R

32. $P = \frac{A}{1+r}$ for r

33. $\frac{V^2}{R^2} = \frac{2g}{R+h}$ for h

34. $v = \frac{d_2-d_1}{t_2-t_1}$ for t_2

Analytic Skills *Solve each problem*

35. The ratio of the weight of an object on the moon to the weight of an object on Earth is 0.16 to 1. How much will an 80-kg astronaut weigh on the moon?

36. A rope is 24 meters long. How can the rope be cut in such a way that the ratio of the resulting two segments is 3 to 5?

37. Walking 4 miles in 2 hours will use up 650 calories. Walking at the same rate, how many miles would a person need to walk to lose 1 lb? (Burning 3500 calories is equivalent to losing 1 pound.) *Round to the nearest hundredth.*

38. On a map of Canada, the linear distance between Vancouver and Calgary is 1.8 cm. The airline distance between the two cities is about 675 kilometers. On this same map, what would be the linear distance between Calgary and Montreal if the airline distance between the two cities is approximately 3000 kilometers?

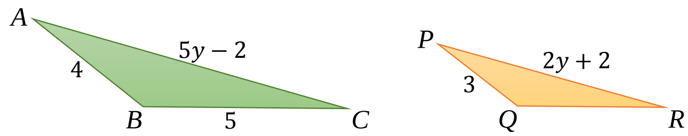
39. To estimate the deer population of a forest preserve, wildlife biologists caught, tagged, and then released 42 deer. A month later, they returned and caught a sample of 75 deer and found that 15 of them were tagged. Based on this experiment, approximately how many deer lived in the forest preserve?

40. Biologists tagged 500 fish in a lake on January 1. On February 1, they returned and collected a random sample of 400 fish, 8 of which had been previously tagged. On the basis of this experiment, approximately how many fish does the lake have?

41. Twenty-two bald eagles are tagged and released into the wilderness. Later, an observed sample of 56 bald eagles contains 7 eagles that are tagged. Estimate the bald eagle population in this wilderness area.

42. A 6–foot tall person casts a 4–foot long shadow. If a nearby tree casts a 44–foot long shadow, estimate the height of the tree.

43. Suppose the following triangles are similar. Find y and the lengths of the unknown sides of each triangle.



44. A rectangle has sides of 9 cm and 14 cm. In a similar rectangle the longer side is 8 cm. What is the length of the shorter side?

45. What is the average speed if Shane runs 18 kilometers per hour for the first half of a race and 22 kilometers per hour for the second half of the race?

46. If you average x kilometers per hour during the first half of a trip, find the speed y in kilometers per hour needed during the second half of the trip to reach 80 kilometers per hour as an overall average.

47. Allen's boat goes 12 mph. Find the rate of the current of the river if he can go 6 mi upstream in the same amount of time he can go 10 mi downstream.

48. A plane averaged 500 mph on a trip going east, but only 350 mph on the return trip. The total flying time in both directions was 8.5 hr. What was the one-way distance?

49. A Boeing 747 flies 2420 mi with the wind. In the same amount of time, it can fly 2140 mi against the wind. The cruising speed (in still air) is 570 mph. Find the speed of the wind.

50. A moving sidewalk moves at a rate of 1.7 ft/sec. Walking on the moving sidewalk, Brenda can travel 120 ft forward in the same time it takes to travel 52 ft in the opposite direction. How fast would Brenda be walking on a non-moving sidewalk?

51. On his drive from Montpelier, Vermont, to Columbia, South Carolina, Arthur averaged 51 mph. If he had been able to average 60 mph, he would have reached his destination 3 hr earlier. What is the driving distance between Montpelier and Columbia?

52. On the first part of a trip to Carmel traveling on the freeway, Margaret averaged 60 mph. On the rest of the trip, which was 10 mi longer than the first part, she averaged 50 mph. Find the total distance to Carmel if the second part of the trip took 30 min more than the first part.

53. Barb is a college professor who lives in an off-campus apartment. On days when she rides her bike to campus, she gets to her first class 36 min faster than when she walks. If her average walking rate is 3 mph and her average biking rate is 12 mph, how far is it from her apartment to her first class?

54. Kate can file all of the daily invoices in 4 hr and Ben can do the same job in 6 hr. If they work together, then what portion of the invoices can they file in 1 hr?

55. Stephanie can paint the entire house in x hours and Tim can do the same job in y hours. Write a rational expression that represents the portion of the house that they can paint in 2 hr working together.

56. John and Nessa are asked to paint a house. John can paint the house by himself in 12 hours and Nessa can paint the house by herself in 16 hours. How long would it take to paint the house if they worked together?

57. Brian, Mark, and Jeff are painting a house. Working together they can paint the house is 6 hours. Working alone Brain can paint the house in 15 hours and Jeff can paint the house in 20 hours. How long would it take Mark to paint the house working alone?

58. An experienced carpenter can frame a house twice as fast as an apprentice. Working together, it takes the carpenters 2 days. How long would it take the apprentice working alone?

59. A tank can be filled in 9 hr and drained in 11 hr. How long will it take to fill the tank if the drain is left open?

60. A cold water faucet can fill the bath tub in 12 minutes, and a hot water faucet can fill the bath tub in 18 minutes. The drain can empty the bath tub in 24 minutes. If both faucets are on and the drain is open, how long would it take to fill the bath tub?

61. Together, a 100-cm wide escalator and a 60-cm wide escalator can empty a 1575-person auditorium in 14 min. The wider escalator moves twice as many people as the narrower one does. How many people per hour does the 60-cm wide escalator move?

Discussion Point

62. At what time after 4:00 will the minute hand overlap the hour hand of a clock for the first time?

63. Michelle drives to work at 50 mph and arrives 1 min late. When she drives to work at 60 mph, she arrives 5 min early. How far does Michelle live from work?