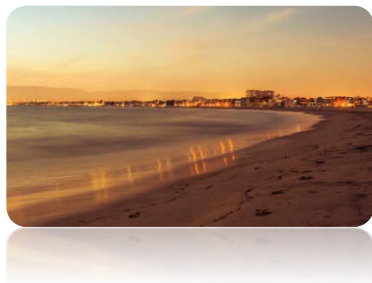


Rational Expressions and Functions



In the previous two chapters we discussed algebraic expressions, equations, and functions related to polynomials. In this chapter, we will examine a broader category of algebraic expressions, *rational expressions*, also referred to as *algebraic fractions*. Similarly as in arithmetic, where a rational number is a quotient of two integers with a denominator that is different than zero, a rational expression is a quotient of two polynomials, also with a denominator that is different than zero.

We start by introducing the related topic of integral exponents, including scientific notation. Then, we discuss operations on algebraic fractions, solving rational equations, and properties and graphs of rational functions with an emphasis on such features as domain, range, and asymptotes. At the end of this chapter, we show examples of applied problems, including work problems, that require solving rational equations.

RT1

Integral Exponents and Scientific Notation

Integral Exponents

In *section P.2*, we discussed the following power rules, using whole numbers for the exponents.

product rule	$a^m \cdot a^n = a^{m+n}$	$(ab)^n = a^n b^n$
quotient rule	$\frac{a^m}{a^n} = a^{m-n}$	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
power rule	$(a^m)^n = a^{mn}$	$a^0 = 1$ for $a \neq 0$ 0^0 is undefined

Observe that these rules gives us the following result.

$$a^{-1} = a^{n-(n+1)} = \frac{a^n}{a^{n+1}} = \frac{a^n}{a^n \cdot a} = \frac{1}{a}$$

↙ quotient rule
↘ product rule

$$\text{Consequently, } a^{-n} = (a^n)^{-1} = \frac{1}{a^n}.$$

↘ power rule

Since $a^{-n} = \frac{1}{a^n}$, then the expression a^n is meaningful for any integral exponent n and a nonzero real base a . So, the above rules of exponents can be extended to include integral exponents.

In practice, to work out the negative sign of an exponent, take the **reciprocal of the base**, or equivalently, “**change the level**” of the power. For example,

$$3^{-2} = \left(\frac{1}{3}\right)^2 = \frac{1^2}{3^2} = \frac{1}{9} \quad \text{and} \quad \frac{2^{-3}}{3^{-1}} = \frac{3^1}{2^3} = \frac{3}{8}.$$

Attention! Exponent refers to the immediate number, letter, or expression in a bracket.

For example,

$$x^{-2} = \frac{1}{x^2}, \quad (-x)^{-2} = \frac{1}{(-x)^2} = \frac{1}{x^2}, \quad \text{but} \quad -x^{-2} = -\frac{1}{x^2}.$$

Example 1 ▶ Evaluating Expressions with Integral Exponents

Evaluate each expression.

a. $3^{-1} + 2^{-1}$

b. $\frac{5^{-2}}{2^{-5}}$

c. $\frac{-2^2}{2^{-7}}$

d. $\frac{-2^{-2}}{3 \cdot 2^{-3}}$

Solution ▶

a. $3^{-1} + 2^{-1} = \frac{1}{3} + \frac{1}{2} = \frac{2}{6} + \frac{3}{6} = \frac{5}{6}$

Caution! $3^{-1} + 2^{-1} \neq (3 + 2)^{-1}$, because the value of $3^{-1} + 2^{-1}$ is $\frac{5}{6}$, as shown in the example, while the value of $(3 + 2)^{-1}$ is $\frac{1}{5}$.

b. $\frac{5^{-2}}{2^{-5}} = \frac{2^5}{5^2} = \frac{32}{25}$

Note: To work out the negative exponent, move the power from the numerator to the denominator or vice versa.

c. $\frac{-2^2}{2^{-7}} = -2^2 \cdot 2^7 = -2^9$

Attention! The role of a negative sign in front of a base number or in front of an exponent is different. To work out the negative in 2^{-7} , we either take the reciprocal of the base, or we change the position of the power to a different level in the fraction. So, $2^{-7} = \left(\frac{1}{2}\right)^7$ or $2^{-7} = \frac{1}{2^7}$. However, the negative sign in -2^2 just means that the number is negative. So, $-2^2 = -4$. **Caution!** $-2^2 \neq \frac{1}{4}$

d. $\frac{-2^{-2}}{3 \cdot 2^{-3}} = \frac{-2^3}{3 \cdot 2^2} = -\frac{2}{3}$

Note: Exponential expressions can be simplified in many ways. For example, to simplify $\frac{2^{-2}}{2^{-3}}$, we can work out the negative exponents first by moving the powers to a different level, $\frac{2^3}{2^2}$, and then reduce the common factors as shown in the example; or we can employ the quotient rule of powers to obtain

$$\frac{2^{-2}}{2^{-3}} = 2^{-2-(-3)} = 2^{-2+3} = 2^1 = 2.$$

Example 2 ▶ **Simplifying Exponential Expressions Involving Negative Exponents**

Simplify the given expression. Leave the answer with only positive exponents.

- a. $4x^{-5}$ b. $(x + y)^{-1}$
 c. $x^{-1} + y^{-1}$ d. $(-2^3x^{-2})^{-2}$
 e. $\frac{x^{-4}y^2}{x^2y^{-5}}$ f. $\left(\frac{-4m^5n^3}{24mn^{-6}}\right)^{-2}$

Solution ▶

a. $4x^{-5} = \frac{4}{x^5}$ exponent -5
refers to x only!

b. $(x + y)^{-1} = \frac{1}{x+y}$ these expressions are
NOT equivalent!

c. $x^{-1} + y^{-1} = \frac{1}{x} + \frac{1}{y}$

d. $(-2^3x^{-2})^{-2} = \left(\frac{-2^3}{x^2}\right)^{-2} = \left(\frac{x^2}{-2^3}\right)^2 = \frac{(x^2)^2}{(-1)^2(2^3)^2} = \frac{x^4}{2^6}$
work out the negative exponents inside the bracket work out the negative exponents outside the bracket power rule – multiply exponents
a “-” sign can be treated as a factor of -1

e. $\frac{x^{-4}y^2}{x^2y^{-5}} = \frac{y^2y^5}{x^2x^4} = \frac{y^7}{x^6}$
product rule – add exponents

f. $\left(\frac{-4m^5n^3}{24mn^{-6}}\right)^{-2} = \left(\frac{-m^4n^3n^6}{6}\right)^{-2} = \left(\frac{(-1)m^4n^9}{6}\right)^{-2} = \left(\frac{6}{(-1)m^4n^9}\right)^2 = \frac{36}{m^8n^{18}}$
(-1)² = 1

Scientific Notation

Integral exponents allow us to record numbers with a very large or very small absolute value in a shorter, more convenient form.

For example, the average distance from the Sun to the Saturn is 1,430,000,000 km, which can be recorded as $1.43 \cdot 10,000,000$ or more concisely as $1.43 \cdot 10^9$.

Similarly, the mass of an electron is 0.000000000000000000000000009 grams, which can be recorded as $9 \cdot 0.000000000000000000000000001$, or more concisely as $9 \cdot 10^{-28}$.

This more concise representation of numbers is called **scientific notation** and it is frequently used in sciences and engineering.

Definition 1.1 ▶ A real number x is written in **scientific notation** iff $x = a \cdot 10^n$, where the coefficient a is such that $|a| \in [1, 10)$, and the exponent n is an integer.

Example 3 ▶ **Converting Numbers to Scientific Notation**

Convert each number to scientific notation.

- a. 520,000 b. -0.000102 c. $12.5 \cdot 10^3$

Solution ▶

an integer has its
decimal dot after
the last digit

- a. To represent 520,000 in scientific notation, we place a decimal point after the first nonzero digit,

$$5.\underline{20000}$$

and then count the number of decimal places needed for the decimal point to move to its original position, which by default was after the last digit. In our example the number of places we need to move the decimal place is 5. This means that 5.2 needs to be multiplied by 10^5 in order to represent the value of 520,000. So, $520,000 = 5.2 \cdot 10^5$.

Note: To comply with the scientific notation format, we always place the decimal point after the first nonzero digit of the given number. This will guarantee that the coefficient a satisfies the condition $1 \leq |a| < 10$.

- b. As in the previous example, to represent -0.000102 in scientific notation, we place a decimal point after the first nonzero digit,

$$-0.\underline{0001}02$$

and then count the number of decimal places needed for the decimal point to move to its original position. In this example, we move the decimal 4 places to the left. So the number 1.02 needs to be divided by 10^4 , or equivalently, multiplied by 10^{-4} in order to represent the value of -0.000102 . So, $-0.000102 = -1.02 \cdot 10^{-4}$.

Observation: Notice that moving the decimal to the **right** corresponds to using a **positive** exponent, as in *Example 3a*, while moving the decimal to the **left** corresponds to using a **negative** exponent, as in *Example 3b*.

- c. Notice that $12.5 \cdot 10^3$ is not in scientific notation as the coefficient 12.5 is not smaller than 10. To convert $12.5 \cdot 10^3$ to scientific notation, first, convert 12.5 to scientific notation and then multiply the powers of 10. So,

$$12.5 \cdot 10^3 = 1.25 \cdot 10 \cdot 10^3 = 1.25 \cdot 10^4$$

multiply powers by
adding exponents

Example 4 ▶ **Converting from Scientific to Decimal Notation**

Convert each number to decimal notation.

a. $-6.57 \cdot 10^6$

b. $4.6 \cdot 10^{-7}$

Solution ▶

- a. The exponent 6 indicates that the decimal point needs to be moved 6 places to the right. So,

$$-6.57 \cdot 10^6 = -6.57\underbrace{}_{\text{fill the empty places by zeros}} = -6,570,000$$

fill the empty places by zeros

- b. The exponent -7 indicates that the decimal point needs to be moved 7 places to the left. So,

$$4.6 \cdot 10^{-7} = 0.\underbrace{}_{\text{fill the empty places by zeros}}4.6 = 0.00000046$$

fill the empty places by zeros

Example 5 ▶ **Using Scientific Notation in Computations**

Evaluate. Leave the answer in scientific notation.

a. $6.5 \cdot 10^7 \cdot 3 \cdot 10^5$

b. $\frac{3.6 \cdot 10^3}{9 \cdot 10^{14}}$

Solution ▶

- a. Since the product of the coefficients $6.5 \cdot 3 = 19.5$ is larger than 10, we convert it to scientific notation and then multiply the remaining powers of 10. So,

$$6.5 \cdot 10^7 \cdot 3 \cdot 10^5 = 19.5 \cdot 10^7 \cdot 10^5 = \underbrace{19.5 \cdot 10}_{\text{convert to scientific notation}} \cdot 10^{12} = 1.95 \cdot 10^{13}$$

- b. Similarly as in the previous example, since the quotient $\frac{3.6}{9} = 0.4$ is smaller than 1, we convert it to scientific notation and then work out the remaining powers of 10. So,

$$\frac{3.6 \cdot 10^3}{9 \cdot 10^{14}} = 0.4 \cdot 10^{-11} = \underbrace{4 \cdot 10^{-1}}_{\text{convert to scientific notation}} \cdot 10^{-11} = 4 \cdot 10^{-12}$$

divide powers by subtracting exponents

Example 6 ▶ **Using Scientific Notation to Solve Problems**

Earth is approximately $1.5 \cdot 10^8$ kilometers from the Sun. Estimate the time in days needed for a space probe moving at an average rate of $2.4 \cdot 10^4$ km/h to reach the Sun? *Assume that the probe moves along a straight line.*

Solution ▶ To find time T needed for the space probe travelling at the rate $R = 2.4 \cdot 10^4$ km/h to reach the Sun that is at the distance $D = 1.5 \cdot 10^8$ km from Earth, first, we solve the motion formula $R \cdot T = D$ for T . Since $T = \frac{D}{R}$, we calculate,

$$T = \frac{1.5 \cdot 10^8}{2.4 \cdot 10^4} = 0.625 \cdot 10^4 = 6.25 \cdot 10^3$$

So, it will take $6.25 \cdot 10^3$ hours = $\frac{6250}{24}$ days \cong **260.4 days** for the space probe to reach the Sun.

RT.1 Exercises

True or false.

1. $\left(\frac{3}{4}\right)^{-2} = \left(\frac{4}{3}\right)^2$
2. $10^{-4} = 0.00001$
3. $(0.25)^{-1} = 4$
4. $-4^5 = \frac{1}{4^5}$
5. $(-2)^{-10} = 4^{-5}$
6. $2 \cdot 2 \cdot 2^{-1} = \frac{1}{8}$
7. $3x^{-2} = \frac{1}{3x^2}$
8. $-2^{-2} = -\frac{1}{4}$
9. $\frac{5^{10}}{5^{-12}} = 5^{-2}$
10. The number $0.68 \cdot 10^{-5}$ is written in scientific notation.
11. $98.6 \cdot 10^7 = 9.86 \cdot 10^6$

12. Match each expression in Row I with the equivalent expression(s) in Row II, if possible.

- | | | | | |
|-------------|-------------------|----------------|-------------------|----------------------|
| a. 5^{-2} | b. -5^{-2} | c. $(-5)^{-2}$ | d. $-(-5)^{-2}$ | e. $-5 \cdot 5^{-2}$ |
| A. 25 | B. $\frac{1}{25}$ | C. -25 | D. $-\frac{1}{5}$ | E. $-\frac{1}{25}$ |

Evaluate each expression.

13. $4^{-6} \cdot 4^3$
14. $-9^3 \cdot 9^{-5}$
15. $\frac{2^{-3}}{2^6}$
16. $\frac{2^{-7}}{2^{-5}}$
17. $\frac{-3^{-4}}{5^{-3}}$
18. $-\left(\frac{3}{2}\right)^{-2}$
19. $2^{-2} + 2^{-3}$
20. $(2^{-1} - 3^{-1})^{-1}$

Simplify each expression, if possible. Leave the answer with only **positive exponents**. Assume that all variables represent nonzero real numbers. Keep large numerical coefficients as powers of prime numbers, if possible.

21. $(-2x^{-3})(7x^{-8})$
22. $(5x^{-2}y^3)(-4x^{-7}y^{-2})$
23. $(9x^{-4n})(-4x^{-8n})$
24. $(-3y^{-4a})(-5y^{-3a})$
25. $-4x^{-3}$
26. $\frac{x^{-4n}}{x^{6n}}$
27. $\frac{3n^5}{nm^{-2}}$
28. $\frac{14a^{-4}b^{-3}}{-8a^8b^{-5}}$
29. $\frac{-18x^{-3}y^3}{-12x^{-5}y^5}$

30. $(2^{-1}p^{-7}q)^{-4}$

31. $(-3a^2b^{-5})^{-3}$

32. $\left(\frac{5x^{-2}}{y^3}\right)^{-3}$

33. $\left(\frac{2x^3y^{-2}}{3y^{-3}}\right)^{-3}$

34. $\left(\frac{-4x^{-3}}{5x^{-1}y^4}\right)^{-4}$

35. $\left(\frac{125x^2y^{-3}}{5x^4y^{-2}}\right)^{-5}$

36. $\left(\frac{-200x^3y^{-5}}{8x^5y^{-7}}\right)^{-4}$

37. $[(-2x^{-4}y^{-2})^{-3}]^{-2}$

38. $\frac{12a^{-2}(a^{-3})^{-2}}{6a^7}$

39. $\frac{(-2k)^2m^{-5}}{(km)^{-3}}$

40. $\left(\frac{2p}{q^2}\right)^3 \left(\frac{3p^4}{q^{-4}}\right)^{-1}$

41. $\left(\frac{-3x^4y^6}{15x^{-6}y^7}\right)^{-3}$

42. $\left(\frac{-4a^3b^2}{12a^6b^{-5}}\right)^{-3}$

43. $\left(\frac{-9^{-2}x^{-4}y}{3^{-3}x^{-3}y^2}\right)^8$

44. $(4^{-x})^{2y}$

45. $(5^a)^{-a}$

46. $x^a x^{-a}$

47. $\frac{9n^{2-x}}{3n^{2-2x}}$

48. $\frac{12x^{a+1}}{-4x^{2-a}}$

49. $(x^{b-1})^3(x^{b-4})^{-2}$

50. $\frac{25x^{a+b}y^{b-a}}{-5x^{a-b}y^{b-a}}$

Convert each number to scientific notation.

51. 26,000,000,000

52. -0.000132

53. 0.0000000105

54. 705.6

Convert each number to decimal notation.

55. $6.7 \cdot 10^8$

56. $5.072 \cdot 10^{-5}$

57. $2 \cdot 10^{12}$

58. $9.05 \cdot 10^{-9}$

59. One megabyte of computer memory equals 2^{20} bytes. Using decimal notation, write the number of bytes in 1 megabyte. Then, using scientific notation, approximate this number by rounding the scientific notation coefficient to two decimals places.

Evaluate. State your answer in scientific notation.

60. $(6.5 \cdot 10^3)(5.2 \cdot 10^{-8})$

61. $(2.34 \cdot 10^{-5})(5.7 \cdot 10^{-6})$

62. $(3.26 \cdot 10^{-6})(5.2 \cdot 10^{-8})$

63. $\frac{4 \cdot 10^{-7}}{8 \cdot 10^{-3}}$

64. $\frac{7.5 \cdot 10^9}{2.5 \cdot 10^4}$

65. $\frac{4 \cdot 10^{-7}}{8 \cdot 10^{-3}}$

66. $\frac{0.05 \cdot 16000}{0.0004}$

67. $\frac{0.003 \cdot 40,000}{0.00012 \cdot 600}$

Solve each problem. State your answer in scientific notation.

68. A light-year is an astronomical unit measuring the distance that light travels in one year. If light travels approximately $3 \cdot 10^5$ kilometers per second, how long is a light-year in kilometers?
69. In 2018, the national debt in Canada was about $6.7 \cdot 10^{11}$ dollars. If the Canadian population in 2018 was approximately $3.7 \cdot 10^7$, what was the share of this debt per person?

70. One of the brightest stars in the night sky, Vega, is about $2.365 \cdot 10^{14}$ kilometers from Earth. If one light-year is approximately $9.46 \cdot 10^{12}$ kilometers, how many light-years is it from Earth to Vega?
71. The Columbia River discharges its water to the Pacific Ocean at approximately 265,000 ft^3/sec . What is the supply of water that comes from the Columbia River in one minute? in one day? *State the answer in scientific notation.*
72. Assuming the current trends continue, the population P of Canada, in millions, can be modelled by the equation $P = 34(1.011)^x$, where x is the number of years passed after the year 2010. According to this model, what is the predicted Canadian population for the years 2025 and 2030?
73. The mass of the Moon is $7.348 \cdot 10^{22}$ kg while the mass of Earth is $5.976 \cdot 10^{24}$ kg. How many times heavier is Earth than the Moon?
74. Most calculators cannot handle operations on numbers outside of the interval $(10^{-100}, 10^{100})$. How can we compute $(5 \cdot 10^{120})^3$ without the use of a calculator?



RT2

Rational Expressions and Functions; Multiplication and Division of Rational Expressions



In arithmetic, a rational number is a quotient of two integers with denominator different than zero. In algebra, a *rational expression*, often called an *algebraic fraction*, is a quotient of two polynomials, also with denominator different than zero. In this section, we will examine rational expressions and functions, paying attention to their domains. Then, we will simplify, multiply, and divide rational expressions, employing the factoring skills developed in *Chapter P*.

Rational Expressions and Functions

Here are some examples of rational expressions:

$$-\frac{x^2}{2xy}, \quad x^{-1}, \quad \frac{x^2-4}{x-2}, \quad \frac{8x^2+6x-5}{4x^2+5x}, \quad \frac{x-3}{3-x}, \quad x^2 - 25, \quad 3x(x-1)^{-2}$$

Definition 2.1 ▶ A **rational expression (algebraic fraction)** is a quotient $\frac{P(x)}{Q(x)}$ of two polynomials $P(x)$ and $Q(x)$, where $Q(x) \neq 0$. Since division by zero is not permitted, a rational expression is defined only for the x -values that make the denominator of the expression different than zero. The set of such x -values is referred to as the **domain** of the expression.

Note 1: Negative exponents indicate hidden fractions and therefore represent rational expressions. For instance, $x^{-1} = \frac{1}{x}$.

Note 2: A single polynomial can also be seen as a rational expression because it can be considered as a fraction with a denominator of 1. For instance, $x^2 - 25 = \frac{x^2-25}{1}$.

Definition 2.2 ▶ A **rational function** is a function defined by a rational expression,

$$f(x) = \frac{P(x)}{Q(x)}.$$

The **domain** of such function consists of all real numbers except for the x -values that make the denominator $Q(x)$ equal to 0. So, the domain $D = \mathbb{R} \setminus \{x | Q(x) = 0\}$

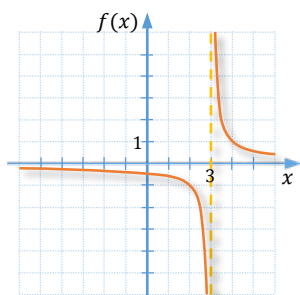


Figure 1

Rational Expressions and Functions

For example, the domain of the rational function $f(x) = \frac{1}{x-3}$ is the set of all real numbers except for 3 because 3 would make the denominator equal to 0. So, we write $D = \mathbb{R} \setminus \{3\}$. Sometimes, to make it clear that we refer to function f , we might denote the domain of f by D_f , rather than just D .

Figure 1 shows a graph of the function $f(x) = \frac{1}{x-3}$. Notice that the graph does not cross the dashed vertical line whose equation is $x = 3$. This is because $f(3)$ is not defined. A closer look at the graphs of rational functions will be given in *Section RT5*.

Example 1 ▶ **Evaluating Rational Expressions or Functions**

Evaluate the given expression or function for $x = -1, 0, 1$. If the value cannot be calculated, write *undefined*.

a. $3x(x - 1)^{-2}$

b. $f(x) = \frac{x}{x^2+x}$

Solution ▶

a. If $x = -1$, then $3x(x - 1)^{-2} = 3(-1)(-1 - 1)^{-2} = -3(-2)^{-2} = \frac{-3}{(-2)^2} = -\frac{3}{4}$.

If $x = 0$, then $3x(x - 1)^{-2} = 3(0)(0 - 1)^{-2} = \mathbf{0}$.

If $x = 1$, then $3x(x - 1)^{-2} = 3(1)(1 - 1)^{-2} = 3 \cdot 0^{-2} = \mathbf{undefined}$, as division by zero is not permitted.

Note: Since the expression $3x(x - 1)^{-2}$ cannot be evaluated at $x = 1$, the number 1 does not belong to its domain.

b. $f(-1) = \frac{-1}{(-1)^2+(-1)} = \frac{-1}{1-1} = \mathbf{undefined}$.

$f(0) = \frac{0}{(0)^2+(0)} = \frac{0}{0} = \mathbf{undefined}$.

$f(1) = \frac{1}{(1)^2+(1)} = \frac{1}{2}$.

Observation: Function $f(x) = \frac{x}{x^2+x}$ is undefined at $x = 0$ and $x = -1$. This is because the denominator $x^2 + x = x(x + 1)$ becomes zero when the x -value is 0 or -1 .

Example 2 ▶ **Finding Domains of Rational Expressions or Functions**

Find the domain of each expression or function.

a. $\frac{4}{2x+5}$

b. $\frac{x-2}{x^2-2x}$

c. $f(x) = \frac{x^2-4}{x^2+4}$

d. $g(x) = \frac{2x-1}{x^2-4x-5}$

Solution ▶

a. The domain of $\frac{4}{2x+5}$ consists of all real numbers except for those that would make the denominator $2x + 5$ equal to zero. To find these numbers, we solve the equation

$$2x + 5 = 0$$

$$2x = -5$$

$$x = -\frac{5}{2}$$

So, the domain of $\frac{4}{2x+5}$ is the set of all real numbers except for $-\frac{5}{2}$. This can be recorded in set notation as $\mathbb{R} \setminus \{-\frac{5}{2}\}$, or in set-builder notation as $\{x \mid x \neq -\frac{5}{2}\}$, or in interval notation as $(-\infty, -\frac{5}{2}) \cup (-\frac{5}{2}, \infty)$.

- b. To find the domain of $\frac{x-2}{x^2-2x}$, we want to exclude from the set of real numbers all the x -values that would make the denominator $x^2 - 2x$ equal to zero. After solving the equation

$$x^2 - 2x = 0$$

via factoring

$$x(x - 2) = 0$$

and zero-product property

$$x = 0 \text{ or } x = 2,$$

we conclude that the domain is the set of all real numbers except for 0 and 2, which can be recorded as $\mathbb{R} \setminus \{0, 2\}$. This is because the x -values of 0 or 2 make the denominator of the expression $\frac{x-2}{x^2-2x}$ equal to zero.

- c. To find the domain of the function $f(x) = \frac{x^2-4}{x^2+4}$, we first look for all the x -values that make the denominator $x^2 + 4$ equal to zero. However, $x^2 + 4$, as a sum of squares, is never equal to 0. So, the domain of function f is the set of all real numbers \mathbb{R} .

- d. To find the domain of the function $g(x) = \frac{2x-1}{x^2-4x-5}$, we first solve the equation $x^2 - 4x - 5 = 0$ to find which x -values make the denominator equal to zero. After factoring, we obtain

$$(x - 5)(x + 1) = 0$$

which results in

$$x = 5 \text{ and } x = -1$$

Thus, the domain of g equals to $D_g = \mathbb{R} \setminus \{-1, 5\}$.

Equivalent Expressions

Definition 2.3 ▶ Two expressions are **equivalent** in the **common domain** iff (if and only if) they produce the same values for every input from the domain.

Consider the expression $\frac{x-2}{x^2-2x}$ from *Example 2b*. Notice that this expression can be simplified to $\frac{x-2}{x(x-2)} = \frac{1}{x}$ by reducing common factors in the numerator and the denominator. However, the domain of the simplified fraction, $\frac{1}{x}$, is the set $\mathbb{R} \setminus \{0\}$, which is different than the domain of the original fraction, $\mathbb{R} \setminus \{0, 2\}$. Notice that for $x = 2$, the expression $\frac{x-2}{x^2-2x}$ is undefined while the value of the expression $\frac{1}{x}$ is $\frac{1}{2}$. So, the two expressions are not equivalent in the set of real numbers. However, if the domain of $\frac{1}{x}$ is

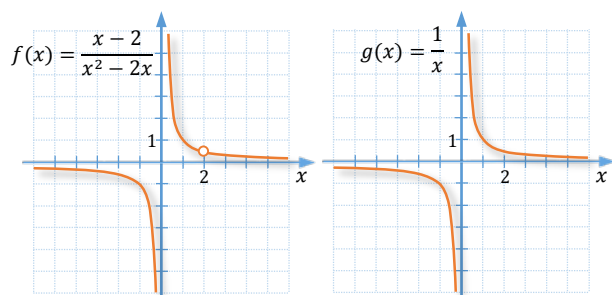


Figure 2

restricted to the set $\mathbb{R} \setminus \{0, 2\}$, then the two expressions produce the same values and as such, they are equivalent. We say that the two expressions are **equivalent** in the **common domain**.

The above situation can be illustrated by graphing the related functions, $f(x) = \frac{x-2}{x^2-2x}$ and $g(x) = \frac{1}{x}$, as in *Figure 2*. The graphs of both functions are exactly the same except for the hole in the graph of f at the point $(2, \frac{1}{2})$.

So, from now on, when writing statements like $\frac{x-2}{x^2-2x} = \frac{1}{x}$, we keep in mind that they apply only to real numbers which make both denominators different than zero. Thus, by saying in short that two **expressions are equivalent**, we really mean that they are **equivalent in the common domain**.

Note: The **domain** of $f(x) = \frac{x-2}{x^2-2x} = \frac{x-2}{x(x-2)} = \frac{1}{x}$ is still $\mathbb{R} \setminus \{0, 2\}$, even though the $(x-2)$ term was simplified.

The process of simplifying expressions involves creating equivalent expressions. In the case of rational expressions, equivalent expressions can be obtained by multiplying or dividing the numerator and denominator of the expression by the same nonzero polynomial. For example,

$$\frac{-x-3}{-5x} = \frac{(-x-3) \cdot (-1)}{(-5x) \cdot (-1)} = \frac{x+3}{5x}$$

$$\frac{x-3}{3-x} = \frac{\cancel{(x-3)}}{-1\cancel{(x-3)}} = \frac{1}{-1} = -1$$

To simplify a rational expression:

- **Factor** the numerator and denominator **completely**.
- **Eliminate all common factors** by following the property of multiplicative identity. *Do not eliminate common terms - they must be factors!*

Example 3 ▶ Simplifying Rational Expressions

Simplify each expression.

a. $\frac{7a^2b^2}{21a^3b-14a^3b^2}$ b. $\frac{x^2-9}{x^2-6x+9}$ c. $\frac{20x-15x^2}{15x^3-5x^2-20x}$

Solution ▶ a. First, we factor the denominator and then reduce the common factors. So,

$$\frac{7a^2b^2}{21a^3b-14a^3b^2} = \frac{\cancel{7}a^2b^2}{\cancel{7}a^3b(3a-2ab)} = \frac{b}{a(3-2b)}$$

b. As before, we factor and then reduce. So,

$$\frac{x^2 - 9}{x^2 - 6x + 9} = \frac{\cancel{(x-3)}(x+3)}{(x-3)^{\cancel{2}1}} = \frac{x+3}{x-3}$$

Neither x nor 3 can be reduced, as they are NOT factors!

c. Factoring and reducing the numerator and denominator gives us

$$\frac{20x - 15x^2}{15x^3 - 5x^2 - 20x} = \frac{5x(4 - 3x)}{5x(3x^2 - x - 4)} = \frac{4 - 3x}{(3x - 4)(x + 1)}$$

Since $\frac{4-3x}{3x-4} = \frac{-(3x-4)}{3x-4} = -1$, the above expression can be reduced further to

$$\frac{\cancel{4-3x}^{-1}}{\cancel{(3x-4)}(x+1)} = \frac{-1}{x+1}$$

Notice: An opposite expression in the numerator and denominator can be reduced to -1 . For example, since $a - b$ is opposite to $b - a$, then

$$\frac{a-b}{b-a} = -1, \text{ as long as } a \neq b.$$

Caution: Note that $a - b$ is NOT opposite to $a + b$!

Multiplication and Division of Rational Expressions

Recall that to multiply common fractions, we multiply their numerators and denominators, and then simplify the resulting fraction. Multiplication of algebraic fractions is performed in a similar way.

To multiply rational expressions:

- **factor** each numerator and denominator **completely**,
- **reduce all common factors** in any of the numerators and denominators,
- **multiply** the remaining expressions by writing the product of their numerators over the product of their denominators.

For instance,

$$\frac{3x}{x^2 + 5x} \cdot \frac{3x + 15}{6x} = \frac{\cancel{3x}}{x(x+5)} \cdot \frac{\cancel{3}(x+5)}{\cancel{6x}_2} = \frac{3}{2x}$$

Example 4 ▶ Multiplying Algebraic Fractions

Multiply and simplify. Assume nonzero denominators.

a. $\frac{2x^2y^3}{3xy^2} \cdot \frac{(2x^3y)^2}{2(xy)^3}$

b. $\frac{x^3 - y^3}{x + y} \cdot \frac{3x + 3y}{x^2 - y^2}$

Solution

- a. To multiply the two algebraic fractions, we use appropriate rules of powers to simplify each fraction, and then reduce all the remaining common factors. So,

$$\frac{2x^2y^3}{3xy^2} \cdot \frac{(2x^3y)^2}{2(xy)^3} = \frac{2xy}{3} \cdot \frac{4x^6y^2}{2x^3y^3} = \frac{2xy \cdot 2x^3}{3 \cdot y} = \frac{4x^4}{3} = \frac{4}{3}x^4$$

equivalent answers

- b. After factoring and simplifying, we have

$$\frac{x^3 - y^3}{x + y} \cdot \frac{3x + 3y}{x^2 - y^2} = \frac{(x - y)(x^2 + xy + y^2)}{x + y} \cdot \frac{3(x + y)}{(x - y)(x + y)} = \frac{3(x^2 + xy + y^2)}{x + y}$$

Recall: $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$
 $x^2 - y^2 = (x + y)(x - y)$

To divide rational expressions, multiply the first, the *dividend*, **by the reciprocal** of the second, the *divisor*.

For instance,

$$\frac{5x - 10}{3x} \div \frac{3x - 6}{2x^2} = \frac{5x - 10}{3x} \cdot \frac{2x^2}{3x - 6} = \frac{5(x - 2)}{3x} \cdot \frac{2x^2}{3(x - 2)} = \frac{10x}{9}$$

multiply by the reciprocal

follow multiplication rules

Example 5**Dividing Algebraic Fractions**

Perform operations and simplify. Assume nonzero denominators.

a. $\frac{2x^2 + 2x}{x - 1} \div (x + 1)$

b. $\frac{x^2 - 25}{x^2 + 5x + 4} \div \frac{x^2 - 10x + 25}{2x^2 + 8x} \cdot \frac{x^2 + x}{4x^2}$

Solution

- a. To divide by $(x + 1)$ we multiply by the reciprocal $\frac{1}{(x + 1)}$. So,

$$\frac{2x^2 + 2x}{x - 1} \div (x + 1) = \frac{2x(x + 1)}{x - 1} \cdot \frac{1}{(x + 1)} = \frac{2x}{x - 1}$$

- b. The order of operations indicates to perform the division first. To do this, we convert the division into multiplication by the reciprocal of the middle expression. Therefore,

$$\begin{aligned} & \frac{x^2 - 25}{x^2 + 5x + 4} \div \frac{x^2 - 10x + 25}{2x^2 + 8x} \cdot \frac{x^2 + x}{4x^2} \\ &= \frac{(x - 5)(x + 5)}{(x + 4)(x + 1)} \cdot \frac{2x^2 + 8x}{x^2 - 10x + 25} \cdot \frac{x(x + 1)}{4x^2} \\ &= \frac{(x - 5)(x + 5)}{(x + 4)} \cdot \frac{2x(x + 4)}{(x - 5)^2} \cdot \frac{1}{4x} = \frac{(x + 5)}{2(x - 5)} \end{aligned}$$

Reduction of common factors can be done gradually, especially if there is many common factors to divide out.

RT.2 Exercises

True or false.

1. $f(x) = \frac{4}{\sqrt{x-4}}$ is a rational function.
2. The domain of $f(x) = \frac{x-2}{4}$ is the set of all real numbers.
3. $\frac{x-3}{4-x}$ is equivalent to $-\frac{x-3}{x-4}$.
4. $\frac{n^2+1}{n^2-1}$ is equivalent to $\frac{n+1}{n-1}$.

Given the rational function f , find $f(-1)$, $f(0)$, and $f(2)$.

5. $f(x) = \frac{x}{x-2}$
6. $f(x) = \frac{5x}{3x-x^2}$
7. $f(x) = \frac{x-2}{x^2+x-6}$

For each rational function, find all numbers that are not in the domain. Then give the **domain**, using both **set notation** and **interval notation**.

8. $f(x) = \frac{x}{x+2}$
9. $g(x) = \frac{x}{x-6}$
10. $h(x) = \frac{2x-1}{3x+7}$
11. $f(x) = \frac{3x+2}{5x-4}$
12. $g(x) = \frac{x+2}{x^2-4}$
13. $h(x) = \frac{x-2}{x^2+4}$
14. $f(x) = \frac{5}{3x-x^2}$
15. $g(x) = \frac{x^2+x-6}{x^2+12x+35}$
16. $h(x) = \frac{7}{|4x-3|}$

17. Which rational expressions are equivalent and what is their simplest form?

- a. $\frac{2x+3}{2x-3}$
- b. $\frac{2x-3}{3-2x}$
- c. $\frac{2x+3}{3+2x}$
- d. $\frac{2x+3}{-2x-3}$
- e. $\frac{3-2x}{2x-3}$

18. Which rational expressions can be simplified?

- a. $\frac{x^2+2}{x^2}$
- b. $\frac{x^2+2}{2}$
- c. $\frac{x^2-x}{x^2}$
- d. $\frac{x^2-y^2}{y^2}$
- e. $\frac{x}{x^2-x}$

Simplify each expression, if possible.

19. $\frac{24a^3b}{3ab^3}$
20. $\frac{-18x^2y^3}{8x^3y}$
21. $\frac{7-x}{x-7}$
22. $\frac{x+2}{x-2}$
23. $\frac{a-5}{-5+a}$
24. $\frac{(3-y)(x+1)}{(y-3)(x-1)}$
25. $\frac{12x-15}{21}$
26. $\frac{18a-2}{22}$
27. $\frac{4y-12}{4y+12}$
28. $\frac{7x+14}{7x-14}$
29. $\frac{6m+18}{7m+21}$
30. $\frac{3z^2+z}{18z+6}$
31. $\frac{m^2-25}{20-4m}$
32. $\frac{9n^2-3}{4-12n^2}$
33. $\frac{t^2-25}{t^2-10t+25}$
34. $\frac{p^2-36}{p^2+12t+36}$

$$35. \frac{x^2-9x+8}{x^2+3x-4} \qquad 36. \frac{p^2+8p-9}{p^2-5p+4} \qquad 37. \frac{x^3-y^3}{x^2-y^2} \qquad 38. \frac{b^2-a^2}{a^3-b^3}$$

Perform operations and simplify. Assume nonzero denominators.

$$39. \frac{18a^4}{5b^2} \cdot \frac{25b^4}{9a^3} \qquad 40. \frac{28}{xy} \div \frac{63x^3}{2y^2} \qquad 41. \frac{12x}{49(xy^2)^3} \cdot \frac{(7xy)^2}{8}$$

$$42. \frac{x+1}{2x-3} \cdot \frac{2x-3}{2x} \qquad 43. \frac{10a}{6a-12} \cdot \frac{20a-40}{30a^3} \qquad 44. \frac{a^2-1}{4a} \cdot \frac{2}{1-a}$$

$$45. \frac{y^2-25}{4y} \cdot \frac{2}{5-y} \qquad 46. (8x-16) \div \frac{3x-6}{10} \qquad 47. (y^2-4) \div \frac{2-y}{8y}$$

$$48. \frac{3n-9}{n^2-9} \cdot (n^3+27) \qquad 49. \frac{x^2-16}{x^2} \cdot \frac{x^2-4x}{x^2-x-12} \qquad 50. \frac{y^2+10y+25}{y^2-9} \cdot \frac{y^2-3y}{y+5}$$

$$51. \frac{b-3}{b^2-4b+3} \div \frac{b^2-b}{b-1} \qquad 52. \frac{x^2-6x+9}{x^2+3x} \div \frac{x^2-9}{x} \qquad 53. \frac{x^2-2x}{3x^2-5x-2} \cdot \frac{9x^2-4}{9x^2-12x+4}$$

$$54. \frac{t^2-49}{t^2+4t-21} \cdot \frac{t^2+8t+15}{t^2-2t-35} \qquad 55. \frac{a^3-b^3}{a^2-b^2} \div \frac{2a-2b}{2a+2b} \qquad 56. \frac{64x^3+1}{4x^2-100} \cdot \frac{4x+20}{64x^2-16x+4}$$

$$57. \frac{x^3y-64y}{x^3y+64y} \div \frac{x^2y^2-16y^2}{x^2y^2-4xy^2+16y^2} \qquad 58. \frac{p^3-27q^3}{p^2+pq-12q^2} \cdot \frac{p^2-2pq-24q^2}{p^2-5pq-6q^2}$$

$$59. \frac{4x^2-9y^2}{8x^3-27y^3} \cdot \frac{4x^2+6xy+9y^2}{4x^2+12xy+9y^2} \qquad 60. \frac{2x^2+x-1}{6x^2+x-2} \div \frac{2x^2+5x+3}{6x^2+13x+6}$$

$$61. \frac{6x^2-13x+6}{14x^2-25x+6} \div \frac{14-21x}{49x^2+7x-6} \qquad 62. \frac{4y^2-12y+36}{27-3y^2} \div (y^3+27)$$

$$63. \frac{3y}{x^2} \div \frac{y^2}{x} \div \frac{y}{5x} \qquad 64. \frac{x+1}{y-2} \div \frac{2x+2}{y-2} \div \frac{x}{y}$$

$$65. \frac{a^2-4b^2}{a+2b} \div (a+2b) \cdot \frac{2b}{a-2b} \qquad 66. \frac{9x^2}{x^2-16y^2} \div \frac{1}{x^2+4xy} \cdot \frac{x-4y}{3x}$$

$$67. \frac{x^2-25}{x-4} \div \frac{x^2-2x-15}{x^2-10x+24} \cdot \frac{x+3}{x^2+10x+25} \qquad 68. \frac{y-3}{y^2-8y+16} \cdot \frac{y^2-16}{y+4} \div \frac{y^2+3y-18}{y^2+11y+30}$$

Given $f(x)$ and $g(x)$, find $f(x) \cdot g(x)$ and $f(x) \div g(x)$.

$$69. f(x) = \frac{x-4}{x^2+x} \text{ and } g(x) = \frac{2x}{x+1} \qquad 70. f(x) = \frac{x^3-3x^2}{x+5} \text{ and } g(x) = \frac{4x^2}{x-3}$$

$$71. f(x) = \frac{x^2-7x+12}{x+3} \text{ and } g(x) = \frac{9-x^2}{x-4} \qquad 72. f(x) = \frac{x+6}{4-x^2} \text{ and } g(x) = \frac{2-x}{x^2+8x+12}$$

RT3

Addition and Subtraction of Rational Expressions



Many real-world applications involve adding or subtracting algebraic fractions. Like in the case of common fractions, to add or subtract algebraic fractions, we first need to change them equivalently to fractions with the same denominator. Thus, we begin by discussing the techniques of finding the least common denominator.

Least Common Denominator

The **least common denominator (LCD)** for fractions with given denominators is the same as the **least common multiple (LCM)** of these denominators. The methods of finding the LCD for fractions with numerical denominators were reviewed in *Section R3*. For example,

$$LCD(4,6,8) = 24,$$

because 24 is a multiple of 4, 6, and 8, and there is no smaller natural number that would be divisible by all three numbers, 4, 6, and 8.

Suppose the denominators of three algebraic fractions are $4(x^2 - y^2)$, $-6(x + y)^2$, and $8x$. The numerical factor of the least common multiple is 24. The variable part of the LCM is built by taking the product of all the different variable factors from each expression, with each factor raised to the **greatest** exponent that occurs in any of the expressions. In our example, since $4(x^2 - y^2) = 4(x + y)(x - y)$, then

$$LCD(4(x + y)(x - y), -6(x + y)^2, 8x) = 24x(x + y)^2(x - y)$$

Notice that we do not worry about the negative sign of the middle expression. This is because a negative sign can always be written in front of a fraction or in the numerator rather than in the denominator. For example,

$$\frac{1}{-6(x + y)^2} = -\frac{1}{6(x + y)^2} = \frac{-1}{6(x + y)^2}$$

In summary, to find the LCD for algebraic fractions, follow the steps:

- **Factor** each denominator **completely**.
 - Build the LCD for the denominators by including the following as factors:
 - **LCD of all numerical coefficients**,
 - all of the **different factors** from each denominator, with each factor **raised to the greatest exponent** that occurs in any of the denominators.
- Note:* Disregard any factor of -1 .

Example 1



Determining the LCM for the Given Expressions

Find the LCM for the given expressions.

- a. $12x^3y$ and $15xy^2(x - 1)$ b. $x^2 - 2x - 8$ and $x^2 + 3x + 2$
- c. $y^2 - x^2$, $2x^2 - 2xy$, and $x^2 + 2xy + y^2$

Solution

- a. Notice that both expressions, $12x^3y$ and $15xy^2(x-1)$, are already in factored form. The $LCM(12,15) = 60$, as

divide by 3

$$\begin{array}{r} 3 \quad 12 \quad 15 \\ \cdot \quad 4 \quad \cdot 5 \\ \hline = 60 \end{array}$$

no more common factors, so we multiply the numbers in the letter L

The highest power of x is 3, the highest power of y is 2, and $(x-1)$ appears in the first power. Therefore,

$$LCM(12x^3y, 15xy^2(x-1)) = 60x^3y^2(x-1)$$

- b. To find the LCM of $x^2 - 2x - 8$ and $x^2 + 3x + 2$, we factor each expression first:

$$\begin{aligned} x^2 - 2x - 8 &= (x-4)(x+2) \\ x^2 + 3x + 2 &= (x+1)(x+2) \end{aligned}$$

There are three different factors in these expressions, $(x-4)$, $(x+2)$, and $(x+1)$. All of these factors appear in the first power, so

$$LCM(x^2 - 2x - 8, x^2 + 3x + 2) = (x-4)(x+2)(x+1)$$

notice that $(x+2)$ is taken only ones!

- c. As before, to find the LCM of $y^2 - x^2$, $2x^2 - 2xy$, and $x^2 + 2xy + y^2$, we factor each expression first:

$$\begin{aligned} y^2 - x^2 &= (y+x)(y-x) = -(x+y)(x-y) \\ 2x^2 - 2xy &= 2x(x-y) \\ x^2 + 2xy + y^2 &= (x+y)^2 \end{aligned}$$

as $y-x = -(x-y)$
and $y+x = x+y$

Since the factor of -1 can be disregarded when finding the LCM, the opposite factors can be treated as the same by factoring the -1 out of one of the expressions. So, there are four different factors to consider, 2, x , $(x+y)$, and $(x-y)$. The highest power of $(x+y)$ is 2 and the other factors appear in the first power. Therefore,

$$LCM(y^2 - x^2, 2x^2 - 2xy, x^2 + 2xy + y^2) = 2x(x-y)(x+y)^2$$

Addition and Subtraction of Rational Expressions

Observe addition and subtraction of common fractions, as review in *Section R3*.

$$\frac{1}{2} + \frac{2}{3} - \frac{5}{6} = \frac{1 \cdot 3 + 2 \cdot 2 - 5}{6} = \frac{3 + 4 - 5}{6} = \frac{2}{6} = \frac{1}{3}$$

work out the numerator
convert fractions to the lowest common denominator
simplify, if possible

- d. To find the LCD, we first factor each denominator. Since

$$y^2 - 7y + 6 = (y - 6)(y - 1) \text{ and } y^2 - 5y - 6 = (y - 6)(y + 1),$$

then $LCD = (y - 6)(y - 1)(y + 1)$ and we calculate

multiply by the missing bracket

$$\begin{aligned} \frac{y+1}{y^2-7y+6} + \frac{y-1}{y^2-5y-6} &= \frac{y+1}{(y-6)(y-1)} + \frac{y-1}{(y-6)(y+1)} = \\ \frac{(y+1) \cdot (y+1) + (y-1) \cdot (y-1)}{(y-6)(y-1)(y+1)} &= \frac{y^2+2y+1+(y^2-1)}{(y-6)(y-1)(y+1)} = \\ \frac{2y^2+2y}{(y-6)(y-1)(y+1)} &= \frac{2y(y+1)}{(y-6)(y-1)(y+1)} = \frac{2y}{(y-6)(y-1)} \end{aligned}$$

- e. As in the previous examples, we first factor the denominators, including factoring out a negative from any opposite expression. So,

$$\frac{2x}{x^2-4} + \frac{5}{2-x} - \frac{1}{2+x} = \frac{2x}{(x-2)(x+2)} + \frac{5}{-(x-2)} - \frac{1}{x+2} =$$

$LCD = (x-2)(x+2)$

$$\begin{aligned} \frac{2x - 5(x+2) - 1(x-2)}{(x-2)(x+2)} &= \frac{2x - 5x - 10 - x + 2}{(x-2)(x+2)} = \\ \frac{-4x - 8}{(x-2)(x+2)} &= \frac{-4(x+2)}{(x-2)(x+2)} = \frac{-4}{(x-2)} \end{aligned}$$

- e. Recall that a negative exponent really represents a hidden fraction. So, we may choose to rewrite the negative powers as fractions, and then add them using techniques as shown in previous examples.

$$3(2x-1)^{-2} + (2x-1)^{-1} = \frac{1}{(2x-1)^2} + \frac{1}{2x-1} = \frac{1+1 \cdot (2x-1)}{(2x-1)^2} =$$

$$\frac{3+2x-1}{(2x-1)^2} = \frac{2x+2}{(2x-1)^2} = \frac{2(x+1)}{(2x-1)^2}$$

nothing to simplify this time

Note: Since addition (or subtraction) of rational expressions results in a rational expression, from now on the term “rational expression” will include sums of rational expressions as well.

Example 3 Adding Rational Expressions in Application Problems

Assume that a boat travels n kilometers up the river and then returns back to the starting point. If the water in the river flows with a constant current of c km/h, the total time for the round-trip can be calculated via the expression $\frac{n}{r+c} + \frac{n}{r-c}$, where r is the speed of the boat in still water in kilometers per hour. Write a single rational expression representing the total time of this trip.

Find the **least common multiple (LCM)** for each group of expressions.

7. $24a^3b^4$, $18a^5b^2$ 8. $6x^2y^2$, $9x^3y$, $15y^3$ 9. $x^2 - 4$, $x^2 + 2x$
 10. $10x^2$, $25(x^2 - x)$ 11. $(x - 1)^2$, $1 - x$ 12. $y^2 - 25$, $5 - y$
 13. $x^2 - y^2$, $xy + y^2$ 14. $5a - 15$, $a^2 - 6a + 9$ 15. $x^2 + 2x + 1$, $x^2 - 4x - 1$
 16. $n^2 - 7n + 10$, $n^2 - 8n + 15$ 17. $2x^2 - 5x - 3$, $2x^2 - x - 1$, $x^2 - 6x + 9$
 18. $1 - 2x$, $2x + 1$, $4x^2 - 1$ 19. $x^5 - 4x^4 + 4x^3$, $12 - 3x^2$, $2x + 4$

True or false? If true, explain why. If false, correct it.

20. $\frac{1}{2x} + \frac{1}{3x} = \frac{1}{5x}$ 21. $\frac{1}{x-3} + \frac{1}{3-x} = 0$ 22. $\frac{1}{x} + \frac{1}{y} = \frac{1}{x+y}$ 23. $\frac{3}{4} + \frac{x}{5} = \frac{3+x}{20}$

Perform the indicated operations and simplify if possible.

24. $\frac{x-2y}{x+y} + \frac{3y}{x+y}$ 25. $\frac{a+3}{a+1} - \frac{a-5}{a+1}$ 26. $\frac{4a+3}{a-3} - 1$
 27. $\frac{n+1}{n-2} + 2$ 28. $\frac{x^2}{x-y} + \frac{y^2}{y-x}$ 29. $\frac{4a-2}{a^2-49} + \frac{5+3a}{49-a^2}$
 30. $\frac{2y-3}{y^2-1} - \frac{4-y}{1-y^2}$ 31. $\frac{a^3}{a-b} + \frac{b^3}{b-a}$ 32. $\frac{1}{x+h} - \frac{1}{x}$
 33. $\frac{x-2}{x+3} + \frac{x+2}{x-4}$ 34. $\frac{x-1}{3x+1} + \frac{2}{x-3}$ 35. $\frac{4xy}{x^2-y^2} + \frac{x-y}{x+y}$
 36. $\frac{x-1}{3x+15} - \frac{x+3}{5x+25}$ 37. $\frac{y-2}{4y+8} - \frac{y+6}{5y+10}$ 38. $\frac{4x}{x-1} - \frac{2}{x+1} - \frac{4}{x^2-1}$
 39. $\frac{-2}{y+2} + \frac{5}{y-2} + \frac{y+3}{y^2-4}$ 40. $\frac{y}{y^2-y-20} + \frac{2}{y+4}$ 41. $\frac{5x}{x^2-6x+8} - \frac{3x}{x^2-x-12}$
 42. $\frac{9x+2}{3x^2-2x-8} + \frac{7}{3x^2+x-4}$ 43. $\frac{3y+2}{2y^2-y-10} + \frac{8}{2y^2-7y+5}$ 44. $\frac{6}{y^2+6y+9} + \frac{5}{y^2-9}$
 45. $\frac{3x-1}{x^2+2x-3} - \frac{x+4}{x^2-9}$ 46. $\frac{1}{x+1} - \frac{x}{x-2} + \frac{x^2+2}{x^2-x-2}$ 47. $\frac{2}{y+3} - \frac{y}{y-1} + \frac{y^2+2}{y^2+2y-3}$
 48. $\frac{4x}{x^2-1} + \frac{3x}{1-x} - \frac{4}{x-1}$ 49. $\frac{5y}{1-2y} - \frac{2y}{2y+1} + \frac{3}{4y^2-1}$ 50. $\frac{x+5}{x-3} - \frac{x+2}{x+1} - \frac{6x+10}{x^2-2x-3}$

Perform the indicated operations and simplify if possible.

51. $2x^{-3} + (3x)^{-1}$ 52. $(x^2 - 9)^{-1} + 2(x - 3)^{-1}$ 53. $\left(\frac{x+1}{3}\right)^{-1} - \left(\frac{x-4}{2}\right)^{-1}$
 54. $\left(\frac{a-3}{a^2} - \frac{a-3}{9}\right) \div \frac{a^2-9}{3a}$ 55. $\frac{x^2-4x+4}{2x+1} \cdot \frac{2x^2+x}{x^3-4x} - \frac{3x-2}{x+1}$ 56. $\frac{2}{x-3} - \frac{x}{x^2-x-6} \cdot \frac{x^2-2x-3}{x^2-x}$

Given $f(x)$ and $g(x)$, find $(f + g)(x)$ and $(f - g)(x)$. Leave the answer in simplified single fraction form.

57. $f(x) = \frac{x}{x+2}$, $g(x) = \frac{4}{x-3}$ 58. $f(x) = \frac{x}{x^2-4}$, $g(x) = \frac{1}{x^2+4x+4}$
 59. $f(x) = \frac{3x}{x^2+2x-3}$, $g(x) = \frac{1}{x^2-2x+1}$ 60. $f(x) = x + \frac{1}{x-1}$, $g(x) = \frac{1}{x+1}$

Solve each problem.

61. There are two part-time waitresses at a restaurant. One waitress works every fourth day, and the other one works every sixth day. Both waitresses were hired and start working on the same day. How often do they both work on the same day?



62. A cylindrical water tank is being filled and drained at the same time. To find the rate of change of the water level one could use the expression $\frac{H}{T_{in}} - \frac{H}{T_{out}}$, where H is the height of the water in the full tank while T_{in} and T_{out} represent the time needed to fill and empty the tank, respectively. Write the rate of change of the water level as a single algebraic fraction.
63. To determine the Canadian population percent growth over the past year, one could use the expression $100 \left(\frac{P_1}{P_0} - 1 \right)$, where P_1 represents the current population and P_0 represents the last year's population. Write this expression as a single algebraic fraction.
64. A boat travels k kilometers against a c km/h current. Assuming the current remains constant, one could calculate the total time, in hours, needed for the entire trip via the expression $\frac{k}{s-c} + \frac{k}{s+c}$, where s represents the speed of the boat in calm water. Write this expression as a single algebraic fraction.



RT4

Complex Fractions



When working with algebraic expressions, sometimes we come across needing to simplify expressions like these:

$$\frac{\frac{x^2 - 9}{x + 1}}{\frac{x + 3}{x^2 - 1}}, \quad 1 + \frac{1}{x}, \quad \frac{1}{x + 2} - \frac{1}{x + h + 2}, \quad \frac{1}{\frac{1}{a} - \frac{1}{b}}$$

A complex fraction is a quotient of rational expressions (including sums of rational expressions) where at least one of these expressions contains a fraction itself. In this section, we will examine two methods of simplifying such fractions.

Simplifying Complex Fractions

Definition 4.1 ▶ A **complex fraction** is a **quotient** of rational expressions (including their sums) that result in a fraction with more than two levels. For example, $\frac{1}{\frac{2}{3}}$ has three levels while $\frac{\frac{2x}{3}}{\frac{4x}{3}}$ has four levels. Such fractions can be **simplified to a single fraction** with only two levels. For example,

$$\frac{1}{\frac{2}{3}} = \frac{1}{2} \cdot \frac{3}{3} = \frac{3}{2}, \quad \text{or} \quad \frac{\frac{1}{2x}}{\frac{3}{4x^2}} = \frac{1}{2x} \cdot \frac{4x^2}{3} = \frac{2x}{3}$$

There are two common methods of simplifying complex fractions.

Method I (*multiplying by the reciprocal of the denominator*)

Replace the main division in the complex fraction with a multiplication of the numerator fraction by the reciprocal of the denominator fraction. We then simplify the resulting fraction if possible. Both examples given in *Definition 4.1* were simplified using this strategy.

Method I is the most convenient to use when both the numerator and the denominator of a complex fraction consist of single fractions. However, if either the numerator or the denominator of a complex fraction contains addition or subtraction of fractions, it is usually easier to use the method shown below.

Method II (*multiplying by LCD*)

Multiply the numerator and denominator of a complex fraction by the least common denominator of all the fractions appearing in the numerator or in the denominator of the complex fraction. Then, simplify the resulting fraction if possible. For example, to simplify $\frac{y + \frac{1}{x}}{x + \frac{1}{y}}$, multiply the numerator $y + \frac{1}{x}$ and the denominator $x + \frac{1}{y}$ by the $LCD \left(\frac{1}{x}, \frac{1}{y}\right) = xy$. So,

$$\frac{\left(y + \frac{1}{x}\right) \cdot xy}{\left(x + \frac{1}{y}\right) \cdot xy} = \frac{xy^2 + y}{x^2y + x} = \frac{y(xy + 1)}{x(xy + 1)} = \frac{y}{x}$$

Example 1 ▶ **Simplifying Complex Fractions**

Use a method of your choice to simplify each complex fraction.

a.
$$\frac{\frac{x^2-x-12}{x^2-2x-15}}{\frac{x^2+8x+12}{x^2-5x-14}}$$

b.
$$\frac{a+b}{\frac{1}{a^3} + \frac{1}{b^3}}$$

c.
$$\frac{x + \frac{1}{5}}{x - \frac{1}{3}}$$

d.
$$\frac{\frac{6}{x^2-4} - \frac{5}{x+2}}{\frac{7}{x^2-4} - \frac{4}{x-2}}$$

Solution ▶

- a. Since the expression $\frac{\frac{x^2-x-12}{x^2-2x-15}}{\frac{x^2+8x+12}{x^2-5x-14}}$ contains a single fraction in both the numerator and denominator, we will simplify it using method I, as below.

$$\frac{\frac{x^2-2x-8}{x^2-2x-15}}{\frac{x^2+8x+12}{x^2-4x-21}} = \frac{(x-4)(x+2)}{(x-5)(x+3)} \cdot \frac{(x-7)(x+3)}{(x+6)(x+2)} = \frac{(x-4)(x-7)}{(x-5)(x+6)}$$

factor and multiply by the reciprocal

- b. $\frac{a+b}{\frac{1}{a^3} + \frac{1}{b^3}}$ can be simplified in the following two ways:

Method I

$$\begin{aligned} \frac{a+b}{\frac{1}{a^3} + \frac{1}{b^3}} &= \frac{a+b}{\frac{b^3+a^3}{a^3b^3}} = \frac{(a+b)a^3b^3}{a^3+b^3} \\ &= \frac{(a+b)a^3b^3}{(a+b)(a^2-ab+b^2)} = \frac{a^3b^3}{a^2-ab+b^2} \end{aligned}$$

Method II

$$\begin{aligned} \frac{a+b}{\frac{1}{a^3} + \frac{1}{b^3}} &= \frac{a+b}{\frac{b^3+a^3}{a^3b^3}} = \frac{(a+b)a^3b^3}{b^3+a^3} \\ &= \frac{(a+b)a^3b^3}{(a+b)(a^2-ab+b^2)} = \frac{a^3b^3}{a^2-ab+b^2} \end{aligned}$$

Caution: In Method II, the factor that we multiply the complex fraction by **must be equal to 1**. This means that **the numerator and denominator of this factor must be exactly the same**.

- c. To simplify $\frac{x + \frac{1}{5}}{x - \frac{1}{3}}$, we will use method II. Multiplying the numerator and denominator by the LCD $(\frac{1}{5}, \frac{1}{3}) = 15$, we obtain

$$\frac{x + \frac{1}{5}}{x - \frac{1}{3}} = \frac{15x + 3}{15x - 5}$$

To simplify this expression, we can multiply the numerator and denominator by the lowest common denominator, which is $(a + h + 1)(a + 1)$. Thus,

$$\frac{\frac{1}{a+h+1} - \frac{1}{a+1}}{h} \cdot \frac{(a+h+1)(a+1)}{(a+h+1)(a+1)} = \frac{a+1 - (a+h+1)}{h(a+h+1)(a+1)}$$

$$= \frac{\cancel{a} + 1 - \cancel{a} - h - 1}{h(a+h+1)(a+1)} = \frac{-h}{h(a+h+1)(a+1)} = \frac{-1}{(a+h+1)(a+1)}$$

This bracket is essential!

keep the denominator in a factored form

RT.4 Exercises

Simplify each complex fraction.

1. $\frac{2 - \frac{1}{3}}{3 + \frac{7}{3}}$

2. $\frac{5 - \frac{3}{4}}{4 + \frac{1}{2}}$

3. $\frac{\frac{3}{8} - 5}{\frac{2}{3} + 6}$

4. $\frac{\frac{2}{3} + \frac{4}{5}}{\frac{3}{4} - \frac{1}{2}}$

Simplify each complex rational expression.

5. $\frac{\frac{x^3}{y}}{\frac{x^2}{y^3}}$

6. $\frac{\frac{n-5}{6n}}{\frac{n-5}{8n^2}}$

7. $\frac{1 - \frac{1}{a}}{4 + \frac{1}{a}}$

8. $\frac{\frac{2}{n} + 3}{\frac{5}{n} - 6}$

9. $\frac{\frac{9-3x}{4x+12}}{\frac{x-3}{6x-24}}$

10. $\frac{\frac{9}{15} \cdot \frac{y}{y}}{\frac{y}{y} - 6}$

11. $\frac{\frac{4}{x} - \frac{2}{y}}{\frac{4}{x} + \frac{2}{y}}$

12. $\frac{\frac{3}{a} + \frac{4}{b}}{\frac{4}{a} - \frac{3}{b}}$

13. $\frac{a - \frac{3a}{b}}{b - \frac{b}{a}}$

14. $\frac{\frac{1}{x} - \frac{1}{y}}{\frac{x^2 - y^2}{xy}}$

15. $\frac{\frac{4}{y} - \frac{y}{x^2}}{\frac{1}{x} - \frac{2}{y}}$

16. $\frac{\frac{5}{p} - \frac{1}{q}}{\frac{1}{5q^2} - \frac{5}{p^2}}$

17. $\frac{\frac{n-12}{n} + n}{n+4}$

18. $\frac{2t-1}{\frac{3t-2}{t} + 2t}$

19. $\frac{\frac{1}{a-h} - \frac{1}{a}}{h}$

20. $\frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$

21. $\frac{4 + \frac{12}{2x-3}}{5 + \frac{15}{2x-3}}$

22. $\frac{1 + \frac{3}{x+2}}{1 + \frac{6}{x-1}}$

23. $\frac{\frac{1}{b^2} - \frac{1}{a^2}}{\frac{1}{b} - \frac{1}{a}}$

24. $\frac{\frac{1}{x^2} - \frac{1}{y^2}}{\frac{1}{x} + \frac{1}{y}}$

25. $\frac{\frac{x+3}{x} - \frac{4}{x-1}}{\frac{x}{x-1} + \frac{1}{x}}$

26. $\frac{\frac{3}{x^2+6x+9} + \frac{3}{x+3}}{\frac{6}{x^2-9} + \frac{6}{3-x}}$

27. $\frac{\frac{1}{a^2} - \frac{1}{b^2}}{\frac{1}{a^3} + \frac{1}{b^3}}$

28. $\frac{\frac{4p^2-12p+9}{2p^2+7p-15}}{\frac{2p^2-15p+18}{p^2-p-30}}$

29. Are the expressions $\frac{x^{-2}+y^{-2}}{x^{-1}+y^{-1}}$ and $\frac{x+y}{x^2+y^2}$ equivalent? Explain why or why not.

Simplify each expression. Leave the answer with only **positive exponents**.

$$30. \frac{1}{a^{-2} - b^{-2}}$$

$$31. \frac{x^{-1} + x^{-2}}{3x^{-1}}$$

$$32. \frac{x^{-2}}{y^{-3} - x^{-3}}$$

$$33. \frac{1 - (2n+1)^{-1}}{1 + (2n+1)^{-1}}$$

Find and simplify the **difference quotient** $\frac{f(a+h)-f(a)}{h}$ for the given function.

$$34. f(x) = \frac{5}{x}$$

$$35. f(x) = \frac{2}{x^2}$$

$$36. f(x) = \frac{1}{1-x}$$

$$37. f(x) = -\frac{1}{x-2}$$

Simplify each **continued fraction**.

$$38. a - \frac{a}{1 - \frac{a}{1-a}}$$

$$39. 3 - \frac{2}{1 - \frac{2}{3 - \frac{2}{x}}}$$

$$40. a + \frac{a}{2 + \frac{1}{1 - \frac{2}{a}}}$$

RT5

Rational Equations and Graphs



In previous sections of this chapter, we worked with rational expressions. If two rational expressions are equated, a *rational equation* arises. Such equations often appear when solving application problems that involve rates of work or amounts of time considered in motion problems. In this section, we will discuss how to solve rational equations, with close attention to their domains. We will also take a look at the graphs of reciprocal functions, their properties and transformations.

Rational Equations

Definition 5.1 ▶ A **rational equation** is an equation involving only rational expressions and containing at least one fractional expression.

Here are some examples of rational equations:

$$\frac{x}{2} - \frac{12}{x} = -1, \quad \frac{x^2}{x-5} = \frac{25}{x-5}, \quad \frac{2x}{x-3} - \frac{6}{x} = \frac{18}{x^2-3x}$$

Attention! A rational equation contains an *equals* sign, while a rational expression does not. An equation can be solved for a given variable, while an expression can only be simplified or evaluated. For example, $\frac{x}{2} - \frac{12}{x}$ is an **expression** to simplify, while $\frac{x}{2} = \frac{12}{x}$ is an **equation** to solve.

When working with algebraic structures, it is essential to identify whether they are equations or expressions before applying appropriate strategies.

By *Definition 5.1*, rational equations contain one or more denominators. Since division by zero is not allowed, we need to pay special attention to the variable values that would make any of these denominators equal to zero. Such values would have to be excluded from the set of possible solutions. For example, neither 0 nor 3 can be solutions to the equation

$$\frac{2x}{x-3} - \frac{6}{x} = \frac{18}{x^2-3x},$$

as it is impossible to evaluate either of its sides for $x = 0$ or 3. So, when solving a rational equation, it is important to find its domain first.

Definition 5.2 ▶ The **domain** of the variable(s) of a rational equation (in short, the **domain of a rational equation**) is the **intersection of the domains** of all rational expressions within the equation.

As stated in *Definition 2.1*, the domain of each single algebraic fraction is the set of all real numbers except for the **zeros of the denominator** (the variable values that would make the denominator equal to zero). Therefore, the **domain of a rational equation** is the set of **all real numbers except for the zeros of all the denominators** appearing in this equation.

Example 1 ▶ **Determining Domains of Rational Equations**

Find the domain of the variable in each of the given equations.

a. $\frac{x}{2} - \frac{12}{x} = -1$

b. $\frac{2x}{x-2} = \frac{-3}{x} + \frac{4}{x-2}$

c. $\frac{2}{y^2-2y-8} - \frac{4}{y^2+6y+8} = \frac{2}{y^2-16}$

Solution ▶

a. The equation $\frac{x}{2} - \frac{12}{x} = -1$ contains two denominators, 2 and x . 2 is never equal to zero and x becomes zero when $x = 0$. Thus, the domain of this equation is $\mathbb{R} \setminus \{0\}$.

b. The equation $\frac{2x}{x-2} = \frac{-3}{x} + \frac{4}{x-2}$ contains two types of denominators, $x - 2$ and x . The $x - 2$ becomes zero when $x = 2$, and x becomes zero when $x = 0$. Thus, the domain of this equation is $\mathbb{R} \setminus \{0, 2\}$.

c. The equation $\frac{2}{y^2-2y-8} - \frac{4}{y^2+6y+8} = \frac{2}{y^2-16}$ contains three different denominators. To find the zeros of these denominators, we solve the following equations by factoring:

$$y^2 - 2y - 8 = 0$$

$$y^2 + 6y + 8 = 0$$

$$y^2 - 16 = 0$$

$$(y - 4)(y + 2) = 0$$

$$(y + 4)(y + 2) = 0$$

$$(y - 4)(y + 4) = 0$$

$$y = 4 \text{ or } y = -2$$

$$y = -4 \text{ or } y = -2$$

$$y = 4 \text{ or } y = -4$$

So, -4 , -2 , and 4 must be excluded from the domain of this equation. Therefore, the domain $D = \mathbb{R} \setminus \{-4, -2, 4\}$.

To solve a rational equation, it is convenient to clear all the fractions first and then solve the resulting polynomial equation. This can be achieved by multiplying all the terms of the equation by the least common denominator.

Caution! Only **equations**, not expressions, can be changed equivalently by **multiplying** both of their sides by the **LCD**.

Multiplying **expressions** by any number other than 1 creates expressions that are **NOT equivalent** to the original ones. So, avoid multiplying rational expressions by the LCD.

Example 2 ▶ **Solving Rational Equations**

Solve each equation.

a. $\frac{x}{2} - \frac{12}{x} = -1$

b. $\frac{2x}{x-2} = \frac{-3}{x} + \frac{4}{x-2}$

c. $\frac{2}{y^2-2y-8} - \frac{4}{y^2+6y+8} = \frac{2}{y^2-16}$

d. $\frac{x-1}{x-3} = \frac{2}{x-3}$

Solution

- a. The domain of the equation $\frac{x}{2} - \frac{12}{x} = -1$ is the set $\mathbb{R} \setminus \{0\}$, as discussed in *Example 1a*. The $LCM(2, x) = 2x$, so we calculate

$$\frac{x}{2} - \frac{12}{x} = -1 \quad / \cdot 2x$$

$$\cancel{2x} \cdot \frac{x}{\cancel{2}} - \cancel{2x} \cdot \frac{12}{\cancel{x}} = -1 \cdot 2x$$

multiply each term by the LCD

$$x^2 - 24 = -2x$$

$$x^2 + 2x - 24 = 0$$

$$(x + 6)(x - 4) = 0$$

factor to find the possible roots

$$x = -6 \text{ or } x = 4$$

Since both of these numbers belong to the domain, the solution set of the original equation is $\{-6, 4\}$.

- b. The domain of the equation $\frac{2x}{x-2} = \frac{-3}{x} + \frac{4}{x-2}$ is the set $\mathbb{R} \setminus \{0, 2\}$, as discussed in *Example 1b*. The $LCM(x-2, x) = x(x-2)$, so we calculate

$$\frac{2x}{x-2} = \frac{-3}{x} + \frac{4}{x-2} \quad / \cdot x(x-2)$$

$$\cancel{x(x-2)} \cdot \frac{2x}{\cancel{x-2}} = \frac{-3}{\cancel{x}} \cdot \cancel{x(x-2)} + \frac{4}{\cancel{x-2}} \cdot \cancel{x(x-2)}$$

$$2x^2 = -3(x-2) + 4x$$

$$2x^2 = -3x + 6 + 4x$$

$$2x^2 - x + 6 = 0$$

$$(2x + 3)(x - 2) = 0$$

factor to find the possible roots

$$x = -\frac{3}{2} \text{ or } x = 2$$

Since 2 is excluded from the domain, there is only one solution to the original equation, $x = -\frac{3}{2}$.

- c. The domain of the equation $\frac{2}{y^2-2y-8} - \frac{4}{y^2+6y+8} = \frac{2}{y^2-16}$ is the set $\mathbb{R} \setminus \{-4, -2, 4\}$, as discussed in *Example 1c*. To find the LCD, it is useful to factor the denominators first. Since

$$y^2 - 2y - 8 = (y - 4)(y + 2),$$

$$y^2 + 6y + 8 = (y + 4)(y + 2), \text{ and}$$

$y^2 - 16 = (y - 4)(y + 4)$, then the LCD needed to clear the fractions in the original equation is $(y - 4)(y + 4)(y + 2)$. So, we calculate

$$\frac{2}{(y-4)(y+2)} - \frac{4}{(y+4)(y+2)} = \frac{2}{(y-4)(y+4)} \quad / \cdot (y-4)(y+4)(y+2)$$

$$\begin{aligned} \frac{\cancel{(y-4)}(y+4)\cancel{(y+2)}}{\cancel{(y-4)}(y+2)} \cdot \frac{2}{\cancel{(y-4)}(y+2)} - \frac{\cancel{(y-4)}(y+4)\cancel{(y+2)}}{\cancel{(y+4)}(y+2)} \cdot \frac{4}{\cancel{(y+4)}(y+2)} \\ = \frac{2}{\cancel{(y-4)}(y+4)} \cdot \cancel{(y-4)}(y+4)(y+2) \end{aligned}$$

$$2(y+4) - 4(y-4) = 2(y+2)$$

$$2y + 8 - 4y + 16 = 2y + 4$$

$$20 = 4y$$

$$y = 5$$

Since 5 is in the domain, this is the true solution.

- d. First, we notice that the domain of the equation $\frac{x-1}{x-3} = \frac{2}{x-3}$ is the set $\mathbb{R} \setminus \{3\}$. To solve this equation, we can multiply it by the *LCD* = $x - 3$, as in the previous examples, or we can apply the method of cross-multiplication, as the equation is a proportion. Here, we show both methods.

Use the method
of your choice
– either one is
fine.

Multiplication by LCD:

$$\frac{x-1}{x-3} = \frac{2}{x-3}$$

$/ \cdot (x-3)$

$$x - 1 = 2$$

$$x = 3$$

this multiplication
is permitted as
 $x - 3 \neq 0$

Cross-multiplication:

$$\frac{x-1}{x-3} = \frac{2}{x-3}$$

$$(x-1)(x-3) = 2(x-3)$$

$$x - 1 = 2$$

$$x = 3$$

$/ \div (x-3)$

this division is
permitted as
 $x - 3 \neq 0$

Since 3 is excluded from the domain, there is **no solution** to the original equation.

Summary of Solving Rational Equations in One Variable

- **Determine the domain** of the variable.
- **Clear** all the **fractions by multiplying** both sides of the equation **by the LCD** of these fractions.
- **Find possible solutions** by solving the resulting equation.
- **Check** the possible solutions **against the domain**. The solution set consists of only these possible solutions that belong to the domain.

Graphs of Basic Rational Functions

So far, we discussed operations on rational expressions and solving rational equations. Now, we will look at rational functions, such as

$$f(x) = \frac{1}{x}, \quad g(x) = \frac{-2}{x+3}, \quad \text{or} \quad h(x) = \frac{x-3}{x-2}.$$

Definition 5.3 ▶ A **rational function** is any function that can be written in the form

$$f(x) = \frac{P(x)}{Q(x)},$$

where P and Q are polynomials and Q is not a zero polynomial.

The **domain** D_f of such function f includes all x -values for which $Q(x) \neq 0$.

Example 3 ▶ **Finding the Domain of a Rational Function**

Find the domain of each function.

a. $g(x) = \frac{-2}{x+3}$

b. $h(x) = \frac{x-3}{x-2}$

Solution ▶

a. Since $x + 3 = 0$ for $x = -3$, the domain of g is the set of all real numbers except for -3 . So, the domain $D_g = \mathbb{R} \setminus \{-3\}$.

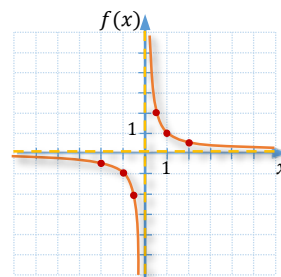
b. Since $x - 2 = 0$ for $x = 2$, the domain of h is the set of all real numbers except for 2 . So, the domain $D_h = \mathbb{R} \setminus \{2\}$.

Note: The subindex f in the notation D_f indicates that the domain is of function f .

To graph a rational function, we usually start by making a table of values. Because the graphs of rational functions are typically nonlinear, it is a good idea to plot at least 3 points on each side of each x -value where the function is undefined. For example, to graph the

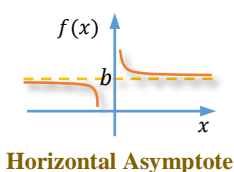
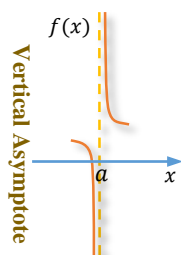
x	$f(x)$
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$
0	undefined
$-\frac{1}{2}$	-2
-1	-1
-2	$-\frac{1}{2}$

basic rational function, $f(x) = \frac{1}{x}$, called the **reciprocal function**, we evaluate f for a few points to the right of zero and to the left of zero. This is because f is undefined at $x = 0$, which means that the graph of f does not cross the y -axis. After plotting the obtained points, we connect them within each group, to the right of zero and to the left of zero, creating two disjoint curves. To see the shape of each curve clearly, we might need to evaluate f at some additional points.



The **domain** of the reciprocal function $f(x) = \frac{1}{x}$ is $\mathbb{R} \setminus \{0\}$, as the denominator x must be different than zero. Projecting the graph of this function onto the y -axis helps us determine the **range**, which is also $\mathbb{R} \setminus \{0\}$.

There is another interesting feature of the graph of the reciprocal function $f(x) = \frac{1}{x}$. Observe that the graph approaches two lines, $y = 0$, the x -axis, and $x = 0$, the y -axis. These lines are called **asymptotes**. They effect the shape of the graph, but they themselves do not belong to the graph. To indicate the fact that asymptotes do not belong to the graph, we use a dashed line when graphing them.



In general, if the y -values of a rational function approach ∞ or $-\infty$ as the x -values approach a real number a , the vertical line $x = a$ is a vertical asymptote of the graph. This can be recorded with the use of arrows, as follows:

$$x = a \text{ is a } \mathbf{vertical\ asymptote} \Leftrightarrow y \rightarrow \infty \text{ (or } -\infty) \text{ when } x \rightarrow a.$$

read: approaches

Also, if the y -values approach a real number b as x -values approach ∞ or $-\infty$, the horizontal line $y = b$ is a horizontal asymptote of the graph. Again, using arrows, we can record this statement as:

$$y = a \text{ is a } \mathbf{horizontal\ asymptote} \Leftrightarrow y \rightarrow b \text{ when } x \rightarrow \infty \text{ (or } -\infty).$$

Example 4

Graphing and Analysing the Graphs of Basic Rational Functions

For each function, state its domain and the equation of the vertical asymptote, graph it, and then state its range and the equation of the horizontal asymptote.

a. $g(x) = \frac{-2}{x+3}$

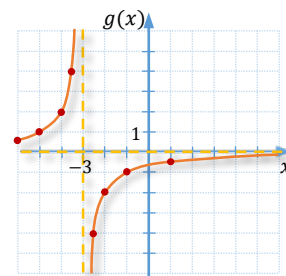
b. $h(x) = \frac{x-3}{x-2}$

Solution

- a. The domain of function $g(x) = \frac{-2}{x+3}$ is $D_g = \mathbb{R} \setminus \{-3\}$, as discussed in *Example 3a*. Since -3 is excluded from the domain, we expect the vertical asymptote to be at $x = -3$.

x	$g(x)$
$-\frac{5}{2}$	-4
-2	-2
-1	-1
1	$-\frac{1}{2}$
-3	<i>undefined</i>
$-\frac{7}{2}$	4
-4	2
-5	1
-6	$\frac{2}{3}$

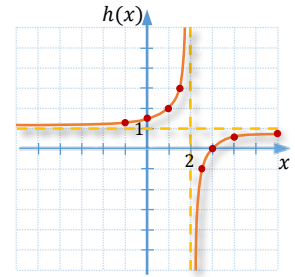
To graph function g , we evaluate it at some points to the right and to the left of -3 . The reader is encouraged to check the values given in the table. Then, we draw the vertical asymptote $x = -3$ and plot and join the obtained points on each side of this asymptote. The graph suggests that the horizontal asymptote is the x -axis. Indeed, the value of zero cannot be attained by the function $g(x) = \frac{-2}{x+3}$, as in order for a fraction to become zero, its numerator would have to be zero. So, the range of function g is $\mathbb{R} \setminus \{0\}$ and $y = 0$ is the equation of the horizontal asymptote.



- b. The domain of function $h(x) = \frac{x-3}{x-2}$ is $D_h = \mathbb{R} \setminus \{2\}$, as discussed in *Example 3b*. Since 2 is excluded from the domain, we expect the vertical asymptote to be at $x = 2$.

x	$h(x)$
-1	$\frac{4}{3}$
0	$\frac{3}{2}$
1	2
$\frac{3}{2}$	3
2	undefined
$\frac{5}{2}$	-1
3	0
4	$\frac{1}{2}$
6	$\frac{3}{4}$

As before, to graph function h , we evaluate it at some points to the right and to the left of 2. Then, we draw the vertical asymptote $x = 2$ and plot and join the obtained points on each side of this asymptote. The graph suggests that the horizontal asymptote is the line $y = 1$. Thus, the range of function h is $\mathbb{R} \setminus \{1\}$.



Notice that $\frac{x-3}{x-2} = \frac{x-2-1}{x-2} = \frac{x-2}{x-2} - \frac{1}{x-2} = 1 - \frac{1}{x-2}$. Since $\frac{1}{x-2}$ is never equal to zero than $1 - \frac{1}{x-2}$ is never equal to 1. This confirms the range and the horizontal asymptote stated above.

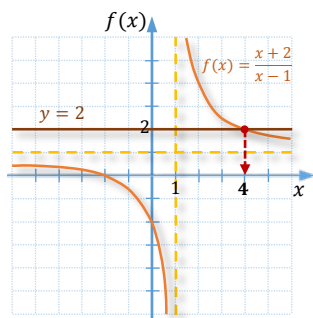
Example 5 ▶ **Connecting the Algebraic and Graphical Solutions of Rational Equations**

Given that $f(x) = \frac{x+2}{x-1}$, find all the x -values for which $f(x) = 2$. Illustrate the situation with a graph.

Solution ▶ To find all the x -values for which $f(x) = 2$, we replace $f(x)$ in the equation $f(x) = \frac{x+2}{x-1}$ with 2 and solve the resulting equation. So, we have

$$\begin{aligned} 2 &= \frac{x+2}{x-1} && / \cdot (x-1) \\ 2x - 2 &= x + 2 && / -x, +2 \\ x &= 4 \end{aligned}$$

Thus, $f(x) = 2$ for $x = 4$.



The geometrical connection can be observed by graphing the function $f(x) = \frac{x+2}{x-1} = \frac{x-1+3}{x-1} = 1 + \frac{3}{x-1}$ and the line $y = 2$ on the same grid, as illustrated by the accompanying graph. The x -coordinate of the intersection of the two graphs is the solution to the equation $2 = \frac{x+2}{x-1}$. This also means that $f(4) = \frac{4+2}{4-1} = 2$. So, we can say that $f(4) = 2$.

Example 6 ▶ **Graphing the Reciprocal of a Linear Function**

Suppose $f(x) = 2x - 3$.

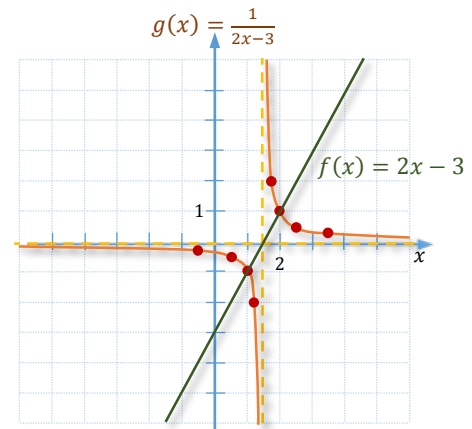
- a. Determine the reciprocal function $g(x) = \frac{1}{f(x)}$ and its domain D_g .

- b. Determine the equation of the vertical asymptote of the reciprocal function g .
- c. Graph the function f and its reciprocal function g on the same grid. Then, describe the relations between the two graphs.

Solution

- a. The reciprocal of $f(x) = 2x - 3$ is the function $g(x) = \frac{1}{2x-3}$. Since $2x - 3 = 0$ for $x = \frac{3}{2}$, then the domain $D_g = \mathbb{R} \setminus \{\frac{3}{2}\}$.
- b. A vertical asymptote of a rational function in simplified form is a vertical line passing through any of the x -values that are excluded from the domain of such a function. So, the equation of the vertical asymptote of function $g(x) = \frac{1}{2x-3}$ is $x = \frac{3}{2}$.
- c. To graph functions f and g , we can use a table of values as below.

x	$f(x)$	$g(x)$
$-\frac{1}{2}$	-4	$-\frac{1}{4}$
$\frac{1}{2}$	-2	$-\frac{1}{2}$
1	-1	-1
$\frac{5}{4}$	$\frac{1}{2}$	2
$\frac{3}{2}$	0	undefined
$\frac{7}{4}$	$-\frac{1}{2}$	-2
2	1	1
$\frac{5}{2}$	2	$\frac{1}{2}$
$\frac{7}{2}$	4	$\frac{1}{4}$



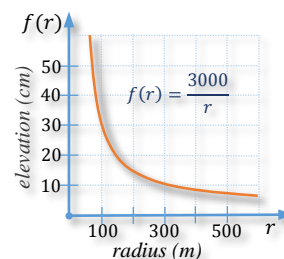
Notice that the vertical asymptote of the reciprocal function comes through the zero of the linear function. Also, the values of both functions are positive to the right of $\frac{3}{2}$ and negative to the left of $\frac{3}{2}$. In addition, $f(2) = g(2) = 1$ and $f(1) = g(1) = -1$. This is because the reciprocal of 1 is 1 and the reciprocal of -1 is -1 . For the rest of the values, observe that the values of the linear function that are very close to zero become very large in the reciprocal function and conversely, the values of the linear function that are very far from zero become very close to zero in the reciprocal function. This suggests the horizontal asymptote at zero.

Example 7**Using Properties of a Rational Function in an Application Problem**

elevation

Elevating the outer rail of a track allows for a safer turn of a train on a circular curve. The elevation depends on the allowable speed of the train and the radius of the curve. Suppose that a circular curve with a radius of r meters is being designed for a train travelling 100 kilometers per hour. Assume that the function $f(r) = \frac{3000}{r}$ can be used to calculate the proper elevation $y = f(r)$, in centimeters, for the outer rail.

- Evaluate $f(300)$ and interpret the result.
- Suppose that the outer rail for a curve is elevated 12 centimeters. Find the radius of this curve.
- Observe the accompanying graph of the function f and discuss how the elevation of the outer rail changes as the radius r increases.



Solution ▶ a. $f(300) = \frac{3000}{300} = 10$. Thus, the outer rail on a curve with a 300-meter radius should be elevated **10 centimeters** for a train to travel through it at 100 km/hr safely.

- b. Since the elevation $y = f(r) = 12$ centimeters, to find the corresponding value of r , we need to solve the equation

$$12 = \frac{3000}{r}.$$

After multiplying this equation by r and dividing it by 12, we obtain

$$r = \frac{3000}{12} = 250$$

So, the radius of this curve should be **250 meters**.

- c. As the radius increases, the outer rail needs less elevation.

RT.5 Exercises

State the **domain** for each equation. There is no need to solve it.

1. $\frac{x+5}{4} - \frac{x+3}{3} = \frac{x}{6}$

2. $\frac{5}{6a} - \frac{a}{4} = \frac{8}{2a}$

3. $\frac{3}{x+4} = \frac{2}{x-9}$

4. $\frac{4}{3x-5} + \frac{2}{x} = \frac{9}{4x+7}$

5. $\frac{4}{y^2-25} - \frac{1}{y+5} = \frac{2}{y-7}$

6. $\frac{x}{2x-6} - \frac{3}{x^2-6x+9} = \frac{x-2}{3x-9}$

Solve each equation.

7. $\frac{3}{8} + \frac{1}{3} = \frac{x}{12}$

8. $\frac{1}{4} - \frac{5}{6} = \frac{1}{y}$

9. $x + \frac{8}{x} = -9$

10. $\frac{4}{3a} - \frac{3}{a} = \frac{10}{3}$

11. $\frac{r}{8} + \frac{r-4}{12} = \frac{r}{24}$

12. $\frac{n-2}{2} - \frac{n}{6} = \frac{4n}{9}$

13. $\frac{5}{r+20} = \frac{3}{r}$

15. $\frac{y+2}{y} = \frac{5}{3}$

17. $\frac{x}{x-1} - \frac{x^2}{x-1} = 5$

19. $\frac{1}{3} - \frac{x-1}{x} = \frac{x}{3}$

21. $\frac{1}{y-1} + \frac{5}{12} = \frac{-2}{3y-3}$

23. $\frac{8}{3k+9} - \frac{8}{15} = \frac{2}{5k+15}$

25. $\frac{3}{y-2} + \frac{2y}{4-y^2} = \frac{5}{y+2}$

27. $\frac{1}{2x+10} = \frac{8}{x^2-25} - \frac{2}{x-5}$

29. $\frac{6}{x^2-4x+3} - \frac{1}{x-3} = \frac{1}{4x-4}$

31. $\frac{5}{x-4} - \frac{3}{x-1} = \frac{x^2-1}{x^2-5x+4}$

33. $\frac{3x}{x+2} + \frac{72}{x^3+8} = \frac{24}{x^2-2x+4}$

35. $\frac{x}{2x-9} - 3x = \frac{10}{9-2x}$

14. $\frac{5}{a+4} = \frac{3}{a-2}$

16. $\frac{x-4}{x+6} = \frac{2x+3}{2x-1}$

18. $3 - \frac{12}{x^2} = \frac{5}{x}$

20. $\frac{1}{x} + \frac{2}{x+10} = \frac{x}{x+10}$

22. $\frac{7}{6x+3} - \frac{1}{3} = \frac{2}{2x+1}$

24. $\frac{6}{m-4} + \frac{5}{m} = \frac{2}{m^2-4m}$

26. $\frac{x}{x-2} + \frac{x}{x^2-4} = \frac{x+3}{x+2}$

28. $\frac{5}{y+3} = \frac{1}{4y^2-36} + \frac{2}{y-3}$

30. $\frac{7}{x-2} - \frac{8}{x+5} = \frac{1}{2x^2+6x-20}$

32. $\frac{y}{y+1} + \frac{3y+5}{y^2+4y+3} = \frac{2}{y+3}$

34. $\frac{4}{x+3} + \frac{7}{x^2-3x+9} = \frac{108}{x^3+27}$

36. $\frac{-2}{x^2+2x-3} - \frac{5}{3-3x} = \frac{4}{3x+9}$

For the given rational function f , find all values of x for which $f(x)$ has the indicated value.

37. $f(x) = 2x - \frac{15}{x}$; $f(x) = 1$

38. $f(x) = \frac{x-5}{x+1}$; $f(x) = \frac{3}{5}$

39. $g(x) = \frac{-3x}{x+3} + x$; $g(x) = 4$

40. $g(x) = \frac{4}{x} + \frac{1}{x-2}$; $g(x) = 3$

Graph each rational function. State its **domain**, **range** and the equations of the **vertical** and **horizontal asymptotes**.

41. $f(x) = \frac{2}{x}$

42. $g(x) = -\frac{1}{x}$

43. $h(x) = \frac{2}{x-3}$

44. $f(x) = \frac{-1}{x+1}$

45. $g(x) = \frac{x-1}{x+2}$

46. $h(x) = \frac{x+2}{x-3}$

For each function f , find its reciprocal function $g(x) = \frac{1}{f(x)}$ and graph both functions on the same grid. Then, state the equations of the **vertical** and **horizontal asymptotes** of function g .

47. $f(x) = \frac{1}{2}x + 1$

48. $f(x) = -x + 2$

49. $f(x) = -2x - 3$

Solve each equation.

$$50. \frac{x}{1 + \frac{1}{x+1}} = x - 3$$

$$51. \frac{2 - \frac{1}{x}}{4 - \frac{1}{x^2}} = 1$$

Solve each problem.

52. Suppose that the number of vehicles searching for a parking place at UFV parking lot is modelled by the function

$$f(x) = \frac{x^2}{2(1-x)},$$

where $0 \leq x < 1$ is a quantity known as **traffic intensity**.



- a. For each traffic intensity, find the number of vehicles searching for a parking place. *Round your answer to the nearest one.*

i. 0.2 ii. 0.8 iii. 0.98

- b. Observing answers to part (a), conclude how does the number of vehicles searching for a parking place changes when the traffic intensity get closer to 1.

53. Suppose that the percent of deaths caused by smoking, called the **incidence rate**, is modelled by the rational function



$$D(x) = \frac{x-1}{x},$$

where x tells us how many times a smoker is more likely to die of lung cancer than a non-smoker.

- a. Find $D(10)$ and interpret it in the context of the problem.
 b. Find the x -value corresponding to the incidence rate of 0.5.
 c. Under what condition would the incidence rate equal to 0?

RT6

Applications of Rational Equations



In previous sections of this chapter, we studied operations on rational expressions, simplifying complex fractions, and solving rational equations. These skills are needed when working with real-world problems that lead to a rational equation. The common types of such problems are motion or work problems. In this section, we first discuss how to solve a rational formula for a given variable, and then present several examples of application problems involving rational equations.

Formulas Containing Rational Expressions

Solving application problems often involves working with formulas. We might need to form a formula, evaluate it, or solve it for a desired variable. The basic strategies used to solve a formula for a variable were shown in *Section L2* and *F4*. Recall the guidelines that we used to isolate the desired variable:

- **Reverse operations** to clear unwanted factors or addends;
Example: To solve $\frac{A+B}{2} = C$ for A , we multiply by 2 and then subtract B .
- **Multiply by the LCD to keep** the desired variable **in the numerator**;
Example: To solve $\frac{A}{1+r} = P$ for r , first, we multiply by $(1+r)$.
- **Take the reciprocal** of both sides of the equation **to keep** the desired variable **in the numerator** (this applies to proportions only);
Example: To solve $\frac{1}{C} = \frac{A+B}{AB}$ for C , we can take the reciprocal of both sides to obtain $C = \frac{AB}{A+B}$.
- **Factor to keep** the desired variable **in one place**.
Example: To solve $P + Prt = A$ for P , we first factor P out.

Below we show how to solve formulas containing rational expressions, using a combination of the above strategies.

Example 1 ▶ Solving Rational Formulas for a Given Variable

Solve each formula for the indicated variable.

a. $\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$, for p

b. $L = \frac{dR}{D-d}$, for D

c. $L = \frac{dR}{D-d}$, for d

Solution ▶

- a. **Solution I:** First, we isolate the term containing p , by ‘moving’ $\frac{1}{q}$ to the other side of the equation. So,

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{q} \quad / -\frac{1}{q}$$

$$\frac{1}{f} - \frac{1}{q} = \frac{1}{p}$$

rewrite from the right to the left,

$$\frac{1}{p} = \frac{q-f}{fq}$$

and perform the subtraction to leave this side as a single fraction

Then, to bring p to the numerator, we can take the reciprocal of both sides of the equation, obtaining

$$p = \frac{fq}{q-f}$$

Caution! This method can be applied only to a proportion (an equation with a **single fraction on each side**).

Solution II: The same result can be achieved by multiplying the original equation by the $LCD = fpq$, as shown below

$$\begin{aligned} \frac{1}{f} &= \frac{1}{p} + \frac{1}{q} && / \cdot fpq \\ pq &= fq + fp && / -fp \\ pq - fp &= fq && \\ \text{factor } p \text{ out} & \left\{ \begin{aligned} p(q-f) &= fq && / \div (q-f) \\ p &= \frac{fq}{q-f} \end{aligned} \right. \end{aligned}$$

- b. To solve $L = \frac{dR}{D-d}$ for D , we may start with multiplying the equation by the denominator to bring the variable D to the numerator. So,

This can be done in one step by interchanging L with $D - d$. The movement of the expressions resembles that of a teeter-totter.

$$\begin{aligned} L &= \frac{dR}{D-d} && / \cdot (D-d) \\ L(D-d) &= d && / \div L \\ D-d &= \frac{dR}{L} && / +d \end{aligned}$$

$$D = \frac{dR}{L} + d = \frac{dR + dL}{L}$$

Both forms are correct answers.

- c. When solving $L = \frac{dR}{D-d}$ for d , we first observe that the variable d appears in both the numerator and denominator. Similarly as in the previous example, we bring the d from the denominator to the numerator by multiplying the formula by the denominator $D - d$. Thus,

$$\begin{aligned} L &= \frac{dR}{D-d} && / \cdot (D-d) \\ L(D-d) &= dR. \end{aligned}$$

Then, to keep the d in one place, we need to expand the bracket, collect terms with d , and finally factor the d out. So, we have

$$LD - Ld = dR \quad /+Ld$$

$$LD = dR + Ld$$

$$LD = d(R + L) \quad /\div (R + L)$$

$$\frac{LD}{R + L} = d$$

Obviously, the final formula can be written starting with d ,

$$d = \frac{LD}{R + L}.$$

Example 2 ▶ Forming and Evaluating a Rational Formula

Suppose a trip consists of two parts of the same distance d .

- a. Given the speed v_1 for the first part of the trip and v_2 for the second part of the trip, find a formula for the average speed v for the whole trip. (*Make sure to leave this formula in the simplified form.*)
- b. Find the average speed v for the whole trip, if the speed for the first part of the trip was 75 km/h and the speed for the second part of the trip was 105 km/h.
- c. How does the v -value from (b) compare to the average of v_1 and v_2 ?

Solution ▶ a. The total distance, D , for the whole trip is $d + d = 2d$. The total time, T , for the whole trip is the sum of the times for the two parts of the trip, t_1 and t_2 . From the relation *rate · time = distance*, we have

$$t_1 = \frac{d}{v_1} \quad \text{and} \quad t_2 = \frac{d}{v_2}.$$

Therefore,

$$t = \frac{d}{v_1} + \frac{d}{v_2},$$

which after substituting to the formula for the average speed, $V = \frac{D}{T}$, gives us

$$V = \frac{2d}{\frac{d}{v_1} + \frac{d}{v_2}}.$$

Since the formula involves a complex fraction, it should be simplified. We can do this by multiplying the numerator and denominator by the *LCD* = $v_1 v_2$. So, we have

$$V = \frac{2d}{\frac{d}{v_1} + \frac{d}{v_2}} \cdot \frac{v_1 v_2}{v_1 v_2}$$

$$V = \frac{2d v_1 v_2}{\frac{d v_1 v_2}{v_1} + \frac{d v_1 v_2}{v_2}}$$

$$V = \frac{2dv_1v_2}{dv_2 + dv_1} \quad \text{factor the } d$$

$$V = \frac{2dv_1v_2}{d(v_2 + v_1)}$$

$$V = \frac{2v_1v_2}{v_2 + v_1}$$

Note 1: The average speed in this formula does not depend on the distance travelled.

Note 2: The average speed for the total trip is not the average (arithmetic mean) of the speeds for each part of the trip. In fact, this formula represents the **harmonic mean** of the two speeds.

- b. Since $v_1 = 75$ km/h and $v_2 = 105$ km/h, using the formula developed in *Example 2a*, we calculate

$$v = \frac{2 \cdot 75 \cdot 105}{75 + 105} = \frac{15750}{180} = \mathbf{87.5 \text{ km/h}}$$

- c. The average speed for the whole trip, $v = 87.5$ km/h, is lower than the average of the speeds for each part of the trip, which is $\frac{75+105}{2} = 90$ km/h.

Applied Problems

Many types of application problems were already introduced in *Sections L3* and *E2*. Some of these types, for example motion problems, may involve solving rational equations. Below we show examples of proportion and motion problems as well as introduce another type of problems, work problems.

Proportion Problems

When forming a proportion,

$$\frac{\text{category I before}}{\text{category II before}} = \frac{\text{category I after}}{\text{category II after}}$$

it is essential that the same type of data is placed in the same row or the same column.

Recall: To solve a proportion

$$\frac{a}{b} = \frac{c}{d},$$

for example, for a , it is enough to multiply the equation by b . This gives us

$$a = \frac{bc}{d}.$$

Similarly, to solve

$$\frac{a}{b} = \frac{c}{d}$$

for b , we can use the cross-multiplication method, which eventually (we encourage the reader to check this) leads us to

$$a = \frac{ad}{c}.$$

Notice that in both cases the desired variable equals the **product** of the blue variables lying **across** each other, **divided by the remaining** purple variable. This is often referred to as the ‘cross multiply and divide’ approach to solving a proportion.

In statistics, proportions are often used to estimate the population by analysing its sample in situations where the exact count of the population is too costly or not possible to obtain.

Example 3 ▶ Estimating Numbers of Wild Animals



To estimate the number of wild horses in a particular area in Nevada, a forest ranger catches 452 wild horses, tags them, and releases them. In a week, he catches 95 horses out of which 10 are found to be tagged. Assuming that the horses mix freely when they are released, estimate the number of wild horses in this region. *Round your answer to the nearest hundreds.*

Solution ▶

Suppose there are x wild horses in region. 452 of them were tagged, so the ratio of the tagged horses to the whole population of the wild horses there is

$$\frac{452}{x}$$

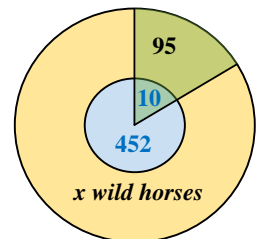
The ratio of the tagged horses found in the sample of 95 horses caught in the later time is

$$\frac{10}{95}$$

So, we form the proportion:

$$\frac{452}{x} = \frac{10}{95}$$

population
sample
tagged horses
all horses



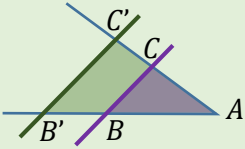
After solving for x , we have

$$x = \frac{452 \cdot 95}{10} = 4294 \approx 4300$$

So, we can say that approximately **4300** wild horses live in this region.

In geometry, proportions are the defining properties of similar figures. One frequently used theorem that involves proportions is the theorem about similar triangles, attributed to the Greek mathematician Thales.

Thales' Theorem ▶ Two triangles are **similar** iff the ratios of the corresponding sides are the same.



$$\triangle ABC \sim \triangle AB'C' \Leftrightarrow \frac{AB}{AB'} = \frac{AC}{AC'} = \frac{BC}{B'C'}$$

Example 4 ▶ **Using Similar Triangles in an Application Problem**

A cross-section of a small storage room is in the shape of a right triangle with a height of 2 meters and a base of 1.2 meters, as shown in *Figure 6.1a*. Find the size of the largest cubic box fitting in this room when placed with its base on the floor.

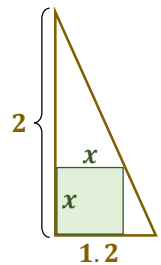


Figure 6.1a

Solution ▶ Suppose that the height of the box is x meters. Since the height of the storage room is 2 meters, the expression $2 - x$ represents the height of the wall above the box, as shown in *Figure 6.1b*.

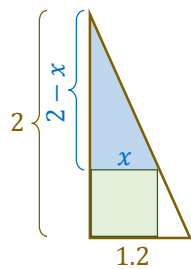


Figure 6.1b

Since the blue and brown triangles are similar, we can use the Thales' Theorem to form the proportion

$$\frac{2 - x}{2} = \frac{x}{1.2}$$

Employing cross-multiplication, we obtain

$$2.4 - 1.2x = 2x$$

$$2.4 = 3.2x$$

$$x = \frac{2.4}{3.2} = \mathbf{0.75}$$

So, the dimensions of the largest cubic box fitting in this storage room are 75 cm by 75 cm by 75 cm.

Motion Problems

Motion problems in which we compare times usually involve solving rational equations. This is because when solving the motion formula $\text{rate } R \cdot \text{time } T = \text{distance } D$ for time, we create a fraction

$$\text{time } T = \frac{\text{distance } D}{\text{rate } R}$$

Example 5 ▶ **Solving a Motion Problem Where Times are the Same**

Two bikers participate in a Cross-Mountain Crusher. One biker is 2 km/h faster than the other. The faster biker travels 35 km in the same amount of time that it takes the slower biker to cover only 30 km. Find the average speed of each biker.

Solution ▶ Let r represent the average speed of the slower biker. Then $r + 2$ represents the average speed of the faster biker. The slower biker travels 30 km, while the faster biker travels 35 km. Now, we can complete the table

	R	\cdot	T	$=$	D
slower biker	r		$\frac{30}{r}$		30
faster biker	$r + 2$		$\frac{35}{r + 2}$		35

To complete the *Time* column, we divide the *Distance* by the *Rate*.

Since the time of travel is the same for both bikers, we form and then solve the equation:

$$\begin{aligned} \frac{30}{r} &= \frac{35}{r + 2} && / \div 5 \\ &&& \text{and cross-multiply} \\ 6(r + 2) &= 7r \\ 6r + 12 &= 7r && / -6r \\ r &= 12 \end{aligned}$$

Thus, the average speed of the slower biker is $r = \mathbf{12}$ km/h and the average speed of the faster biker is $r + 2 = \mathbf{14}$ km/h.

Example 6 ▶ **Solving a Motion Problem Where the Total Time is Given**

Judy and Nathan drive from Abbotsford to Kelowna, a distance of 322 km. Judy's average driving rate is 5 km/h faster than Nathan's. Judy got tired after driving the first 154 kilometers, so Nathan drove the remaining part of the trip. If the total driving time was 3 hours, what was the average rate of each driver?

Solution ▶ Let r represent Nathan's average rate. Then $r + 5$ represents Judy's average rate. Since Judy drove 154 km, Nathan drove $322 - 154 = 168$ km. Now, we can complete the table:

	R	\cdot	T	$=$	D
Judy	$r + 5$		$\frac{154}{r + 5}$		154
Nathan	r		$\frac{168}{r}$		168
total			3		322

Note: In motion problems we may add *times* or *distances* but we usually **do not add rates!**

The equation to solve comes from the **Time** column.

$$\begin{aligned} \frac{154}{r+5} + \frac{168}{r} &= 3 && / \cdot r(r+5) \\ 154r + 168(r+5) &= 3r(r+5) && \text{distribute; then} \\ 154r + 168r + 840 &= 3r^2 + 15r && \text{collect like terms on} \\ &&& \text{one side} \\ 0 &= 3r^2 - 307r - 840 && \text{factor} \\ (3r+8)(r-105) &= 0 \\ r &= -\frac{8}{3} \text{ or } r = 105 \end{aligned}$$

Since a rate cannot be negative, we discard the solution $r = -\frac{8}{3}$. Therefore, Nathan's average rate was $r = \mathbf{105}$ km/h and Judy's average rate was $r + 5 = \mathbf{110}$ km/h.

Work Problems

Notice the similarity to the formula $R \cdot T = D$ used in motion problems.

When solving work problems, refer to the formula

$$\mathbf{Rate\ of\ work \cdot Time = amount\ of\ Job\ completed}$$

and organize data in a table like this:

	R	\cdot	T	$=$	J
worker I					
worker II					
together					

Note: In work problems we usually add rates but do not add times!

Example 7 ▶ Solving a Work Problem Involving Addition of Rates

Adam can trim the shrubs at Centralia College in 8 hr. Bruce can do the same job in 6 hr. To the nearest minute, how long would it take them to complete the same trimming job if they work together?



Solution ▶ Let t be the time needed to trim the shrubs when Adam and Bruce work together. Since trimming the shrubs at Centralia College is considered to be the whole one job to complete, then the rate R in which this work is done equals

$$R = \frac{\text{Job}}{\text{Time}} = \frac{1}{\text{Time}}$$

To organize the information, we can complete the table below.

	R	T	$= J$
Adam	$\frac{1}{8}$	8	1
Bruce	$\frac{1}{6}$	6	1
together	$\frac{1}{t}$	t	1

The job column is often equal to **1**, although sometimes other values might need to be used.

To complete the *Rate* column, we divide the *Job* by the *Time*.

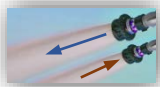
Since the rate of work when both Adam and Bruce trim the shrubs is the sum of rates of individual workers, we form and solve the equation

$$\begin{aligned} \frac{1}{8} + \frac{1}{6} &= \frac{1}{t} && / \cdot 24t \\ 3t + 4t &= 24 \\ 7t &= 24 && / \div 7 \\ t &= \frac{24}{7} \approx 3.43 \end{aligned}$$

So, if both Adam and Bruce work together, the amount of time needed to complete the job is approximately 3.43 hours \approx **3 hours 26 minutes**.

Note: The time needed for both workers is **shorter** than either of the individual times.

Example 8 ▶ Solving a Work Problem Involving Subtraction of Rates



The inlet pipe can fill a swimming pool in 4 hours, while the outlet pipe can empty the pool in 5 hours. If both pipes were left open, how long would it take to fill the pool?

Solution ▶ Suppose t is the time needed to fill the pool when both pipes are left open. If filling the pool is the whole one job to complete, then emptying the pool corresponds to -1 job. This is because when emptying the pool, we reverse the filling job.

To organize the information given in the problem, we complete the following table.

	R	T	$= J$
inlet pipe	$\frac{1}{4}$	4	1
outlet pipe	$-\frac{1}{5}$	5	-1
both pipes	$\frac{1}{t}$	t	1

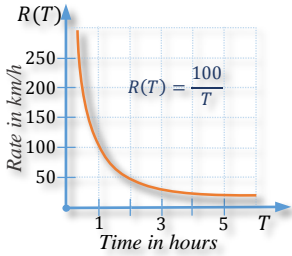
The equation to solve comes from the **Rate** column.

$$\begin{aligned} \frac{1}{4} - \frac{1}{5} &= \frac{1}{t} && / \cdot 20t \\ 5t - 4t &= 20 \\ t &= 20 \end{aligned}$$

So, it will take **20 hours** to fill the pool when both pipes are left open.

Inverse and Combined Variation

When two quantities vary in such a way that their **product remains constant**, we say that they are **inversely proportional**. For example, consider rate R and time T of a moving object that covers a constant distance D . In particular, if $D = 100$ km, we have



$$R = \frac{100}{T} = 100 \cdot \frac{1}{T}$$

This relation tells us that the rate is 100 times larger than the reciprocal of time. Observe though that when the time doubles, the rate is half as large. When the time triples, the rate is three times smaller, and so on. One can observe that the rate decreases proportionally to the increase of time. Such a **reciprocal relation** between the two quantities is called an **inverse variation**.

Definition 6.1 ▶ Two quantities, x and y , are **inversely proportional** to each other (there is an **inverse variation** between them) iff there is a real constant $k \neq 0$, such that

$$y = \frac{k}{x}$$

We say that y **varies inversely** as x with the **variation constant k** .
(or equivalently: y is **inversely proportional to x** with the **proportionality constant k** .)

Example 9 ▶ Solving Inverse Variation Problems

The volume V of a gas is inversely proportional to the pressure P of the gas. If a pressure of 30 kg/cm^2 corresponds to a volume of 240 cm^3 , find the following:

- a. The equation that relates V and P ,
- b. The pressure needed to produce a volume of 150 cm^3 .

Solution ▶ a. To find the inverse variation equation that relates V and P , we need to find the variation constant k first. This can be done by substituting $V = 240$ and $P = 30$ into the equation $V = \frac{k}{P}$. So, we obtain

$$\begin{aligned} 240 &= \frac{k}{30} && / \cdot 30 \\ k &= 7200. \end{aligned}$$

Therefore, our equation is $V = \frac{7200}{P}$.

- b. The required pressure can be found by substituting $V = 150$ into the inverse variation equation,

This gives us

$$150 = \frac{7200}{P} \quad / \cdot P, \div 150$$

'swap' 150 and P

$$P = \frac{7200}{150} = 48.$$

So, the pressure of the gas that assumes the volume of 150 cm^3 is **48 kg/cm²**.

Extension: We say that **y varies inversely** as the **n-th power** of **x** iff $y = \frac{k}{x^n}$, for some nonzero constant **k**.

Example 10 ▶ Solving an Inverse Variation Problem Involving the Square of a Variable



The intensity of light varies inversely as the square of the distance from the light source. If 4 meters from the source the intensity of light is 9 candelas, what is the intensity of this light 3 meters from the source?

Solution ▶ Let I represents the intensity of the light and d the distance from the source of this light. Since I varies inversely as d^2 , we set the equation

$$I = \frac{k}{d^2}$$

After substituting the data given in the problem, we find the value of k :

$$9 = \frac{k}{4^2} \quad / \cdot 16$$

$$k = 9 \cdot 16 = 144$$

So, the inverse variation equation is $I = \frac{144}{d^2}$. Hence, the light intensity at 3 meters from the source is $I = \frac{144}{3^2} = \mathbf{16 \text{ candelas}}$.

Recall from *Section L2* that two variables, say x and y , vary **directly** with a proportionality constant $k \neq 0$ if $y = kx$. Also, we say that one variable, say z , varies **jointly** as other variables, say x and y , with a proportionality constant $k \neq 0$ if $z = kxy$.

Definition 6.2 ▶ A combination of the **direct** or **joint** variation with the **inverse** variation is called a **combined variation**.

Example:

w may vary **jointly** as x and y and **inversely** as the square of z . This means that there is a real constant $k \neq 0$, such that

$$w = \frac{kxy}{z^2}.$$

Example 11 ▶ **Solving Combined Variation Problems**

The resistance of a cable varies directly as its length and inversely as the square of its diameter. A 20-meter cable with a diameter of 1.2 cm has a resistance of 0.2 ohms. A 50-meter cable with a diameter of 0.6 cm is made out of the same material. What would be its resistance?

Solution ▶ Let R , l , and d represent respectively the resistance, length, and diameter of a cable. Since R varies directly as l and inversely as d^2 , we set the combined variation equation

$$R = \frac{kl}{d^2}.$$

Substituting the data given in the problem, we have

$$0.2 = \frac{k \cdot 20}{1.2^2}, \quad / \cdot 1.44, \quad \div 20$$

which gives us

$$k = \frac{0.2 \cdot 1.44}{20} = 0.0144$$

So, the combined variation equation is $R = \frac{0.0144l}{d^2}$. Therefore, the resistance of a 50-meter cable with the diameter of 0.6 cm is $R = \frac{0.0144 \cdot 50}{0.6^2} = \mathbf{2 \text{ ohms}}$.

RT.6 Exercises

- Using the formula $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, find q if $r = 6$ and $p = 10$.
- The gravitational force between two masses is given by the formula $F = \frac{GMm}{d^2}$. Find M if $F = 20$, $G = 6.67 \cdot 10^{-11}$, $m = 1$, and $d = 4 \cdot 10^{-6}$. Round your answer to one decimal place.
- What is the first step in solving the formula $ka + kb = a - b$ for k ?
- What is the first step in solving the formula $A = \frac{pq}{q-p}$ for p ?

Solve each formula for the specified variable.

5. $m = \frac{F}{a}$ for a

6. $l = \frac{E}{R}$ for R

7. $\frac{W_1}{W_2} = \frac{d_1}{d_2}$ for d_1

8. $F = \frac{GMm}{d^2}$ for m

9. $s = \frac{(v_1+v_2)t}{2}$ for t

10. $s = \frac{(v_1+v_2)t}{2}$ for v_1

11. $\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2}$ for R

12. $\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2}$ for r_1

13. $\frac{1}{p} + \frac{1}{q} = \frac{1}{f}$ for q

14. $\frac{t}{a} + \frac{t}{b} = 1$ for a

15. $\frac{PV}{T} = \frac{pv}{t}$ for v

16. $\frac{PV}{T} = \frac{pv}{t}$ for T

17. $A = \frac{h(a+b)}{2}$ for b

18. $a = \frac{V-v}{t}$ for V

19. $R = \frac{gs}{g+s}$ for s

20. $I = \frac{2V}{V+2r}$ for V

21. $I = \frac{nE}{E+nr}$ for n

22. $\frac{E}{e} = \frac{R+r}{r}$ for e

23. $\frac{E}{e} = \frac{R+r}{r}$ for r

24. $S = \frac{H}{m(t_1-t_2)}$ for t_1

25. $V = \frac{\pi h^2(3R-h)}{3}$ for R

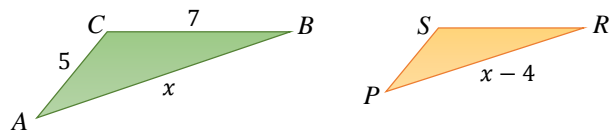
26. $P = \frac{A}{1+r}$ for r

27. $\frac{V^2}{R^2} = \frac{2g}{R+h}$ for h

28. $v = \frac{d_2-d_1}{t_2-t_1}$ for t_2

Solve each problem.

29. The ratio of the weight of an object on Earth to the weight of an object on the moon is 200 to 33. What would be the weight of a 75-kg astronaut on the moon?
30. A 30-meter long ribbon is cut into two sections. How long are the two sections if the ratio of their lengths is 5 to 7?
31. Assume that burning 7700 calories causes a decrease of 1 kilogram in body mass. If walking 7 kilometers in 2 hours burns 700 calories, how many kilometers would a person need to walk at the same rate to lose 1 kg?
32. On a map of Canada, the linear distance between Vancouver and Calgary is 1.8 cm. The flight distance between the two cities is about 675 kilometers. On this same map, what would be the linear distance between Calgary and Montreal if the flight distance between the two cities is approximately 3000 kilometers?
33. To estimate the population of Cape Mountain Zebras in South Africa, biologists caught, tagged, and then released 68 Cape Mountain Zebras. In a month, they caught a random sample of 84 of this type of zebras. It turned out that 5 of them were tagged. Assuming that zebras mixed freely, approximately how many Cape Mountain Zebras lived in South Africa?
34. To estimate the number of white bass fish in a particular lake, biologists caught, tagged, and then released 300 of this fish. In two weeks, they returned and collected a random sample of 196 white bass fish. This sample contained 12 previously tagged fish. Approximately how many white bass fish does the lake have?
35. Eighteen white-tailed eagles are tagged and released into the wilderness. In a few weeks, a sample of 43 white-tailed eagles was examined, and 5 of them were tagged. Estimate the white-tailed eagle population in this wilderness area.
36. A meter stick casts a 64 cm long shadow. At the same time, a 15-year old cottonwood tree casts an 18-meter long shadow. To the nearest meter, how tall is the tree?
37. The ratio of corresponding sides of similar triangles is 5 to 3. The two shorter sides of the larger triangle are 5 and 7 units long, correspondingly. Find the length of each side of the smaller triangle if its longest side is 4 units shorter than the corresponding side of the larger triangle.



38. The width of a rectangle is the same as the length of a similar rectangle. If the dimensions of the smaller rectangle are 7 cm by 12 cm, what are the dimensions of the larger rectangle?
39. Justin runs twice around a park. He averages 20 kilometers per hour during the first round and only 16 kilometers per hour during the second round. What is his average speed for the whole run? *Round your answer to one decimal place.*
40. Robert runs twice around a stadium. He averages 18 km/h during the first round. What should his average speed be during the second round to have an overall average of 20 km/h for the whole run?



41. Jim's boat moves at 20 km/h in still water. Suppose it takes the same amount of time for Jim to travel by his boat either 15 km downriver or 10 km upriver. Find the rate of the current.

42. The average speed of a plane flying west was 880 km/h. On the return trip, the same plane averaged only 620 km/h. If the total flying time in both directions was 6 hours, what was the one-way distance?

43. A plane flies 3800 kilometers with the wind, while only 3400 kilometers against the same wind. If the airplane speed in still air is 900 km/h, find the speed of the wind.

44. Walking on a moving sidewalk, Sarah could travel 40 meters forward in the same time it would take her to travel 15 meters in the opposite direction. If the rate of the moving sidewalk was 35 m/min, what was Sarah's rate of walking?



45. Arthur travelled by car from Madrid to Paris. He usually averages 100 km/h on such trips. This time, due to heavier traffic and few stops, he averaged only 85 km/h, and he reached his destination 2 hours 15 minutes later than expected. How far did Arthur travel?

46. Tony averaged 100 km/h on the first part of his trip to Lillooet, BC. The second part of his trip was 20 kilometers longer than the first, and his average speed was only 80 km/h. If the second part of the trip took him 30 minutes longer than the first part, what was the overall distance travelled by Tony?

47. Page is a college student who lives in a near-campus apartment. When she rides her bike to campus, she gets there 24 min faster than when she walks. If her average walking rate is 4 km/h and her average biking rate is 20 km/h, how far does she live from the campus?

48. Sonia can respond to all the daily e-mails in 2 hours. Betty needs 3 hours to do the same job. If they both work on responding to e-mails, what portion of this daily job can be done in 1 hour? How much more time would they need to complete the job?

49. Brenda can paint a deck in x hours, while Tony can do the same job in y hours. Write a rational expression that represents the portion of the deck that can be painted by both of them in 4 hours.



50. Aaron and Ben plan to paint a house. Aaron needs 24 hours to paint the house by himself. Ben needs 18 to do the same job. To the nearest minute, how long would it take them to paint the house if they work together?

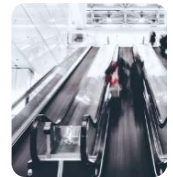
51. When working together, Adam and Brian can paint a house in 6 hours. Brian could paint this house on his own in 10 hours. How long would it take Adam to paint the house working alone?

52. An experienced floor installer can install a parquet floor twice as fast as an apprentice. Working together, it takes the two workers 2 days to install the floor in a particular house. How long would it take the apprentice to do the same job on his own?
53. A pool can be filled in 8 hr and drained in 12 hr. On one occasion, when filling the pool, the drain was accidentally left open. How long did it take to fill this pool?
54. One inlet pipe can fill a hot tub in 15 minutes. Another inlet pipe can fill the tub in 10 minutes. An outlet pipe can drain the hot tub in 18 minutes. How long would it take to fill the hot tub if all three pipes are left open?
55. Two different width escalators can empty a 1470-people auditorium in 12 min. If the wider escalator can move twice as many people as the narrower one, how many people per hour can the narrower escalator move?



56. At what times between 3:00 and 4:00 are the minute and hour hands perfectly lined up?

57. If Miranda drives to work at an average speed of 60 km/h, she is 1 min late. When she drives at an average speed of 75 km/h, she is 3 min early. How far is Miranda's workplace from her home?



58. The current in an electrical circuit at a constant potential varies inversely as the resistance of the circuit. Suppose that the current I is 9 amperes when the resistance R is 10 ohms. Find the current when the resistance is 6 ohms.
59. Assuming the same rate of work for all workers, the number of workers needed for a job varies inversely as the time required to complete the job. If it takes 3 hours for 8 workers to build a deck, how long would it take two workers to build the same deck?
60. The length of a guitar string is inversely proportional to the frequency of the string vibrations. Suppose a 60-cm long string vibrates at a frequency of 500 Hz (*1 hertz = one cycle per second*). What is the frequency of the same string when it is shortened to 50 centimeters?
61. A musical tone's pitch is inversely proportional to its wavelength. If a wavelength of 2.2 meters corresponds to a pitch of 420 vibrations per second, find the wavelength of a tone with a pitch of 660 vibrations per second.
62. The intensity, I , of a television signal is inversely proportional to the square of the distance, d , from a transmitter. If 2 km away from the transmitter the intensity is 25 W/m^2 (watts per square meter), how far from the transmitter is a TV set that receives a signal with the intensity of 2.56 W/m^2 ?
63. The weight W of an object is inversely proportional to the square of the distance D from the center of Earth. To the nearest kilometer, how high above the surface of Earth must a 60-kg astronaut be to weigh half as much? Assume the radius of Earth to be 6400 km.
64. The number of long-distance phone calls between two cities during a specified period in time varies jointly as the populations of the cities, P_1 and P_2 , and inversely as the distance between them. Suppose 80,000 calls are made between two cities that are 400 km apart and have populations of 70,000 and 100,000. How many calls are made between Vancouver and Abbotsford that are 70 km apart and have populations of 630,000 and 140,000, respectively?

65. The force that keeps a car from skidding on a curve is inversely proportional to the radius of the curve and jointly proportional to the weight of the car and the square of its speed. Knowing that a force of 880 N (Newtons) keeps an 800-kg car moving at 50 km/h from skidding on a curve of radius 160 m, estimate the force that would keep the same car moving at 80 km/h from skidding on a curve of radius 200 meters.
66. Suppose that the renovation time is inversely proportional to the number of workers hired for the job. Will the renovation time decrease more when hiring additional 2 workers in a 4-worker company or a 6-worker company? Justify your answer.

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