

Rational Expressions and Functions



In the previous two chapters we discussed algebraic expressions, equations, and functions related to polynomials. In this chapter, we will examine a broader category of algebraic expressions, *rational expressions*, also referred to as *algebraic fractions*. Similarly as in arithmetic, where a rational number is a quotient of two integers with a denominator that is different than zero, a rational expression is a quotient of two polynomials, also with a denominator that is different than zero.

We start by introducing the related topic of integral exponents, including scientific notation. Then, we discuss operations on algebraic fractions, solving rational equations, and properties and graphs of rational functions with an emphasis on such features as domain, range, and asymptotes. At the end of this chapter, we show examples of applied problems, including work problems, that require solving rational equations.

RT.1

Integral Exponents and Scientific Notation

Integral Exponents

In *section P.2*, we discussed the following power rules, using whole numbers for the exponents.

product rule	$a^m \cdot a^n = a^{m+n}$	$(ab)^n = a^n b^n$
quotient rule	$\frac{a^m}{a^n} = a^{m-n}$	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
power rule	$(a^m)^n = a^{mn}$	$a^0 = 1$ for $a \neq 0$ 0^0 is undefined

Observe that these rules gives us the following result.

$$a^{-1} = a^{n-(n+1)} = \frac{a^n}{a^{n+1}} = \frac{a^n}{a^n \cdot a} = \frac{1}{a}$$

quotient rule
product rule

$$\text{Consequently, } a^{-n} = (a^n)^{-1} = \frac{1}{a^n}.$$

power rule

Since $a^{-n} = \frac{1}{a^n}$, then the expression a^n is meaningful for any integral exponent n and a nonzero real base a . So, the above rules of exponents can be extended to include integral exponents.

In practice, to work out the negative sign of an exponent, take the **reciprocal of the base**, or equivalently, “**change the level**” of the power. For example,

$$3^{-2} = \left(\frac{1}{3}\right)^2 = \frac{1^2}{3^2} = \frac{1}{9} \quad \text{and} \quad \frac{2^{-3}}{3^{-1}} = \frac{3^1}{2^3} = \frac{3}{8}.$$

Attention! Exponent refers to the immediate number, letter, or expression in a bracket.

For example,

$$x^{-2} = \frac{1}{x^2}, \quad (-x)^{-2} = \frac{1}{(-x)^2} = \frac{1}{x^2}, \quad \text{but} \quad -x^{-2} = -\frac{1}{x^2}.$$

Example 1 ▶ Evaluating Expressions with Integral Exponents

Evaluate each expression.

a. $3^{-1} + 2^{-1}$

b. $\frac{5^{-2}}{2^{-5}}$

c. $\frac{-2^2}{2^{-7}}$

d. $\frac{-2^{-2}}{3 \cdot 2^{-3}}$

Solution ▶

a. $3^{-1} + 2^{-1} = \frac{1}{3} + \frac{1}{2} = \frac{2}{6} + \frac{3}{6} = \frac{5}{6}$

Caution! $3^{-1} + 2^{-1} \neq (3 + 2)^{-1}$, because the value of $3^{-1} + 2^{-1}$ is $\frac{5}{6}$, as shown in the example, while the value of $(3 + 2)^{-1}$ is $\frac{1}{5}$.

b. $\frac{5^{-2}}{2^{-5}} = \frac{2^5}{5^2} = \frac{32}{25}$

Note: To work out the negative exponent, move the power from the numerator to the denominator or vice versa.

c. $\frac{-2^2}{2^{-7}} = -2^2 \cdot 2^7 = -2^9$

Attention! The role of a negative sign in front of a base number or in front of an exponent is different. To work out the negative in 2^{-7} , we either take the reciprocal of the base, or we change the position of the power to a different level in the fraction. So, $2^{-7} = \left(\frac{1}{2}\right)^7$ or $2^{-7} = \frac{1}{2^7}$. However, the negative sign in -2^2 just means that the number is negative. So, $-2^2 = -4$. **Caution!** $-2^2 \neq \frac{1}{4}$

d. $\frac{-2^{-2}}{3 \cdot 2^{-3}} = \frac{-2^3}{3 \cdot 2^2} = -\frac{2}{3}$

Note: Exponential expressions can be simplified in many ways. For example, to simplify $\frac{2^{-2}}{2^{-3}}$, we can work out the negative exponents first by moving the powers to a different level, $\frac{2^3}{2^2}$, and then reduce the common factors as shown in the example; or we can employ the quotient rule of powers to obtain

$$\frac{2^{-2}}{2^{-3}} = 2^{-2-(-3)} = 2^{-2+3} = 2^1 = 2.$$

Example 2 ▶ **Simplifying Exponential Expressions Involving Negative Exponents**

Simplify the given expression. Leave the answer with only positive exponents.

- a. $4x^{-5}$ b. $(x + y)^{-1}$
 c. $x^{-1} + y^{-1}$ d. $(-2^3x^{-2})^{-2}$
 e. $\frac{x^{-4}y^2}{x^2y^{-5}}$ f. $\left(\frac{-4m^5n^3}{24mn^{-6}}\right)^{-2}$

Solution ▶

a. $4x^{-5} = \frac{4}{x^5}$ exponent -5
refers to x only!

b. $(x + y)^{-1} = \frac{1}{x+y}$ these expressions are
NOT equivalent!

c. $x^{-1} + y^{-1} = \frac{1}{x} + \frac{1}{y}$

d. $(-2^3x^{-2})^{-2} = \left(\frac{-2^3}{x^2}\right)^{-2} = \left(\frac{x^2}{-2^3}\right)^2 = \frac{(x^2)^2}{(-1)^2(2^3)^2} = \frac{x^4}{2^6}$
work out the negative exponents inside the bracket work out the negative exponents outside the bracket a “-” sign can be treated as a factor of -1 power rule – multiply exponents

e. $\frac{x^{-4}y^2}{x^2y^{-5}} = \frac{y^2y^5}{x^2x^4} = \frac{y^7}{x^6}$
product rule – add exponents

f. $\left(\frac{-4m^5n^3}{24mn^{-6}}\right)^{-2} = \left(\frac{-m^4n^3n^6}{6}\right)^{-2} = \left(\frac{(-1)m^4n^9}{6}\right)^{-2} = \left(\frac{6}{(-1)m^4n^9}\right)^2 = \frac{36}{m^8n^{18}}$
(-1)² = 1

Scientific Notation

Integral exponents allow us to record numbers with a very large or very small absolute value in a shorter, more convenient form.

For example, the average distance from the Sun to the Saturn is 1,430,000,000 km, which can be recorded as $1.43 \cdot 10,000,000$ or more concisely as $1.43 \cdot 10^9$.

Similarly, the mass of an electron is 0.000000000000000000000000009 grams, which can be recorded as $9 \cdot 0.0000000000000000000000000001$, or more concisely as $9 \cdot 10^{-28}$.

This more concise representation of numbers is called **scientific notation** and it is frequently used in sciences and engineering.

Definition 1.1 ▶ A real number x is written in **scientific notation** iff $x = a \cdot 10^n$, where the coefficient a is such that $|a| \in [1, 10)$, and the exponent n is an integer.

Example 3 ▶ **Converting Numbers to Scientific Notation**

Convert each number to scientific notation.

- a. 520,000 b. -0.000102 c. $12.5 \cdot 10^3$

Solution ▶

an integer has its decimal dot after the last digit

- a. To represent 520,000 in scientific notation, we place a decimal point after the first nonzero digit,

$$5.20000$$

and then count the number of decimal places needed for the decimal point to move to its original position, which by default was after the last digit. In our example the number of places we need to move the decimal place is 5. This means that 5.2 needs to be multiplied by 10^5 in order to represent the value of 520,000. So, $520,000 = 5.2 \cdot 10^5$.

Note: To comply with the scientific notation format, we always place the decimal point after the first nonzero digit of the given number. This will guarantee that the coefficient a satisfies the condition $1 \leq |a| < 10$.

- b. As in the previous example, to represent -0.000102 in scientific notation, we place a decimal point after the first nonzero digit,

$$-0.000102$$

and then count the number of decimal places needed for the decimal point to move to its original position. In this example, we move the decimal 4 places to the left. So the number 1.02 needs to be divided by 10^4 , or equivalently, multiplied by 10^{-4} in order to represent the value of -0.000102 . So, $-0.000102 = -1.02 \cdot 10^{-4}$.

Observation: Notice that moving the decimal to the **right** corresponds to using a **positive** exponent, as in *Example 3a*, while moving the decimal to the **left** corresponds to using a **negative** exponent, as in *Example 3b*.

- c. Notice that $12.5 \cdot 10^3$ is not in scientific notation as the coefficient 12.5 is not smaller than 10. To convert $12.5 \cdot 10^3$ to scientific notation, first, convert 12.5 to scientific notation and then multiply the powers of 10. So,

$$12.5 \cdot 10^3 = 1.25 \cdot 10 \cdot 10^3 = 1.25 \cdot 10^4$$

multiply powers by adding exponents

Example 4 ▶ **Converting from Scientific to Decimal Notation**

Convert each number to decimal notation.

a. $-6.57 \cdot 10^6$

b. $4.6 \cdot 10^{-7}$

Solution ▶

- a. The exponent 6 indicates that the decimal point needs to be moved 6 places to the right. So,

$$-6.57 \cdot 10^6 = -6.\underline{57} = -6,570,000$$

fill the empty
places by zeros

- b. The exponent -7 indicates that the decimal point needs to be moved 7 places to the left. So,

$$4.6 \cdot 10^{-7} = 0.\underline{46} = 0.0000046$$

fill the empty
places by zeros

Example 5 ▶ **Using Scientific Notation in Computations**

Evaluate. Leave the answer in scientific notation.

a. $6.5 \cdot 10^7 \cdot 3 \cdot 10^5$

b. $\frac{3.6 \cdot 10^3}{9 \cdot 10^{14}}$

Solution ▶

- a. Since the product of the coefficients $6.5 \cdot 3 = 19.5$ is larger than 10, we convert it to scientific notation and then multiply the remaining powers of 10. So,

$$6.5 \cdot 10^7 \cdot 3 \cdot 10^5 = 19.5 \cdot 10^7 \cdot 10^5 = \underline{1.95 \cdot 10} \cdot 10^{12} = \mathbf{1.95 \cdot 10^{13}}$$

- b. Similarly as in the previous example, since the quotient $\frac{3.6}{9} = 0.4$ is smaller than 1, we convert it to scientific notation and then work out the remaining powers of 10. So,

$$\frac{3.6 \cdot 10^3}{9 \cdot 10^{14}} = 0.4 \cdot 10^{-11} = \underline{4 \cdot 10^{-1}} \cdot 10^{-11} = \mathbf{4 \cdot 10^{-12}}$$

divide powers by
subtracting exponents

Example 6 ▶ **Using Scientific Notation to Solve Problems**

The average distance from Earth to the sun is $1.5 \cdot 10^8$ km. How long would it take a rocket, traveling at $4.7 \cdot 10^3$ km/h, to reach the sun?

Solution ▶ To find time T needed for the rocket traveling at the rate $R = 4.7 \cdot 10^3$ km/h to reach the sun that is at the distance $D = 1.5 \cdot 10^8$ km from Earth, first, we solve the motion formula $R \cdot T = D$ for T . Since $T = \frac{D}{R}$, we calculate,

$$T = \frac{1.5 \cdot 10^8}{4.7 \cdot 10^3} \cong 0.32 \cdot 10^5 = 3.2 \cdot 10^4$$

So, it will take approximately $3.2 \cdot 10^4$ hours for the rocket to reach the sun.

RT.1 Exercises

Vocabulary Check Complete each blank with the most appropriate term or phrase from the given list: *numerator, opposite, reciprocal, scientific.*

- To raise a base to a negative exponent, raise the _____ of the base to the opposite exponent.
- A power in a numerator of a fraction can be written equivalently as a power with the _____ exponent in the denominator of this fraction. Similarly, a power in a denominator of a fraction can be written equivalently as a power with the opposite exponent in the _____ of this fraction.
- A number with a very large or very small absolute value is often written in the _____ notation.

Concept Check True or false.

- | | | |
|--|---|---|
| 4. $\left(\frac{3}{4}\right)^{-2} = \left(\frac{4}{3}\right)^2$ | 5. $10^{-4} = 0.00001$ | 6. $(0.25)^{-1} = 4$ |
| 7. $-4^5 = \frac{1}{4^5}$ | 8. $(-2)^{-10} = 4^{-5}$ | 9. $2 \cdot 2 \cdot 2^{-1} = \frac{1}{8}$ |
| 10. $3x^{-2} = \frac{1}{3x^2}$ | 11. $-2^{-2} = -\frac{1}{4}$ | 12. $\frac{5^{10}}{5^{-12}} = 5^{-2}$ |
| 13. The number $0.68 \cdot 10^{-5}$ is written in scientific notation. | 14. $98.6 \cdot 10^7 = 9.86 \cdot 10^6$ | |

Concept Check

15. Match the expression in Row I with its equivalent expression in Row II. Choices may be used once, more than once, or not at all.

- | | | | | |
|-------------|-------------------|----------------|-------------------|----------------------|
| a. 5^{-2} | b. -5^{-2} | c. $(-5)^{-2}$ | d. $-(-5)^{-2}$ | e. $-5 \cdot 5^{-2}$ |
| A. 25 | B. $\frac{1}{25}$ | C. -25 | D. $-\frac{1}{5}$ | E. $-\frac{1}{25}$ |

Concept Check Evaluate each expression.

- | | | | |
|------------------------|-------------------------|--------------------------|-----------------------------|
| 16. $4^{-6} \cdot 4^3$ | 17. $-9^3 \cdot 9^{-5}$ | 18. $\frac{2^{-3}}{2^6}$ | 19. $\frac{2^{-7}}{2^{-5}}$ |
|------------------------|-------------------------|--------------------------|-----------------------------|

20. $\frac{-3^{-4}}{5^{-3}}$

21. $-\left(\frac{3}{2}\right)^{-2}$

22. $2^{-2} + 2^{-3}$

23. $(2^{-1} - 3^{-1})^{-1}$

Concept Check Simplify each expression, if possible. Leave the answer with only **positive exponents**. Assume that all variables represent nonzero real numbers. Keep large numerical coefficients as powers of prime numbers, if possible.

24. $(-2x^{-3})(7x^{-8})$

25. $(5x^{-2}y^3)(-4x^{-7}y^{-2})$

26. $(9x^{-4n})(-4x^{-8n})$

27. $(-3y^{-4a})(-5y^{-3a})$

28. $-4x^{-3}$

29. $\frac{x^{-4n}}{x^{6n}}$

30. $\frac{3n^5}{nm^{-2}}$

31. $\frac{14a^{-4}b^{-3}}{-8a^8b^{-5}}$

32. $\frac{-18x^{-3}y^3}{-12x^{-5}y^5}$

33. $(2^{-1}p^{-7}q)^{-4}$

34. $(-3a^2b^{-5})^{-3}$

35. $\left(\frac{5x^{-2}}{y^3}\right)^{-3}$

36. $\left(\frac{2x^3y^{-2}}{3y^{-3}}\right)^{-3}$

37. $\left(\frac{-4x^{-3}}{5x^{-1}y^4}\right)^{-4}$

38. $\left(\frac{125x^2y^{-3}}{5x^4y^{-2}}\right)^{-5}$

39. $\left(\frac{-200x^3y^{-5}}{8x^5y^{-7}}\right)^{-4}$

40. $[(-2x^{-4}y^{-2})^{-3}]^{-2}$

41. $\frac{12a^{-2}(a^{-3})^{-2}}{6a^7}$

42. $\frac{(-2k)^2m^{-5}}{(km)^{-3}}$

43. $\left(\frac{2p}{q^2}\right)^3 \left(\frac{3p^4}{q^{-4}}\right)^{-1}$

44. $\left(\frac{-3x^4y^6}{15x^{-6}y^7}\right)^{-3}$

45. $\left(\frac{-4a^3b^2}{12a^6b^{-5}}\right)^{-3}$

46. $\left(\frac{-9^{-2}x^{-4}y}{3^{-3}x^{-3}y^2}\right)^8$

47. $(4^{-x})^{2y}$

48. $(5^a)^{-a}$

49. $x^a x^{-a}$

50. $\frac{9n^{2-x}}{3n^{2-2x}}$

51. $\frac{12x^{a+1}}{-4x^{2-a}}$

52. $(x^{b-1})^3(x^{b-4})^{-2}$

53. $\frac{25x^{a+b}y^{b-a}}{-5x^{a-b}y^{b+a}}$

Convert each number to scientific notation.

54. 26,000,000,000

55. -0.000132

56. 0.0000000105

57. 705.6

Convert each number to decimal notation.

58. $6.7 \cdot 10^8$

59. $5.072 \cdot 10^{-5}$

60. $2 \cdot 10^{12}$

61. $9.05 \cdot 10^{-9}$

62. One gigabyte of computer memory equals 2^{30} bytes. Write the number of bytes in 1 gigabyte, using decimal notation. Then, using scientific notation, approximate this number by rounding the scientific notation coefficient to two decimal places.

Evaluate. State your answer in scientific notation.

63. $(6.5 \cdot 10^3)(5.2 \cdot 10^{-8})$

64. $(2.34 \cdot 10^{-5})(5.7 \cdot 10^{-6})$

65. $(3.26 \cdot 10^{-6})(5.2 \cdot 10^{-8})$

66. $\frac{4 \cdot 10^{-7}}{8 \cdot 10^{-3}}$

67. $\frac{7.5 \cdot 10^9}{2.5 \cdot 10^4}$

68. $\frac{4 \cdot 10^{-7}}{8 \cdot 10^{-3}}$

69. $\frac{0.05 \cdot 16000}{0.0004}$

70. $\frac{0.003 \cdot 40,000}{0.00012 \cdot 600}$

Analytic Skills Solve each problem. State your answer in scientific notation.

71. A *light-year* is the distance that light travels in one year. Find the number of kilometers in a light-year if light travels approximately $3 \cdot 10^5$ kilometers per second.
72. In 2017, the national debt in Canada was about $6.4 \cdot 10^{11}$ dollars. If Canadian population in 2017 was approximately $3.56 \cdot 10^7$, what was the share of this debt per person?
73. The brightest star in the night sky, Sirius, is about $4.704 \cdot 10^{13}$ miles from Earth. If one light-year is approximately $5.88 \cdot 10^{12}$ miles, how many light-years is it from Earth to Sirius?
74. The average discharge at the mouth of the Amazon River is 4,200,000 cubic feet per second. How much water is discharged from the Amazon River in one hour? in one year?
75. If current trends continue, world population P in billions may be modeled by the equation $P = 6(1.014)^x$, where x is in years and $x = 0$ corresponds to the year 2000. Estimate the world population in 2020 and 2025.
76. The mass of the sun is $1.989 \cdot 10^{30}$ kg and the mass of the earth is $5.976 \cdot 10^{24}$ kg. How many times larger in mass is the sun than the earth?



Discussion Point

77. Many calculators can only handle scientific notation for powers of 10 between -99 and 99 . How can we compute $(4 \cdot 10^{220})^2$?

RT.2

Rational Expressions and Functions; Multiplication and Division of Rational Expressions



In arithmetic, a rational number is a quotient of two integers with denominator different than zero. In algebra, a *rational expression*, often called an *algebraic fraction*, is a quotient of two polynomials, also with denominator different than zero. In this section, we will examine rational expressions and functions, paying attention to their domains. Then, we will simplify, multiply, and divide rational expressions, employing the factoring skills developed in *Chapter P*.

Rational Expressions and Functions

Here are some examples of rational expressions:

$$-\frac{x^2}{2xy}, \quad x^{-1}, \quad \frac{x^2-4}{x-2}, \quad \frac{8x^2+6x-5}{4x^2+5x}, \quad \frac{x-3}{3-x}, \quad x^2 - 25, \quad 3x(x-1)^{-2}$$

Definition 2.1 ▶ A **rational expression (algebraic fraction)** is a quotient $\frac{P(x)}{Q(x)}$ of two polynomials $P(x)$ and $Q(x)$, where $Q(x) \neq 0$. Since division by zero is not permitted, a rational expression is defined only for the x -values that make the denominator of the expression different than zero. The set of such x -values is referred to as the **domain** of the expression.

Note 1: Negative exponents indicate hidden fractions and therefore represent rational expressions. For instance, $x^{-1} = \frac{1}{x}$.

Note 2: A single polynomial can also be seen as a rational expression because it can be considered as a fraction with a denominator of 1.

$$\text{For instance, } x^2 - 25 = \frac{x^2-25}{1}.$$

Definition 2.2 ▶ A **rational function** is a function defined by a rational expression,

$$f(x) = \frac{P(x)}{Q(x)}.$$

The **domain** of such function consists of all real numbers except for the x -values that make the denominator $Q(x)$ equal to 0. So, the domain $D = \mathbb{R} \setminus \{x \mid Q(x) = 0\}$

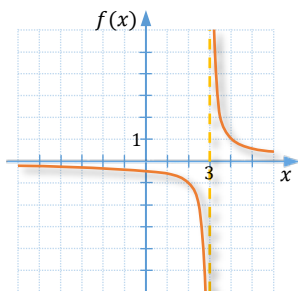


Figure 1

For example, the domain of the rational function $f(x) = \frac{1}{x-3}$ is the set of all real numbers except for 3 because 3 would make the denominator equal to 0. So, we write $D = \mathbb{R} \setminus \{3\}$. Sometimes, to make it clear that we refer to function f , we might denote the domain of f by D_f , rather than just D .

Figure 1 shows a graph of the function $f(x) = \frac{1}{x-3}$. Notice that the graph does not cross the dashed vertical line whose equation is $x = 3$. This is because $f(3)$ is not defined. A closer look at the graphs of rational functions will be given in *Section RT.5*.

Example 1 ▶ **Evaluating Rational Expressions or Functions**

Evaluate the given expression or function for $x = -1, 0, 1$. If the value cannot be calculated, write *undefined*.

a. $3x(x - 1)^{-2}$

b. $f(x) = \frac{x}{x^2+x}$

Solution ▶

a. If $x = -1$, then $3x(x - 1)^{-2} = 3(-1)(-1 - 1)^{-2} = -3(-2)^{-2} = \frac{-3}{(-2)^2} = -\frac{3}{4}$.

If $x = 0$, then $3x(x - 1)^{-2} = 3(0)(0 - 1)^{-2} = \mathbf{0}$.

If $x = 1$, then $3x(x - 1)^{-2} = 3(1)(1 - 1)^{-2} = 3 \cdot 0^{-2} = \mathbf{undefined}$, as division by zero is not permitted.

Note: Since the expression $3x(x - 1)^{-2}$ cannot be evaluated at $x = 1$, the number 1 does not belong to its domain.

b. $f(-1) = \frac{-1}{(-1)^2+(-1)} = \frac{-1}{1-1} = \mathbf{undefined}$.

$f(0) = \frac{0}{(0)^2+(0)} = \frac{0}{0} = \mathbf{undefined}$.

$f(1) = \frac{1}{(1)^2+(1)} = \frac{1}{2}$.

Observation: Function $f(x) = \frac{x}{x^2+x}$ is undefined at $x = 0$ and $x = -1$. This is because the denominator $x^2 + x = x(x + 1)$ becomes zero when the x -value is 0 or -1 .

Example 2 ▶ **Finding Domains of Rational Expressions or Functions**

Find the domain of each expression or function.

a. $\frac{4}{2x+5}$

b. $\frac{x-2}{x^2-2x}$

c. $f(x) = \frac{x^2-4}{x^2+4}$

d. $g(x) = \frac{2x-1}{x^2-4x-5}$

Solution ▶

a. The domain of $\frac{4}{2x+5}$ consists of all real numbers except for those that would make the denominator $2x + 5$ equal to zero. To find these numbers, we solve the equation

$$2x + 5 = 0$$

$$2x = -5$$

$$x = -\frac{5}{2}$$

So, the domain of $\frac{4}{2x+5}$ is the set of all real numbers except for $-\frac{5}{2}$. This can be recorded in set notation as $\mathbb{R} \setminus \{-\frac{5}{2}\}$, or in set-builder notation as $\{x \mid x \neq -\frac{5}{2}\}$, or in interval notation as $(-\infty, -\frac{5}{2}) \cup (-\frac{5}{2}, \infty)$.

- b. To find the domain of $\frac{x-2}{x^2-2x}$, we want to exclude from the set of real numbers all the x -values that would make the denominator $x^2 - 2x$ equal to zero. After solving the equation

$$x^2 - 2x = 0$$

via factoring

$$x(x - 2) = 0$$

and zero-product property

$$x = 0 \text{ or } x = 2,$$

we conclude that the domain is the set of all real numbers except for 0 and 2, which can be recorded as $\mathbb{R} \setminus \{0, 2\}$. This is because the x -values of 0 or 2 make the denominator of the expression $\frac{x-2}{x^2-2x}$ equal to zero.

- c. To find the domain of the function $f(x) = \frac{x^2-4}{x^2+4}$, we first look for all the x -values that make the denominator $x^2 + 4$ equal to zero. However, $x^2 + 4$, as a sum of squares, is never equal to 0. So, the domain of function f is the set of all real numbers \mathbb{R} .

- d. To find the domain of the function $g(x) = \frac{2x-1}{x^2-4x-5}$, we first solve the equation $x^2 - 4x - 5 = 0$ to find which x -values make the denominator equal to zero. After factoring, we obtain

$$(x - 5)(x + 1) = 0$$

which results in

$$x = 5 \text{ and } x = -1$$

Thus, the domain of g equals to $D_g = \mathbb{R} \setminus \{-1, 5\}$.

Equivalent Expressions

Definition 2.3 ▶ Two expressions are **equivalent** in the **common domain** iff (if and only if) they produce the same values for every input from the domain.

Consider the expression $\frac{x-2}{x^2-2x}$ from *Example 2b*. Notice that this expression can be simplified to $\frac{x-2}{x(x-2)} = \frac{1}{x}$ by reducing common factors in the numerator and the denominator. However, the domain of the simplified fraction, $\frac{1}{x}$, is the set $\mathbb{R} \setminus \{0\}$, which is different than the domain of the original fraction, $\mathbb{R} \setminus \{0, 2\}$. Notice that for $x = 2$, the expression $\frac{x-2}{x^2-2x}$ is undefined while the value of the expression $\frac{1}{x}$ is $\frac{1}{2}$. So, the two expressions are not equivalent in the set of real numbers. However, if the domain of $\frac{1}{x}$ is

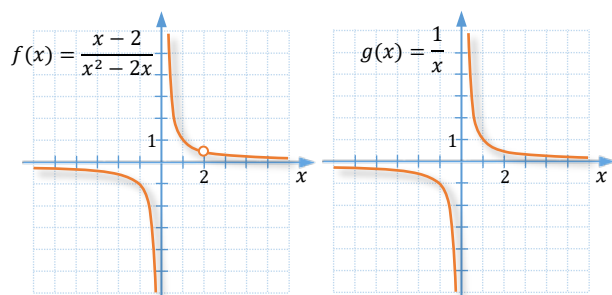


Figure 2

restricted to the set $\mathbb{R} \setminus \{0, 2\}$, then the two expressions produce the same values and as such, they are equivalent. We say that the two expressions are **equivalent** in the **common domain**.

The above situation can be illustrated by graphing the related functions, $f(x) = \frac{x-2}{x^2-2x}$ and $g(x) = \frac{1}{x}$, as in *Figure 2*. The graphs of both functions are exactly the same except for the hole in the graph of f at the point $(2, \frac{1}{2})$.

So, from now on, when writing statements like $\frac{x-2}{x^2-2x} = \frac{1}{x}$, we keep in mind that they apply only to real numbers which make both denominators different than zero. Thus, by saying in short that two **expressions are equivalent**, we really mean that they are **equivalent in the common domain**.

Note: The **domain** of $f(x) = \frac{x-2}{x^2-2x} = \frac{x-2}{x(x-2)} = \frac{1}{x}$ is still $\mathbb{R} \setminus \{0, 2\}$, even though the $(x-2)$ term was simplified.

The process of simplifying expressions involve creating equivalent expressions. In the case of rational expressions, equivalent expressions can be obtained by multiplying or dividing the numerator and denominator of the expression by the same nonzero polynomial. For example,

$$\frac{-x-3}{-5x} = \frac{(-x-3) \cdot (-1)}{(-5x) \cdot (-1)} = \frac{x+3}{5x}$$

$$\frac{x-3}{3-x} = \frac{\cancel{(x-3)}}{-1\cancel{(x-3)}} = \frac{1}{-1} = -1$$

To simplify a rational expression:

- **Factor** the numerator and denominator **completely**.
- **Eliminate all common factors** by following the property of multiplicative identity. *Do not eliminate common terms - they must be factors!*

Example 3 ▶ **Simplifying Rational Expressions**

Simplify each expression.

a. $\frac{7a^2b^2}{21a^3b-14a^3b^2}$ b. $\frac{x^2-9}{x^2-6x+9}$ c. $\frac{20x-15x^2}{15x^3-5x^2-20x}$

Solution ▶ a. First, we factor the denominator and then reduce the common factors. So,

$$\frac{7a^2b^2}{21a^3b-14a^3b^2} = \frac{\cancel{7}a^2b^2}{\cancel{7}a^3b(3a-2ab)} = \frac{b}{a(3-2b)}$$

b. As before, we factor and then reduce. So,

$$\frac{x^2 - 9}{x^2 - 6x + 9} = \frac{\cancel{(x-3)}(x+3)}{(x-3)^{\cancel{2}1}} = \frac{x+3}{x-3}$$

Neither x nor 3 can be reduced, as they are NOT factors!

c. Factoring and reducing the numerator and denominator gives us

$$\frac{20x - 15x^2}{15x^3 - 5x^2 - 20x} = \frac{5x(4 - 3x)}{5x(3x^2 - x - 4)} = \frac{4 - 3x}{(3x - 4)(x + 1)}$$

Since $\frac{4-3x}{3x-4} = \frac{-(3x-4)}{3x-4} = -1$, the above expression can be reduced further to

$$\frac{\cancel{4-3x}^{-1}}{\cancel{(3x-4)}(x+1)} = \frac{-1}{x+1}$$

Notice: An opposite expression in the numerator and denominator can be reduced to -1 . For example, since $a - b$ is opposite to $b - a$, then

$$\frac{a-b}{b-a} = -1, \text{ as long as } a \neq b.$$

Caution: Note that $a - b$ is NOT opposite to $a + b$!

Multiplication and Division of Rational Expressions

Recall that to multiply common fractions, we multiply their numerators and denominators, and then simplify the resulting fraction. Multiplication of algebraic fractions is performed in a similar way.

To multiply rational expressions:

- **factor** each numerator and denominator **completely**,
- **reduce all common factors** in any of the numerators and denominators,
- **multiply** the remaining expressions by writing the product of their numerators over the product of their denominators.

For instance,

$$\frac{3x}{x^2 + 5x} \cdot \frac{3x + 15}{6x} = \frac{\cancel{3x}}{x(x+5)} \cdot \frac{\cancel{3}(x+5)}{\cancel{6}2x} = \frac{3}{2x}$$

Example 4 ▶ Multiplying Algebraic Fractions

Multiply and simplify. Assume non-zero denominators.

a. $\frac{2x^2y^3}{3xy^2} \cdot \frac{(2x^3y)^2}{2(xy)^3}$

b. $\frac{x^3 - y^3}{x + y} \cdot \frac{3x + 3y}{x^2 - y^2}$

- Solution** ▶ a. To multiply the two algebraic fractions, we use appropriate rules of powers to simplify each fraction, and then reduce all the remaining common factors. So,

$$\frac{2x^2y^3}{3xy^2} \cdot \frac{(2x^3y)^2}{2(xy)^3} = \frac{2xy}{3} \cdot \frac{4x^6y^2}{2x^3y^3} = \frac{2xy \cdot 2x^3}{3 \cdot y} = \frac{4x^4}{3} = \frac{4}{3}x^4$$

equivalent answers

- b. After factoring and simplifying, we have

$$\frac{x^3 - y^3}{x + y} \cdot \frac{3x + 3y}{x^2 - y^2} = \frac{(x - y)(x^2 + xy + y^2)}{x + y} \cdot \frac{3(x + y)}{(x - y)(x + y)} = \frac{3(x^2 + xy + y^2)}{x + y}$$

Recall: $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$
 $x^2 - y^2 = (x + y)(x - y)$

To divide rational expressions, multiply the first, the *dividend*, **by the reciprocal** of the second, the *divisor*.

For instance,

$$\frac{5x - 10}{3x} \div \frac{3x - 6}{2x^2} = \frac{5x - 10}{3x} \cdot \frac{2x^2}{3x - 6} = \frac{5(x - 2)}{3x} \cdot \frac{2x^2}{3(x - 2)} = \frac{10x}{9}$$

multiply by the reciprocal follow multiplication rules

Example 5 ▶ Dividing Algebraic Fractions

Perform operations and simplify. Assume non-zero denominators.

a. $\frac{2x^2 + 2x}{x - 1} \div (x + 1)$ b. $\frac{x^2 - 25}{x^2 + 5x + 4} \div \frac{x^2 - 10x + 25}{2x^2 + 8x} \cdot \frac{x^2 + x}{4x^2}$

- Solution** ▶ a. To divide by $(x + 1)$ we multiply by the reciprocal $\frac{1}{(x + 1)}$. So,

$$\frac{2x^2 + 2x}{x - 1} \div (x + 1) = \frac{2x(x + 1)}{x - 1} \cdot \frac{1}{(x + 1)} = \frac{2x}{x - 1}$$

- b. The order of operations indicates to perform the division first. To do this, we convert the division into multiplication by the reciprocal of the middle expression. Therefore,

$$\begin{aligned} & \frac{x^2 - 25}{x^2 + 5x + 4} \div \frac{x^2 - 10x + 25}{2x^2 + 8x} \cdot \frac{x^2 + x}{4x^2} \\ &= \frac{(x - 5)(x + 5)}{(x + 4)(x + 1)} \cdot \frac{2x^2 + 8x}{x^2 - 10x + 25} \cdot \frac{x(x + 1)}{4x^2} \\ &= \frac{(x - 5)(x + 5)}{(x + 4)} \cdot \frac{2x(x + 4)}{(x - 5)^2} \cdot \frac{1}{4x} = \frac{(x + 5)}{2(x - 5)} \end{aligned}$$

Reduction of common factors can be done gradually, especially if there is many common factors to divide out.

RT.2 Exercises

Vocabulary Check Complete each blank with the most appropriate term or phrase from the given list: **rational, zero, common, factor, reciprocal**.

- If $P(x)$ and $Q(x)$ are polynomials, then $f(x) = \frac{P(x)}{Q(x)}$ is a _____ function provided that $Q(x)$ is not the zero polynomial.
- The domain of a rational function consists of all real numbers except for the x -values that will make the denominator equal to _____.
- Two rational expressions are equivalent if they produce the same values for any inputs from their _____ domain.
- To simplify, multiply, or divide rational expressions, we first _____ each numerator and denominator.
- To divide by a rational expression, we multiply by its _____.

Concept Check True or false.

- $f(x) = \frac{4}{\sqrt{x}-4}$ is a rational function.
- The domain of $f(x) = \frac{x-2}{4}$ is the set of all real numbers.
- $\frac{x-3}{4-x}$ is equivalent to $-\frac{x-3}{x-4}$.
- $\frac{n^2+1}{n^2-1}$ is equivalent to $\frac{n+1}{n-1}$.

Concept Check Given the rational function f , find $f(-1)$, $f(0)$, and $f(2)$.

- $f(x) = \frac{x}{x-2}$
- $f(x) = \frac{5x}{3x-x^2}$
- $f(x) = \frac{x-2}{x^2+x-6}$

Concept Check For each rational function, find all numbers that are not in the domain. Then give the domain, using both set notation and interval notation.

- $f(x) = \frac{x}{x+2}$
- $g(x) = \frac{x}{x-6}$
- $h(x) = \frac{2x-1}{3x+7}$
- $f(x) = \frac{3x+2}{5x-4}$
- $g(x) = \frac{x+2}{x^2-4}$
- $h(x) = \frac{x-2}{x^2+4}$
- $f(x) = \frac{5}{3x-x^2}$
- $g(x) = \frac{x^2+x-6}{x^2+12x+35}$
- $h(x) = \frac{7}{|4x-3|}$

Discussion Point

- Is there a rational function f such that 2 is not in its domain and $f(0) = 5$? If yes, how many such functions are there? If not, explain why not.

Concept Check

23. Which rational expressions are equivalent and what is the simplest expression that they are equivalent to?

a. $\frac{2x+3}{2x-3}$ b. $\frac{2x-3}{3-2x}$ c. $\frac{2x+3}{3+2x}$ d. $\frac{2x+3}{-2x-3}$ e. $\frac{3-2x}{2x-3}$

Concept Check

24. Which rational expressions can be simplified?

a. $\frac{x^2+2}{x^2}$ b. $\frac{x^2+2}{2}$ c. $\frac{x^2-x}{x^2}$ d. $\frac{x^2-y^2}{y^2}$ e. $\frac{x}{x^2-x}$

Simplify each expression, if possible.

25. $\frac{24a^3b}{3ab^3}$ 26. $\frac{-18x^2y^3}{8x^3y}$ 27. $\frac{7-x}{x-7}$ 28. $\frac{x+2}{x-2}$
 29. $\frac{a-5}{-5+a}$ 30. $\frac{(3-y)(x+1)}{(y-3)(x-1)}$ 31. $\frac{12x-15}{21}$ 32. $\frac{18a-2}{22}$
 33. $\frac{4y-12}{4y+12}$ 34. $\frac{7x+14}{7x-14}$ 35. $\frac{6m+18}{7m+21}$ 36. $\frac{3z^2+z}{18z+6}$
 37. $\frac{m^2-25}{20-4m}$ 38. $\frac{9n^2-3}{4-12n^2}$ 39. $\frac{t^2-25}{t^2-10t+25}$ 40. $\frac{p^2-36}{p^2+12t+36}$
 41. $\frac{x^2-9x+8}{x^2+3x-4}$ 42. $\frac{p^2+8p-9}{p^2-5p+4}$ 43. $\frac{x^3-y^3}{x^2-y^2}$ 44. $\frac{b^2-a^2}{a^3-b^3}$

Perform operations and simplify. Assume non-zero denominators.

45. $\frac{18a^4}{5b^2} \cdot \frac{25b^4}{9a^3}$ 46. $\frac{28}{xy} \div \frac{63x^3}{2y^2}$ 47. $\frac{12x}{49(xy^2)^3} \cdot \frac{(7xy)^2}{8}$
 48. $\frac{x+1}{2x-3} \cdot \frac{2x-3}{2x}$ 49. $\frac{10a}{6a-12} \cdot \frac{20a-40}{30a^3}$ 50. $\frac{a^2-1}{4a} \cdot \frac{2}{1-a}$
 51. $\frac{y^2-25}{4y} \cdot \frac{2}{5-y}$ 52. $(8x-16) \div \frac{3x-6}{10}$ 53. $(y^2-4) \div \frac{2-y}{8y}$
 54. $\frac{3n-9}{n^2-9} \cdot (n^3+27)$ 55. $\frac{x^2-16}{x^2} \cdot \frac{x^2-4x}{x^2-x-12}$ 56. $\frac{y^2+10y+25}{y^2-9} \cdot \frac{y^2-3y}{y+5}$
 57. $\frac{b-3}{b^2-4b+3} \div \frac{b^2-b}{b-1}$ 58. $\frac{x^2-6x+9}{x^2+3x} \div \frac{x^2-9}{x}$ 59. $\frac{x^2-2x}{3x^2-5x-2} \cdot \frac{9x^2-4}{9x^2-12x+4}$
 60. $\frac{t^2-49}{t^2+4t-21} \cdot \frac{t^2+8t+15}{t^2-2t-35}$ 61. $\frac{a^3-b^3}{a^2-b^2} \div \frac{2a-2b}{2a+2b}$ 62. $\frac{64x^3+1}{4x^2-100} \cdot \frac{4x+20}{64x^2-16x+4}$

$$63. \frac{x^3y-64y}{x^3y+64y} \div \frac{x^2y^2-16y^2}{x^2y^2-4xy^2+16y^2}$$

$$65. \frac{4x^2-9y^2}{8x^3-27y^3} \cdot \frac{4x^2+6xy+9y^2}{4x^2+12xy+9y^2}$$

$$67. \frac{6x^2-13x+6}{14x^2-25x+6} \div \frac{14-21x}{49x^2+7x-6}$$

$$69. \frac{3y}{x^2} \div \frac{y^2}{x} \div \frac{y}{5x}$$

$$71. \frac{a^2-4b^2}{a+2b} \div (a+2b) \cdot \frac{2b}{a-2b}$$

$$73. \frac{x^2-25}{x-4} \div \frac{x^2-2x-15}{x^2-10x+24} \cdot \frac{x+3}{x^2+10x+25}$$

$$64. \frac{p^3-27q^3}{p^2+pq-12q^2} \cdot \frac{p^2-2pq-24q^2}{p^2-5pq-6q^2}$$

$$66. \frac{2x^2+x-1}{6x^2+x-2} \div \frac{2x^2+5x+3}{6x^2+13x+6}$$

$$68. \frac{4y^2-12y+36}{27-3y^2} \div (y^3+27)$$

$$70. \frac{x+1}{y-2} \div \frac{2x+2}{y-2} \div \frac{x}{y}$$

$$72. \frac{9x^2}{x^2-16y^2} \div \frac{1}{x^2+4xy} \cdot \frac{x-4y}{3x}$$

$$74. \frac{y-3}{y^2-8y+16} \cdot \frac{y^2-16}{y+4} \div \frac{y^2+3y-18}{y^2+11y+30}$$

Given $f(x)$ and $g(x)$, find $f(x) \cdot g(x)$ and $f(x) \div g(x)$.

$$75. f(x) = \frac{x-4}{x^2+x} \text{ and } g(x) = \frac{2x}{x+1}$$

$$76. f(x) = \frac{x^3-3x^2}{x+5} \text{ and } g(x) = \frac{4x^2}{x-3}$$

$$77. f(x) = \frac{x^2-7x+12}{x+3} \text{ and } g(x) = \frac{9-x^2}{x-4}$$

$$78. f(x) = \frac{x+6}{4-x^2} \text{ and } g(x) = \frac{2-x}{x^2+8x+12}$$

RT.3

Addition and Subtraction of Rational Expressions



Many real-world applications involve adding or subtracting algebraic fractions. Similarly as in the case of common fractions, to add or subtract algebraic fractions, we first need to change them equivalently to fractions with the same denominator. Thus, we begin by discussing the techniques of finding the least common denominator.

Least Common Denominator

The **least common denominator (LCD)** for fractions with given denominators is the same as the **least common multiple (LCM)** of these denominators. The methods of finding the LCD for fractions with numerical denominators were reviewed in *section R3*. For example,

$$LCD(4,6,8) = 24,$$

because 24 is a multiple of 4, 6, and 8, and there is no smaller natural number that would be divisible by all three numbers, 4, 6, and 8.

Suppose the denominators of three algebraic fractions are $4(x^2 - y^2)$, $-6(x + y)^2$, and $8x$. The numerical factor of the least common multiple is 24. The variable part of the LCM is built by taking the product of all the different variable factors from each expression, with each factor raised to the **greatest** exponent that occurs in any of the expressions. In our example, since $4(x^2 - y^2) = 4(x + y)(x - y)$, then

$$LCD(4(x + y)(x - y), -6(x + y)^2, 8x) = 24x(x + y)^2(x - y)$$

Notice that we do not worry about the negative sign of the middle expression. This is because a negative sign can always be written in front of a fraction or in the numerator rather than in the denominator. For example,

$$\frac{1}{-6(x + y)^2} = -\frac{1}{6(x + y)^2} = \frac{-1}{6(x + y)^2}$$

In summary, to find the LCD for algebraic fractions, follow the steps:

- **Factor** each denominator **completely**.
- Build the LCD for the denominators by including the following as factors:
 - **LCD of all numerical coefficients**,
 - all of the **different factors** from each denominator, with each factor **raised to the greatest exponent** that occurs in any of the denominators.

Note: Disregard any factor of -1 .

Example 1

Determining the LCM for the Given Expressions

Find the LCM for the given expressions.

- a. $12x^3y$ and $15xy^2(x - 1)$ b. $x^2 - 2x - 8$ and $x^2 + 3x + 2$
- c. $y^2 - x^2$, $2x^2 - 2xy$, and $x^2 + 2xy + y^2$

Solution

- a. Notice that both expressions, $12x^3y$ and $15xy^2(x-1)$, are already in factored form. The $LCM(12,15) = 60$, as

divide by 3

$$\begin{array}{r} 3 \quad 12 \quad 15 \\ \cdot \quad 4 \quad \cdot 5 \\ \hline = 60 \end{array}$$

no more common factors, so we multiply the numbers in the letter L

The highest power of x is 3, the highest power of y is 2, and $(x-1)$ appears in the first power. Therefore,

$$LCM(12x^3y, 15xy^2(x-1)) = 60x^3y^2(x-1)$$

- b. To find the LCM of $x^2 - 2x - 8$ and $x^2 + 3x + 2$, we factor each expression first:

$$\begin{aligned} x^2 - 2x - 8 &= (x-4)(x+2) \\ x^2 + 3x + 2 &= (x+1)(x+2) \end{aligned}$$

There are three different factors in these expressions, $(x-4)$, $(x+2)$, and $(x+1)$. All of these factors appear in the first power, so

$$LCM(x^2 - 2x - 8, x^2 + 3x + 2) = (x-4)(x+2)(x+1)$$

notice that $(x+2)$ is taken only ones!

- c. As before, to find the LCM of $y^2 - x^2$, $2x^2 - 2xy$, and $x^2 + 2xy + y^2$, we factor each expression first:

$$\begin{aligned} y^2 - x^2 &= (y+x)(y-x) = -(x+y)(x-y) \\ 2x^2 - 2xy &= 2x(x-y) \\ x^2 + 2xy + y^2 &= (x+y)^2 \end{aligned}$$

as $y-x = -(x-y)$
and $y+x = x+y$

Since the factor of -1 can be disregarded when finding the LCM, the opposite factors can be treated as the same by factoring the -1 out of one of the expressions. So, there are four different factors to consider, 2 , x , $(x+y)$, and $(x-y)$. The highest power of $(x+y)$ is 2 and the other factors appear in the first power. Therefore,

$$LCM(y^2 - x^2, 2x^2 - 2xy, x^2 + 2xy + y^2) = 2x(x-y)(x+y)^2$$

Addition and Subtraction of Rational Expressions

Observe addition and subtraction of common fractions, as review in *section R3*.

$$\frac{1}{2} + \frac{2}{3} - \frac{5}{6} = \frac{1 \cdot 3 + 2 \cdot 2 - 5}{6} = \frac{3 + 4 - 5}{6} = \frac{2}{6} = \frac{1}{3}$$

work out the numerator

convert fractions to the lowest common denominator

simplify, if possible

To add or subtract algebraic fractions, follow the steps:

- Step 1:** Factor the denominators of all algebraic fractions **completely**.
- Step 2:** Find the **LCD** of all the denominators.
- Step 3:** Convert each algebraic fraction to the lowest common denominator found in Step 2 and write the sum (or difference) as a single fraction.
- Step 4:** Simplify the numerator and the whole fraction, if possible.

Example 2 Adding and Subtracting Rational Expressions

Perform the operations and simplify if possible.

a. $\frac{a}{5} - \frac{3b}{2a}$

b. $\frac{x}{x-y} + \frac{y}{y-x}$

c. $\frac{3x^2+3xy}{x^2-y^2} - \frac{2-3x}{x-y}$

d. $\frac{y+1}{y^2-7y+6} + \frac{y+2}{y^2-5y-6}$

e. $\frac{2x}{x^2-4} + \frac{5}{2-x} - \frac{1}{2+x}$

f. $(2x-1)^{-2} + (2x-1)^{-1}$

Solution

Multiplying the numerator and denominator of a fraction by the same factor is equivalent to multiplying the whole fraction by 1, which does not change the value of the fraction.

- a. Since $LCM(5, 2a) = 10a$, we would like to rewrite expressions, $\frac{a}{5}$ and $\frac{3b}{2a}$, so that they have a denominator of $10a$. This can be done by multiplying the numerator and denominator of each expression by the factors of $10a$ that are missing in each denominator. So, we obtain

$$\frac{a}{5} - \frac{3b}{2a} = \frac{a}{5} \cdot \frac{2a}{2a} - \frac{3b}{2a} \cdot \frac{5}{5} = \frac{2a^2 - 15b}{10a}$$

- b. Notice that the two denominators, $x - y$ and $y - x$, are opposite expressions. If we write $y - x$ as $-(x - y)$, then

$$\frac{x}{x-y} + \frac{y}{y-x} = \frac{x}{x-y} + \frac{y}{-(x-y)} = \frac{x}{x-y} - \frac{y}{x-y} = \frac{x-y}{x-y} = 1$$

combine the signs

- c. To find the LCD, we begin by factoring $x^2 - y^2 = (x - y)(x + y)$. Since this expression includes the second denominator as a factor, the LCD of the two fractions is $(x - y)(x + y)$. So, we calculate

keep the bracket after a “-” sign

$$\begin{aligned} \frac{3x^2+3xy}{x^2-y^2} - \frac{2-3x}{x-y} &= \frac{(3x^2+3xy) \cdot 1 + (2-3x) \cdot (x+y)}{(x-y)(x+y)} = \\ \frac{3x^2+3xy - (2x+2y+3x^2+3xy)}{(x-y)(x+y)} &= \frac{3x^2+3xy-2x-2y-3x^2-3xy}{(x-y)(x+y)} = \\ \frac{-2x-2y}{(x-y)(x+y)} &= \frac{-2(x+y)}{(x-y)(x+y)} = \frac{-2}{x-y} \end{aligned}$$

- d. To find the LCD, we first factor each denominator. Since

$$y^2 - 7y + 6 = (y - 6)(y - 1) \text{ and } y^2 - 5y - 6 = (y - 6)(y + 1),$$

then $LCD = (y - 6)(y - 1)(y + 1)$ and we calculate

multiply by the missing bracket

$$\begin{aligned} \frac{y+1}{y^2-7y+6} + \frac{y-1}{y^2-5y-6} &= \frac{y+1}{(y-6)(y-1)} + \frac{y-1}{(y-6)(y+1)} = \\ \frac{(y+1) \cdot (y+1) + (y-1) \cdot (y-1)}{(y-6)(y-1)(y+1)} &= \frac{y^2+2y+1+(y^2-1)}{(y-6)(y-1)(y+1)} = \\ \frac{2y^2+2y}{(y-6)(y-1)(y+1)} &= \frac{2y(y+1)}{(y-6)(y-1)(y+1)} = \frac{2y}{(y-6)(y-1)} \end{aligned}$$

- e. As in the previous examples, we first factor the denominators, including factoring out a negative from any opposite expression. So,

$$\frac{2x}{x^2-4} + \frac{5}{2-x} - \frac{1}{2+x} = \frac{2x}{(x-2)(x+2)} + \frac{5}{-(x-2)} - \frac{1}{x+2} =$$

$LCD = (x-2)(x+2)$

$$\begin{aligned} \frac{2x - 5(x+2) - 1(x-2)}{(x-2)(x+2)} &= \frac{2x - 5x - 10 - x + 2}{(x-2)(x+2)} = \\ \frac{-4x - 8}{(x-2)(x+2)} &= \frac{-4(x+2)}{(x-2)(x+2)} = \frac{-4}{(x-2)} \end{aligned}$$

- e. Recall that a negative exponent really represents a hidden fraction. So, we may choose to rewrite the negative powers as fractions, and then add them using techniques as shown in previous examples.

$$3(2x-1)^{-2} + (2x-1)^{-1} = \frac{1}{(2x-1)^2} + \frac{1}{2x-1} = \frac{1+1 \cdot (2x-1)}{(2x-1)^2} =$$

$$\frac{3+2x-1}{(2x-1)^2} = \frac{2x+2}{(2x-1)^2} = \frac{2(x+1)}{(2x-1)^2} \quad \text{nothing to simplify this time}$$

Note: Since addition (or subtraction) of rational expressions results in a rational expression, from now on the term “rational expression” will include sums of rational expressions as well.

Example 3 ▶ Adding Rational Expressions in Application Problems

An airplane flies m miles with a windspeed of w mph. On the return flight, the airplane flies against the same wind. The expression $\frac{m}{s+w} + \frac{m}{s-w}$, where s is the speed of the airplane in still air, represents the total time in hours that it takes to make the round-trip flight. Write a single rational expression representing the total time.

4. The LCD of rational expressions is the product of the different factors in the denominators, where the power of each factor is the _____ number of times that it occurs in any single denominator.
5. A polynomial expression raised to a negative exponent represents a _____ expression.

Concept Check

6. a. What is the LCM for 6 and 9? b. What is the LCD for $\frac{1}{6}$ and $\frac{1}{9}$?
7. a. What is the LCM for $x^2 - 25$ and $x + 5$? b. What is the LCD for $\frac{1}{x^2-25}$ and $\frac{1}{x+5}$?

Concept Check Find the LCD and then perform the indicated operations. Simplify the resulting fraction.

8. $\frac{5}{12} + \frac{13}{18}$ 9. $\frac{11}{30} - \frac{19}{75}$ 10. $\frac{3}{4} + \frac{7}{30} - \frac{1}{16}$ 11. $\frac{5}{8} - \frac{7}{12} + \frac{11}{40}$

Concept Check Find the least common multiple (LCM) for each group of expressions.

12. $24a^3b^4$, $18a^5b^2$ 13. $6x^2y^2$, $9x^3y$, $15y^3$ 14. $x^2 - 4$, $x^2 + 2x$
15. $10x^2$, $25(x^2 - x)$ 16. $(x - 1)^2$, $1 - x$ 17. $y^2 - 25$, $5 - y$
18. $x^2 - y^2$, $xy + y^2$ 19. $5a - 15$, $a^2 - 6a + 9$ 20. $x^2 + 2x + 1$, $x^2 - 4x - 1$
21. $n^2 - 7n + 10$, $n^2 - 8n + 15$ 22. $2x^2 - 5x - 3$, $2x^2 - x - 1$, $x^2 - 6x + 9$
23. $1 - 2x$, $2x + 1$, $4x^2 - 1$ 24. $x^5 - 4x^4 + 4x^3$, $12 - 3x^2$, $2x + 4$

Concept Check True or false? If true, explain why. If false, correct it.

25. $\frac{1}{2x} + \frac{1}{3x} = \frac{1}{5x}$ 26. $\frac{1}{x-3} + \frac{1}{3-x} = 0$ 27. $\frac{1}{x} + \frac{1}{y} = \frac{1}{x+y}$ 28. $\frac{3}{4} + \frac{x}{5} = \frac{3+x}{20}$

Perform the indicated operations and simplify if possible.

29. $\frac{x-2y}{x+y} + \frac{3y}{x+y}$ 30. $\frac{a+3}{a+1} - \frac{a-5}{a+1}$ 31. $\frac{4a+3}{a-3} - 1$
32. $\frac{n+1}{n-2} + 2$ 33. $\frac{x^2}{x-y} + \frac{y^2}{y-x}$ 34. $\frac{4a-2}{a^2-49} + \frac{5+3a}{49-a^2}$
35. $\frac{2y-3}{y^2-1} - \frac{4-y}{1-y^2}$ 36. $\frac{a^3}{a-b} + \frac{b^3}{b-a}$ 37. $\frac{1}{x+h} - \frac{1}{x}$
38. $\frac{x-2}{x+3} + \frac{x+2}{x-4}$ 39. $\frac{x-1}{3x+1} + \frac{2}{x-3}$ 40. $\frac{4xy}{x^2-y^2} + \frac{x-y}{x+y}$
41. $\frac{x-1}{3x+15} - \frac{x+3}{5x+25}$ 42. $\frac{y-2}{4y+8} - \frac{y+6}{5y+10}$ 43. $\frac{4x}{x-1} - \frac{2}{x+1} - \frac{4}{x^2-1}$
44. $\frac{-2}{y+2} + \frac{5}{y-2} + \frac{y+3}{y^2-4}$ 45. $\frac{y}{y^2-y-20} + \frac{2}{y+4}$ 46. $\frac{5x}{x^2-6x+8} - \frac{3x}{x^2-x-12}$
47. $\frac{9x+2}{3x^2-2x-8} + \frac{7}{3x^2+x-4}$ 48. $\frac{3y+2}{2y^2-y-10} + \frac{8}{2y^2-7y+5}$ 49. $\frac{6}{y^2+6y+9} + \frac{5}{y^2-9}$
50. $\frac{3x-1}{x^2+2x-3} - \frac{x+4}{x^2-9}$ 51. $\frac{1}{x+1} - \frac{2x}{x-2} + \frac{x^2+2}{x^2-x-2}$ 52. $\frac{2}{y+3} - \frac{y}{y-1} + \frac{y^2+2}{y^2+2y-3}$

53. $\frac{4x}{x^2-1} + \frac{3x}{1-x} - \frac{4}{x-1}$

54. $\frac{5y}{1-2y} - \frac{2y}{2y+1} + \frac{3}{4y^2-1}$

55. $\frac{x+5}{x-3} - \frac{x+2}{x+1} - \frac{6x+10}{x^2-2x-3}$

Discussion Point

56. Consider the following calculation

$$\frac{x}{x-2} - \frac{4x-1}{x^2-4} = \frac{x(x+2) - 4x - 1}{x^2-4} = \frac{x^2 + 2x - 4x - 1}{x^2-4} = \frac{x^2 - 2x - 1}{x^2-4}$$

Is this correct? If *yes*, check if the result can be simplified. If *no*, correct it.

Perform the indicated operations and simplify if possible.

57. $2x^{-3} + (3x)^{-1}$

58. $(x^2 - 9)^{-1} + 2(x - 3)^{-1}$

59. $\left(\frac{x+1}{3}\right)^{-1} - \left(\frac{x-4}{2}\right)^{-1}$

60. $\left(\frac{a-3}{a^2} - \frac{a-3}{9}\right) \div \frac{a^2-9}{3a}$

61. $\frac{x^2-4x+4}{2x+1} \cdot \frac{2x^2+x}{x^3-4x} - \frac{3x-2}{x+1}$

62. $\frac{2}{x-3} - \frac{x}{x^2-x-6} \cdot \frac{x^2-2x-3}{x^2-x}$

Given $f(x)$ and $g(x)$, find $(f + g)(x)$ and $(f - g)(x)$. Leave the answer in simplified single fraction form.

63. $f(x) = \frac{x}{x+2}, g(x) = \frac{4}{x-3}$

64. $f(x) = \frac{x}{x^2-4}, g(x) = \frac{1}{x^2+4x+4}$

65. $f(x) = \frac{3x}{x^2+2x-3}, g(x) = \frac{1}{x^2-2x+1}$

66. $f(x) = x + \frac{1}{x-1}, g(x) = \frac{1}{x+1}$

Solve each problem.

67. Two friends work part-time at a store. The first person works every sixth day and the second person works every tenth day. If they are both working today, how many days pass before they both work on the same day again?



68. Suppose a cylindrical water tank is being drained. The change in the water level can be found using the expression $\frac{V_1}{\pi r^2} - \frac{V_2}{\pi r^2}$, where V_1 and V_2 represent the original and new volume, respectively, and r is the radius of the tank. Write the change in water level as a single algebraic fraction.

69. To determine the percent growth in sales from the previous year, the owner of a company uses the expression $100\left(\frac{S_1}{S_0} - 1\right)$, where S_1 represents the current year's sales and S_0 represents last year's sales. Write this expression as a single algebraic fraction.

70. A boat travels d mi against the current whose rate is c mph. On the return trip, the boat travels with the same current. The expression $\frac{d}{r-c} + \frac{d}{r+c}$, where r is the speed of the boat in calm water, represents the total amount of time in hours it takes for the entire boating trip. Represent this amount of time as a single algebraic fraction.



RT.4

Complex Fractions



When working with algebraic expressions, sometimes we come across needing to simplify expressions like these:

$$\frac{\frac{x^2 - 9}{x + 1}}{\frac{x + 3}{x^2 - 1}}, \quad 1 + \frac{1}{x}, \quad \frac{1}{x + 2} - \frac{1}{x + h + 2}, \quad \frac{1}{\frac{1}{a} - \frac{1}{b}}$$

A complex fraction is a quotient of rational expressions (including sums of rational expressions) where at least one of these expressions contains a fraction itself. In this section, we will examine two methods of simplifying such fractions.

Simplifying Complex Fractions

Definition 4.1 ▶ A **complex fraction** is a **quotient** of rational expressions (including their sums) that result in a fraction with more than two levels. For example, $\frac{1}{\frac{2}{3}}$ has three levels while $\frac{\frac{2x}{3}}{\frac{4x}{3}}$ has four levels. Such fractions can be **simplified to a single fraction** with only two levels. For example,

$$\frac{\frac{1}{\frac{2}{3}}}{\frac{1}{\frac{1}{3}}} = \frac{1}{\frac{2}{2}} \cdot \frac{1}{\frac{1}{3}} = \frac{1}{6}, \quad \text{or} \quad \frac{\frac{1}{\frac{2x}{3}}}{\frac{4x^2}{3}} = \frac{1}{\frac{2x}{2x}} \cdot \frac{4x^2}{3} = \frac{2x}{3}$$

There are two common methods of simplifying complex fractions.

Method I (multiplying by the reciprocal of the denominator)

Replace the main division in the complex fraction with a multiplication of the numerator fraction by the reciprocal of the denominator fraction. We then simplify the resulting fraction if possible. Both examples given in *Definition 4.1* were simplified using this strategy.

Method I is the most convenient to use when both the numerator and the denominator of a complex fraction consist of single fractions. However, if either the numerator or the denominator of a complex fraction contains addition or subtraction of fractions, it is usually easier to use the method shown below.

Method II (multiplying by LCD)

Multiply the numerator and denominator of a complex fraction by the least common denominator of all the fractions appearing in the numerator or in the denominator of the complex fraction. Then, simplify the resulting fraction if possible. For example, to simplify $\frac{y + \frac{1}{x}}{x + \frac{1}{y}}$, multiply the numerator $y + \frac{1}{x}$ and the denominator $x + \frac{1}{y}$ by the LCD $(\frac{1}{x}, \frac{1}{y}) = xy$. So,

$$\frac{\left(y + \frac{1}{x}\right) \cdot xy}{\left(x + \frac{1}{y}\right) \cdot xy} = \frac{xy^2 + y}{x^2y + x} = \frac{y(xy + 1)}{x(xy + 1)} = \frac{y}{x}$$

Example 1 ▶ **Simplifying Complex Fractions**

Use a method of your choice to simplify each complex fraction.

a.
$$\frac{\frac{x^2-x-12}{x^2-2x-15}}{\frac{x^2+8x+12}{x^2-5x-14}}$$

b.
$$\frac{a+b}{\frac{1}{a^3} + \frac{1}{b^3}}$$

c.
$$\frac{x + \frac{1}{5}}{x - \frac{1}{3}}$$

d.
$$\frac{\frac{6}{x^2-4} - \frac{5}{x+2}}{\frac{7}{x^2-4} - \frac{4}{x-2}}$$

Solution ▶

- a. Since the expression $\frac{\frac{x^2-x-12}{x^2-2x-15}}{\frac{x^2+8x+12}{x^2-5x-14}}$ contains a single fraction in both the numerator and denominator, we will simplify it using method I, as below.

$$\frac{\frac{x^2-2x-8}{x^2-2x-15}}{\frac{x^2+8x+12}{x^2-4x-21}} = \frac{(x-4)(x+2)}{(x-5)(x+3)} \cdot \frac{(x-7)(x+3)}{(x+6)(x+2)} = \frac{(x-4)(x-7)}{(x-5)(x+6)}$$

factor and multiply by the reciprocal

- b. $\frac{a+b}{\frac{1}{a^3} + \frac{1}{b^3}}$ can be simplified in the following two ways:

Method I

$$\begin{aligned} \frac{a+b}{\frac{1}{a^3} + \frac{1}{b^3}} &= \frac{a+b}{\frac{b^3+a^3}{a^3b^3}} = \frac{(a+b)a^3b^3}{a^3+b^3} \\ &= \frac{(a+b)a^3b^3}{(a+b)(a^2-ab+b^2)} = \frac{a^3b^3}{a^2-ab+b^2} \end{aligned}$$

Method II

$$\begin{aligned} \frac{a+b}{\frac{1}{a^3} + \frac{1}{b^3}} &= \frac{a+b}{\frac{b^3+a^3}{a^3b^3}} = \frac{(a+b)a^3b^3}{b^3+a^3} \\ &= \frac{(a+b)a^3b^3}{(a+b)(a^2-ab+b^2)} = \frac{a^3b^3}{a^2-ab+b^2} \end{aligned}$$

Caution: In Method II, the factor that we multiply the complex fraction by **must be equal to 1**. This means that **the numerator and denominator of this factor must be exactly the same**.

- c. To simplify $\frac{x + \frac{1}{5}}{x - \frac{1}{3}}$, we will use method II. Multiplying the numerator and denominator by the LCD $(\frac{1}{5}, \frac{1}{3}) = 15$, we obtain

$$\frac{x + \frac{1}{5}}{x - \frac{1}{3}} = \frac{15x + 3}{15x - 5}$$

To simplify this expression, we can multiply the numerator and denominator by the lowest common denominator, which is $(a + h + 1)(a + 1)$. Thus,

$$\frac{\frac{1}{a+h+1} - \frac{1}{a+1}}{h} \cdot \frac{(a+h+1)(a+1)}{(a+h+1)(a+1)} = \frac{a+1 - (a+h+1)}{h(a+h+1)(a+1)}$$

$$= \frac{\cancel{a} + \cancel{1} - \cancel{a} - h - \cancel{1}}{h(a+h+1)(a+1)} = \frac{-h}{h(a+h+1)(a+1)} = \frac{-1}{(a+h+1)(a+1)}$$

This bracket is essential!

keep the denominator in a factored form

RT.4 Exercises

Vocabulary Check Complete each blank with the most appropriate term from the given list: **complex, LCD, multiply, reciprocal, same, single**.

- A quotient of two rational expressions that results in a fraction with more than two levels is called a _____ algebraic fraction.
- To simplify a complex fraction means to find an equivalent _____ fraction $\frac{P}{Q}$, where P and Q are polynomials with no essential common factors.
- To simplify a complex rational expression using the _____ method, first write both the numerator and the denominator as single fractions in simplified form.
- To simplify a complex fraction using the _____ method, first find the LCD of all rational expressions within the complex fraction. Then, _____ the numerator and denominator by the _____ LCD expression.

Concept Check Simplify each complex fraction.

5. $\frac{2 - \frac{1}{3}}{3 + \frac{7}{3}}$

6. $\frac{5 - \frac{3}{4}}{4 + \frac{1}{2}}$

7. $\frac{\frac{3}{8} - 5}{\frac{2}{3} + 6}$

8. $\frac{\frac{2}{3} + \frac{4}{5}}{\frac{3}{4} - \frac{1}{2}}$

Simplify each complex rational expression.

9. $\frac{\frac{x^3}{y}}{\frac{x^2}{y^3}}$

10. $\frac{\frac{n-5}{6n}}{\frac{n-5}{8n^2}}$

11. $\frac{1 - \frac{1}{a}}{4 + \frac{1}{a}}$

12. $\frac{\frac{2}{5} + 3}{\frac{5}{n} - 6}$

13. $\frac{\frac{9-3x}{4x+12}}{\frac{x-3}{6x-24}}$

14. $\frac{\frac{9}{15} - \frac{y}{6}}{y}$

15. $\frac{\frac{4}{4} - \frac{2}{2}}{\frac{x}{4} + \frac{y}{2}}$

16. $\frac{\frac{3}{a} + \frac{4}{b}}{\frac{4}{a} - \frac{3}{b}}$

17. $\frac{a - \frac{3a}{b}}{b - \frac{b}{a}}$

18. $\frac{\frac{1}{x} - \frac{1}{y}}{x^2 - y^2}$

19. $\frac{\frac{4}{y} - \frac{y}{x^2}}{\frac{1}{x} - \frac{2}{y}}$

20. $\frac{\frac{5}{p} - \frac{1}{q}}{\frac{1}{5q^2} - \frac{5}{p^2}}$

21. $\frac{\frac{n-12}{n} + n}{n+4}$

22. $\frac{2t-1}{\frac{3t-2}{t} + 2t}$

23. $\frac{\frac{1}{a-h} - \frac{1}{a}}{h}$

24. $\frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$

25. $\frac{4 + \frac{12}{2x-3}}{5 + \frac{15}{2x-3}}$

26. $\frac{1 + \frac{3}{x+2}}{1 + \frac{6}{x-1}}$

27. $\frac{\frac{1}{b^2} - \frac{1}{a^2}}{\frac{1}{b} - \frac{1}{a}}$

28. $\frac{\frac{1}{x^2} - \frac{1}{y^2}}{\frac{1}{x} + \frac{1}{y}}$

29. $\frac{\frac{x+3}{x} - \frac{4}{x-1}}{\frac{x}{x-1} + \frac{1}{x}}$

30. $\frac{\frac{3}{x^2+6x+9} + \frac{3}{x+3}}{\frac{6}{x^2-9} + \frac{6}{3-x}}$

31. $\frac{\frac{1}{a^2} - \frac{1}{b^2}}{\frac{1}{a^3} + \frac{1}{b^3}}$

32. $\frac{\frac{4p^2-12p+9}{2p^2+7p-15}}{\frac{2p^2-15p+18}{p^2-p-30}}$

Discussion Point

33. Are the expressions
- $\frac{x^{-1}+y^{-1}}{x^{-2}+y^{-2}}$
- and
- $\frac{x^2+y^2}{x+y}$
- equivalent? Explain why or why not.

Simplify each expression. Leave your answer with only positive exponents.

34. $\frac{1}{a^{-2} - b^{-2}}$

35. $\frac{x^{-1} + x^{-2}}{3x^{-1}}$

36. $\frac{x^{-2}}{y^{-3} - x^{-3}}$

37. $\frac{1 - (2n+1)^{-1}}{1 + (2n+1)^{-1}}$

Analytic Skills Find and simplify the difference quotient $\frac{f(a+h)-f(a)}{h}$ for the given function.

38. $f(x) = \frac{5}{x}$

39. $f(x) = \frac{2}{x^2}$

40. $f(x) = \frac{1}{1-x}$

41. $f(x) = -\frac{1}{x-2}$

Analytic Skills Simplify each continued fraction.

42. $a - \frac{a}{1 - \frac{a}{1-a}}$

43. $3 - \frac{2}{1 - \frac{2}{3 - \frac{2}{x}}}$

44. $a + \frac{a}{2 + \frac{1}{1 - \frac{2}{a}}}$

RT.5

Rational Equations and Graphs



In previous sections of this chapter, we worked with rational expressions. If two rational expressions are equated, a *rational equation* arises. Such equations often appear when solving application problems that involve rates of work or amounts of time considered in motion problems. In this section, we will discuss how to solve rational equations, with close attention to their domains. We will also take a look at the graphs of reciprocal functions, their properties and transformations.

Rational Equations

Definition 5.1 ▶ A **rational equation** is an equation involving only rational expressions and containing at least one fractional expression.

Here are some examples of rational equations:

$$\frac{x}{2} - \frac{12}{x} = -1, \quad \frac{x^2}{x-5} = \frac{25}{x-5}, \quad \frac{2x}{x-3} - \frac{6}{x} = \frac{18}{x^2-3x}$$

Attention! A rational equation contains an *equals* sign, while a rational expression does not. An equation can be solved for a given variable, while an expression can only be simplified or evaluated. For example, $\frac{x}{2} - \frac{12}{x}$ is an **expression** to simplify, while $\frac{x}{2} = \frac{12}{x}$ is an **equation** to solve.

When working with algebraic structures, it is essential to identify whether they are equations or expressions before applying appropriate strategies.

By *Definition 5.1*, rational equations contain one or more denominators. Since division by zero is not allowed, we need to pay special attention to the variable values that would make any of these denominators equal to zero. Such values would have to be excluded from the set of possible solutions. For example, neither 0 nor 3 can be solutions to the equation

$$\frac{2x}{x-3} - \frac{6}{x} = \frac{18}{x^2-3x},$$

as it is impossible to evaluate either of its sides for $x = 0$ or 3. So, when solving a rational equation, it is important to find its domain first.

Definition 5.2 ▶ The **domain** of the variable(s) of a rational equation (in short, the **domain of a rational equation**) is the **intersection of the domains** of all rational expressions within the equation.

As stated in *Definition 2.1*, the domain of each single algebraic fraction is the set of all real numbers except for the **zeros of the denominator** (the variable values that would make the denominator equal to zero). Therefore, the **domain of a rational equation is the set of all real numbers except for the zeros of all the denominators** appearing in this equation.

Example 1 ▶ **Determining Domains of Rational Equations**

Find the domain of the variable in each of the given equations.

a. $\frac{x}{2} - \frac{12}{x} = -1$

b. $\frac{2x}{x-2} = \frac{-3}{x} + \frac{4}{x-2}$

c. $\frac{2}{y^2-2y-8} - \frac{4}{y^2+6y+8} = \frac{2}{y^2-16}$

Solution ▶

a. The equation $\frac{x}{2} - \frac{12}{x} = -1$ contains two denominators, 2 and x . 2 is never equal to zero and x becomes zero when $x = 0$. Thus, the domain of this equation is $\mathbb{R} \setminus \{0\}$.

b. The equation $\frac{2x}{x-2} = \frac{-3}{x} + \frac{4}{x-2}$ contains two types of denominators, $x - 2$ and x . The $x - 2$ becomes zero when $x = 2$, and x becomes zero when $x = 0$. Thus, the domain of this equation is $\mathbb{R} \setminus \{0, 2\}$.

c. The equation $\frac{2}{y^2-2y-8} - \frac{4}{y^2+6y+8} = \frac{2}{y^2-16}$ contains three different denominators. To find the zeros of these denominators, we solve the following equations by factoring:

$$y^2 - 2y - 8 = 0$$

$$y^2 + 6y + 8 = 0$$

$$y^2 - 16 = 0$$

$$(y - 4)(y + 2) = 0$$

$$(y + 4)(y + 2) = 0$$

$$(y - 4)(y + 4) = 0$$

$$y = 4 \text{ or } y = -2$$

$$y = -4 \text{ or } y = -2$$

$$y = 4 \text{ or } y = -4$$

So, -4 , -2 , and 4 must be excluded from the domain of this equation. Therefore, the domain $D = \mathbb{R} \setminus \{-4, -2, 4\}$.

To solve a rational equation, it is convenient to clear all the fractions first and then solve the resulting polynomial equation. This can be achieved by multiplying all the terms of the equation by the least common denominator.

Caution! Only **equations**, not expressions, can be changed equivalently by **multiplying** both of their sides by the **LCD**.

Multiplying **expressions** by any number other than 1 creates expressions that are **NOT equivalent** to the original ones. So, avoid multiplying rational expressions by the LCD.

Example 2 ▶ **Solving Rational Equations**

Solve each equation.

a. $\frac{x}{2} - \frac{12}{x} = -1$

b. $\frac{2x}{x-2} = \frac{-3}{x} + \frac{4}{x-2}$

c. $\frac{2}{y^2-2y-8} - \frac{4}{y^2+6y+8} = \frac{2}{y^2-16}$

d. $\frac{x-1}{x-3} = \frac{2}{x-3}$

Solution

- ▶ a. The domain of the equation $\frac{x}{2} - \frac{12}{x} = -1$ is the set $\mathbb{R} \setminus \{0\}$, as discussed in *Example 1a*. The $LCM(2, x) = 2x$, so we calculate

$$\frac{x}{2} - \frac{12}{x} = -1 \quad / \cdot 2x$$

$$\cancel{2x} \cdot \frac{x}{\cancel{2}} - \cancel{2x} \cdot \frac{12}{\cancel{x}} = -1 \cdot 2x$$

multiply each term by the LCD

$$x^2 - 24 = -2x$$

$$x^2 + 2x - 24 = 0$$

$$(x + 6)(x - 4) = 0$$

factor to find the possible roots

$$x = -6 \text{ or } x = 4$$

Since both of these numbers belong to the domain, the solution set of the original equation is $\{-6, 4\}$.

- b. The domain of the equation $\frac{2x}{x-2} = \frac{-3}{x} + \frac{4}{x-2}$ is the set $\mathbb{R} \setminus \{0, 2\}$, as discussed in *Example 1b*. The $LCM(x-2, x) = x(x-2)$, so we calculate

$$\frac{2x}{x-2} = \frac{-3}{x} + \frac{4}{x-2} \quad / \cdot x(x-2)$$

$$\cancel{x(x-2)} \cdot \frac{2x}{\cancel{x-2}} = \frac{-3}{\cancel{x}} \cdot \cancel{x(x-2)} + \frac{4}{\cancel{x-2}} \cdot \cancel{x(x-2)}$$

$$2x^2 = -3(x-2) + 4x$$

$$2x^2 = -3x + 6 + 4x$$

$$2x^2 - x + 6 = 0$$

$$(2x + 3)(x - 2) = 0$$

factor to find the possible roots

$$x = -\frac{3}{2} \text{ or } x = 2$$

Since 2 is excluded from the domain, there is only one solution to the original equation, $x = -\frac{3}{2}$.

- c. The domain of the equation $\frac{2}{y^2-2y-8} - \frac{4}{y^2+6y+8} = \frac{2}{y^2-16}$ is the set $\mathbb{R} \setminus \{-4, -2, 4\}$, as discussed in *Example 1c*. To find the LCD, it is useful to factor the denominators first. Since

$$y^2 - 2y - 8 = (y - 4)(y + 2),$$

$$y^2 + 6y + 8 = (y + 4)(y + 2), \text{ and}$$

$y^2 - 16 = (y - 4)(y + 4)$, then the LCD needed to clear the fractions in the original equation is $(y - 4)(y + 4)(y + 2)$. So, we calculate

$$\frac{2}{(y-4)(y+2)} - \frac{4}{(y+4)(y+2)} = \frac{2}{(y-4)(y+4)} \quad / \cdot (y-4)(y+4)(y+2)$$

$$\begin{aligned} \cancel{(y-4)}(y+4)\cancel{(y+2)} \cdot \frac{2}{\cancel{(y-4)}(y+2)} - \cancel{(y-4)}\cancel{(y+4)}\cancel{(y+2)} \cdot \frac{4}{\cancel{(y+4)}(y+2)} \\ = \frac{2}{\cancel{(y-4)}(y+4)} \cdot \cancel{(y-4)}\cancel{(y+4)}(y+2) \end{aligned}$$

$$2(y+4) - 4(y-4) = 2(y+2)$$

$$2y + 8 - 4y + 16 = 2y + 4$$

$$20 = 4y$$

$$y = 5$$

Since 5 is in the domain, this is the true solution.

- d. First, we notice that the domain of the equation $\frac{x-1}{x-3} = \frac{2}{x-3}$ is the set $\mathbb{R} \setminus \{3\}$. To solve this equation, we can multiply it by the *LCD* = $x - 3$, as in the previous examples, or we can apply the method of cross-multiplication, as the equation is a proportion. Here, we show both methods.

Use the method
of your choice
– either one is
fine.

Multiplication by LCD:

$$\frac{x-1}{x-3} = \frac{2}{x-3}$$

$/ \cdot (x-3)$

$$x - 1 = 2$$

$$x = 3$$

this multiplication
is permitted as
 $x - 3 \neq 0$

Cross-multiplication:

$$\frac{x-1}{x-3} = \frac{2}{x-3}$$

$$(x-1)(x-3) = 2(x-3)$$

$$x - 1 = 2$$

$$x = 3$$

$/ \div (x-3)$

this division is
permitted as
 $x - 3 \neq 0$

Since 3 is excluded from the domain, there is **no solution** to the original equation.

Summary of Solving Rational Equations in One Variable

- Determine the domain** of the variable.
- Clear** all the **fractions by multiplying** both sides of the equation **by the LCD** of these fractions.
- Find possible solutions** by solving the resulting equation.
- Check** the possible solutions **against the domain**. The solution set consists of only these possible solutions that belong to the domain.

Graphs of Basic Rational Functions

So far, we discussed operations on rational expressions and solving rational equations. Now, we will look at rational functions, such as

$$f(x) = \frac{1}{x}, \quad g(x) = \frac{-2}{x+3}, \quad \text{or} \quad h(x) = \frac{x-3}{x-2}.$$

Definition 5.3 ▶ A **rational function** is any function that can be written in the form

$$f(x) = \frac{P(x)}{Q(x)},$$

where P and Q are polynomials and Q is not a zero polynomial.

The **domain** D_f of such function f includes all x -values for which $Q(x) \neq 0$.

Example 3 ▶ **Finding the Domain of a Rational Function**

Find the domain of each function.

a. $g(x) = \frac{-2}{x+3}$

b. $h(x) = \frac{x-3}{x-2}$

Solution ▶

a. Since $x + 3 = 0$ for $x = -3$, the domain of g is the set of all real numbers except for -3 . So, the domain $D_g = \mathbb{R} \setminus \{-3\}$.

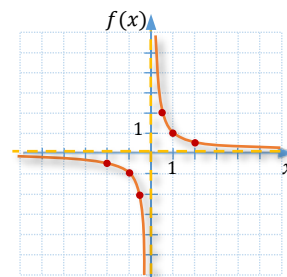
b. Since $x - 2 = 0$ for $x = 2$, the domain of h is the set of all real numbers except for 2. So, the domain $D_h = \mathbb{R} \setminus \{2\}$.

Note: The subindex f in the notation D_f indicates that the domain is of function f .

To graph a rational function, we usually start by making a table of values. Because the graphs of rational functions are typically nonlinear, it is a good idea to plot at least 3 points on each side of each x -value where the function is undefined. For example, to graph the

x	$f(x)$
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$
0	undefined
$-\frac{1}{2}$	-2
-1	-1
-2	$-\frac{1}{2}$

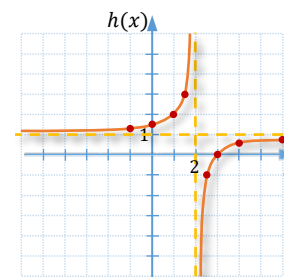
basic rational function, $f(x) = \frac{1}{x}$, called the **reciprocal function**, we evaluate f for a few points to the right of zero and to the left of zero. This is because f is undefined at $x = 0$, which means that the graph of f does not cross the y -axis. After plotting the obtained points, we connect them within each group, to the right of zero and to the left of zero, creating two disjoint curves. To see the shape of each curve clearly, we might need to evaluate f at some additional points.



The **domain** of the reciprocal function $f(x) = \frac{1}{x}$ is $\mathbb{R} \setminus \{0\}$, as the denominator x must be different than zero. Projecting the graph of this function onto the y -axis helps us determine the **range**, which is also $\mathbb{R} \setminus \{0\}$.

x	$h(x)$
-1	$\frac{4}{3}$
0	$\frac{3}{2}$
1	2
$\frac{3}{2}$	3
2	undefined
$\frac{5}{2}$	-1
3	0
4	$\frac{1}{2}$
6	$\frac{3}{4}$

As before, to graph function h , we evaluate it at some points to the right and to the left of 2. Then, we draw the vertical asymptote $x = 2$ and plot and join the obtained points on each side of this asymptote. The graph suggests that the horizontal asymptote is the line $y = 1$. Thus, the range of function h is $\mathbb{R} \setminus \{1\}$.



Notice that $\frac{x-3}{x-2} = \frac{x-2-1}{x-2} = \frac{x-2}{x-2} - \frac{1}{x-2} = 1 - \frac{1}{x-2}$. Since $\frac{1}{x-2}$ is never equal to zero than $1 - \frac{1}{x-2}$ is never equal to 1. This confirms the range and the horizontal asymptote stated above.

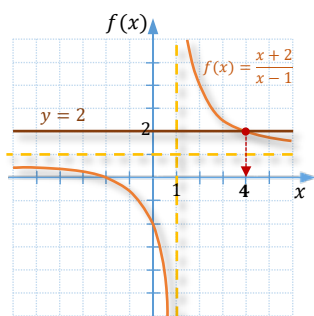
Example 5 ▶ Connecting the Algebraic and Graphical Solutions of Rational Equations

Given that $f(x) = \frac{x+2}{x-1}$, find all the x -values for which $f(x) = 2$. Illustrate the situation with a graph.

Solution ▶ To find all the x -values for which $f(x) = 2$, we replace $f(x)$ in the equation $f(x) = \frac{x+2}{x-1}$ with 2 and solve the resulting equation. So, we have

$$\begin{aligned} 2 &= \frac{x+2}{x-1} && / \cdot (x-1) \\ 2x - 2 &= x + 2 && / -x, +2 \\ x &= 4 \end{aligned}$$

Thus, $f(x) = 2$ for $x = 4$.



The geometrical connection can be observed by graphing the function $f(x) = \frac{x+2}{x-1} = \frac{x-1+3}{x-1} = 1 + \frac{3}{x-1}$ and the line $y = 2$ on the same grid, as illustrated by the accompanying graph. The x -coordinate of the intersection of the two graphs is the solution to the equation $2 = \frac{x+2}{x-1}$. This also means that $f(4) = \frac{4+2}{4-1} = 2$. So, we can say that $f(4) = 2$.

Example 6 ▶ Graphing the Reciprocal of a Linear Function

Suppose $f(x) = 2x - 3$.

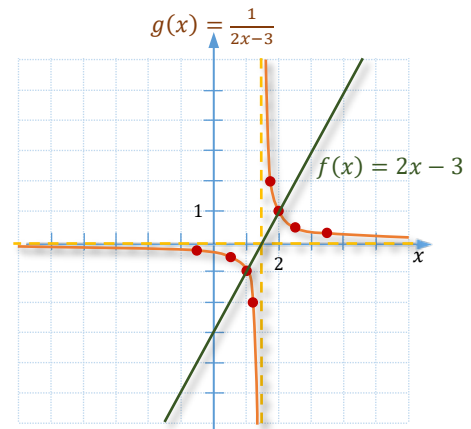
- a. Determine the reciprocal function $g(x) = \frac{1}{f(x)}$ and its domain D_g .

- b. Determine the equation of the vertical asymptote of the reciprocal function g .
- c. Graph the function f and its reciprocal function g on the same grid. Then, describe the relations between the two graphs.

Solution

- a. The reciprocal of $f(x) = 2x - 3$ is the function $g(x) = \frac{1}{2x-3}$. Since $2x - 3 = 0$ for $x = \frac{3}{2}$, then the domain $D_g = \mathbb{R} \setminus \{\frac{3}{2}\}$.
- b. A vertical asymptote of a rational function in simplified form is a vertical line passing through any of the x -values that are excluded from the domain of such a function. So, the equation of the vertical asymptote of function $g(x) = \frac{1}{2x-3}$ is $x = \frac{3}{2}$.
- c. To graph functions f and g , we can use a table of values as below.

x	$f(x)$	$g(x)$
$-\frac{1}{2}$	-4	$-\frac{1}{4}$
$\frac{1}{2}$	-2	$-\frac{1}{2}$
1	-1	-1
$\frac{5}{4}$	$\frac{1}{2}$	2
$\frac{3}{2}$	0	undefined
$\frac{7}{4}$	$-\frac{1}{2}$	-2
2	1	1
$\frac{5}{2}$	2	$\frac{1}{2}$
$\frac{7}{2}$	4	$\frac{1}{4}$



Notice that the vertical asymptote of the reciprocal function comes through the zero of the linear function. Also, the values of both functions are positive to the right of $\frac{3}{2}$ and negative to the left of $\frac{3}{2}$. In addition, $f(2) = g(2) = 1$ and $f(1) = g(1) = -1$. This is because the reciprocal of 1 is 1 and the reciprocal of -1 is -1 . For the rest of the values, observe that the values of the linear function that are very close to zero become very large in the reciprocal function and conversely, the values of the linear function that are very far from zero become very close to zero in the reciprocal function. This suggests the horizontal asymptote at zero.

Example 7

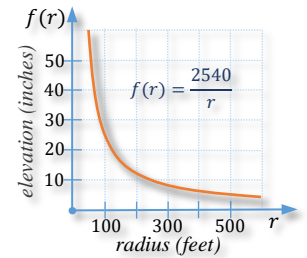
Using Properties of a Rational Function in an Application Problem



elevation

When curves are designed for train tracks, the outer rail is usually elevated so that a locomotive and its cars can safely take the curve at a higher speed than if the tracks were at the same level. Suppose that a circular curve with a radius of r feet is being designed for a train traveling 60 miles per hour. The function $f(r) = \frac{2540}{r}$ calculates the proper elevation $y = f(r)$, in inches, for the outer rail.

- a. Evaluate $f(300)$ and interpret the result.
- b. Suppose that the outer rail for a curve is elevated 6 inches. What should the radius of the curve be?
- c. Observe the accompanying graph of the function f and discuss how the elevation of the outer rail changes as the radius r increases.



Solution

- ▶ a. $f(300) = \frac{2540}{300} \approx 8.5$. Thus, the outer rail on a curve with radius 300 ft should be elevated about **8.5 inches** for a train to safely travel through it at 60 miles per hour.
- b. Since the elevation $y = f(r) = 6$ inches, to find the corresponding value of r , we need to solve the equation

$$6 = \frac{2540}{r}.$$

After multiplying this equation by r and dividing it by 6, we obtain

$$r = \frac{2540}{6} \approx 423$$

So, the radius of the curve should be about **423 feet**.

- c. As the radius increases, the outer rail needs less elevation.

RT.5 Exercises

Vocabulary Check Complete each blank with one of the suggested words, or with the most appropriate term from the given list: *asymptote*, *cross-product*, *domain*, *equations*, *LCD*, *numerator*, *rational*.

1. A _____ equation involves fractional expressions.
2. To solve a rational equation, first find the _____ of all the rational expressions.
3. Multiplication by any quantity other than zero can be used to create equivalent _____.
4. When simplifying rational expressions, we _____ multiply by LCD.
can / cannot
5. The _____ of a rational function is the set of all real numbers except for the variable values that would make the denominator of this function equal to zero.
6. An _____ is a line that a given graph approaches in a long run.
7. A fractional expression is equal to zero when its _____ is equal to zero.
8. To solve a rational proportion we can use either the LCD method or the _____ method.

Concept Check State the **domain** for each equation. There is no need to solve it.

9. $\frac{x+5}{4} - \frac{x+3}{3} = \frac{x}{6}$

10. $\frac{5}{6a} - \frac{a}{4} = \frac{8}{2a}$

11. $\frac{3}{x+4} = \frac{2}{x-9}$

12. $\frac{4}{3x-5} + \frac{2}{x} = \frac{9}{4x+7}$

13. $\frac{4}{y^2-25} - \frac{1}{y+5} = \frac{2}{y-7}$

14. $\frac{x}{2x-6} - \frac{3}{x^2-6x+9} = \frac{x-2}{3x-9}$

Solve each equation.

15. $\frac{3}{8} + \frac{1}{3} = \frac{x}{12}$

16. $\frac{1}{4} - \frac{5}{6} = \frac{1}{y}$

17. $x + \frac{8}{x} = -9$

18. $\frac{4}{3a} - \frac{3}{a} = \frac{10}{3}$

19. $\frac{r}{8} + \frac{r-4}{12} = \frac{r}{24}$

20. $\frac{n-2}{2} - \frac{n}{6} = \frac{4n}{9}$

21. $\frac{5}{r+20} = \frac{3}{r}$

22. $\frac{5}{a+4} = \frac{3}{a-2}$

23. $\frac{y+2}{y} = \frac{5}{3}$

24. $\frac{x-4}{x+6} = \frac{2x+3}{2x-1}$

25. $\frac{x}{x-1} - \frac{x^2}{x-1} = 5$

26. $3 - \frac{12}{x^2} = \frac{5}{x}$

27. $\frac{1}{3} - \frac{x-1}{x} = \frac{x}{3}$

28. $\frac{1}{x} + \frac{2}{x+10} = \frac{x}{x+10}$

29. $\frac{1}{y-1} + \frac{5}{12} = \frac{-2}{3y-3}$

30. $\frac{7}{6x+3} - \frac{1}{3} = \frac{2}{2x+1}$

31. $\frac{8}{3k+9} - \frac{8}{15} = \frac{2}{5k+15}$

32. $\frac{6}{m-4} + \frac{5}{m} = \frac{2}{m^2-4m}$

33. $\frac{3}{y-2} + \frac{2y}{4-y^2} = \frac{5}{y+2}$

34. $\frac{x}{x-2} + \frac{x}{x^2-4} = \frac{x+3}{x+2}$

35. $\frac{1}{2x+10} = \frac{8}{x^2-25} - \frac{2}{x-5}$

36. $\frac{5}{y+3} = \frac{1}{4y^2-36} + \frac{2}{y-3}$

37. $\frac{6}{x^2-4x+3} - \frac{1}{x-3} = \frac{1}{4x-4}$

38. $\frac{7}{x-2} - \frac{8}{x+5} = \frac{1}{2x^2+6x-20}$

39. $\frac{5}{x-4} - \frac{3}{x-1} = \frac{x^2-1}{x^2-5x+4}$

40. $\frac{y}{y+1} + \frac{3y+5}{y^2+4y+3} = \frac{2}{y+3}$

41. $\frac{3x}{x+2} + \frac{72}{x^3+8} = \frac{24}{x^2-2x+4}$

42. $\frac{4}{x+3} + \frac{7}{x^2-3x+9} = \frac{108}{x^3+27}$

43. $\frac{x}{2x-9} - 3x = \frac{10}{9-2x}$

44. $\frac{-2}{x^2+2x-3} - \frac{5}{3-3x} = \frac{4}{3x+9}$

For the given rational function f , find all values of x for which $f(x)$ has the indicated value.

45. $f(x) = 2x - \frac{15}{x}$; $f(x) = 1$

46. $f(x) = \frac{x-5}{x+1}$; $f(x) = \frac{3}{5}$

47. $g(x) = \frac{-3x}{x+3} + x$; $g(x) = 4$

48. $g(x) = \frac{4}{x} + \frac{1}{x-2}$; $g(x) = 3$

Graph each rational function. State its **domain**, **range** and the equations of the **vertical** and **horizontal asymptotes**.

49. $f(x) = \frac{2}{x}$

50. $g(x) = -\frac{1}{x}$

51. $h(x) = \frac{2}{x-3}$

52. $f(x) = \frac{-1}{x+1}$

53. $g(x) = \frac{x-1}{x+2}$

54. $h(x) = \frac{x+2}{x-3}$

Analytic Skills For each function f , find its reciprocal function $g(x) = \frac{1}{f(x)}$ and graph both functions on the same grid. Then, state the equations of the vertical and horizontal asymptotes of function g .

55. $f(x) = \frac{1}{2}x + 1$

56. $f(x) = -x + 2$

57. $f(x) = -2x - 3$

Analytic Skills Solve each equation.

58. $\frac{x}{1 + \frac{1}{x+1}} = x - 3$

59. $\frac{2 - \frac{1}{x}}{4 - \frac{1}{x^2}} = 1$

Solve each problem.

60. The average number of vehicles waiting in line to enter a parking area is modeled by the function defined by

$$w(x) = \frac{x^2}{2(1-x)},$$

where x is a quantity between 0 and 1 known as the **traffic intensity**.

- a. For each traffic intensity, find the average number of vehicles waiting. Round the answer to the nearest one.

i. 0.1 ii. 0.8 iii. 0.9

- b. What happens to the number of vehicles waiting in line as traffic intensity increases?

61. The percent of deaths caused by smoking, called the **incidence rate**, is modeled by the rational function



$$p(x) = \frac{x-1}{x},$$

where x is the number of times a smoker is more likely to die of lung cancer than a non-smoker is. For example, $x = 10$ means that a smoker is 10 times more likely than a non-smoker to die from lung cancer.

- a. Find $p(x)$ if x is 10.
 b. For what values of x is $p(x) = 80\%$? (Hint: Change 80% to a decimal.)
 c. Can the incidence rate equal 0? Explain.
 d. Can the incidence rate equal 1? Explain.



RT.6

Applications of Rational Equations



In previous sections of this chapter, we studied operations on rational expressions, simplifying complex fractions, and solving rational equations. These skills are needed when working with real-world problems that lead to a rational equation. The common types of such problems are motion or work problems. In this section, we first discuss how to solve a rational formula for a given variable, and then present several examples of application problems involving rational equations.

Formulas Containing Rational Expressions

Solving application problems often involves working with formulas. We might need to form a formula, evaluate it, or solve it for a desired variable. The basic strategies used to solve a formula for a variable were shown in *section L2* and *F4*. Recall the guidelines that we used to isolate the desired variable:

- **Reverse operations** to clear unwanted factors or addends;
Example: To solve $\frac{A+B}{2} = C$ for A , we multiply by 2 and then subtract B .
- **Multiply by the LCD to keep the desired variable in the numerator;**
Example: To solve $\frac{A}{1+r} = P$ for r , first, we multiply by $(1+r)$.
- **Take the reciprocal of both sides of the equation to keep the desired variable in the numerator** (this applies to proportions only);
Example: To solve $\frac{1}{C} = \frac{A+B}{AB}$ for C , we can take the reciprocal of both sides to obtain $C = \frac{AB}{A+B}$.
- **Factor to keep the desired variable in one place.**
Example: To solve $P + Prt = A$ for P , we first factor P out.

Below we show how to solve formulas containing rational expressions, using a combination of the above strategies.

Example 1 ▶ Solving Rational Formulas for a Given Variable

Solve each formula for the indicated variable.

a. $\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$, for p

b. $L = \frac{dR}{D-d}$, for D

c. $L = \frac{dR}{D-d}$, for d

Solution ▶

- a. **Solution I:** First, we isolate the term containing p , by ‘moving’ $\frac{1}{q}$ to the other side of the equation. So,

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{q} \quad / -\frac{1}{q}$$

$$\frac{1}{f} - \frac{1}{q} = \frac{1}{p}$$

$$\frac{1}{p} = \frac{q-f}{fq}$$

rewrite from the right to the left,

and perform the subtraction to leave this side as a single fraction

Then, to bring p to the numerator, we can take the reciprocal of both sides of the equation, obtaining

$$p = \frac{fq}{q-f}$$

Caution! This method can be applied only to a proportion (an equation with a **single fraction on each side**).

Solution II: The same result can be achieved by multiplying the original equation by the $LCD = fpq$, as shown below

$$\begin{aligned} \frac{1}{f} &= \frac{1}{p} + \frac{1}{q} && / \cdot fpq \\ pq &= fq + fp && / -fp \\ pq - fp &= fq && \\ \text{factor } p \text{ out} & p(q-f) = fq && / \div (q-f) \\ p &= \frac{fq}{q-f} \end{aligned}$$

- b. To solve $L = \frac{dR}{D-d}$ for D , we may start with multiplying the equation by the denominator to bring the variable D to the numerator. So,

This can be done in one step by interchanging L with $D-d$. The movement of the expressions resembles that of a teeter-totter.

$$\begin{aligned} L &= \frac{dR}{D-d} && / \cdot (D-d) \\ L(D-d) &= dR && / \div L \\ D-d &= \frac{dR}{L} && / +d \\ D &= \frac{dR}{L} + d = \frac{dR + dL}{L} \end{aligned}$$

Both forms are correct answers.

- c. When solving $L = \frac{dR}{D-d}$ for d , we first observe that the variable d appears in both the numerator and denominator. Similarly as in the previous example, we bring the d from the denominator to the numerator by multiplying the formula by the denominator $D-d$. Thus,

$$\begin{aligned} L &= \frac{dR}{D-d} && / \cdot (D-d) \\ L(D-d) &= dR. \end{aligned}$$

Then, to keep the d in one place, we need to expand the bracket, collect terms with d , and finally factor the d out. So, we have

$$LD - Ld = dR \quad /+Ld$$

$$LD = dR + Ld$$

$$LD = d(R + L) \quad /\div (R + L)$$

$$\frac{LD}{R + L} = d$$


Obviously, the final formula can be written starting with d ,

$$d = \frac{LD}{R + L}.$$

Example 2 Forming and Evaluating a Rational Formula

Suppose a trip consists of two parts, each of length x .

- Find a formula for determining the average speed v for the whole trip, given the speed v_1 for the first part of the trip and v_2 for the second part of the trip.
- Find the average speed v for the whole trip, if the speed for the first part of the trip was 60 km/h and the speed for the second part of the trip was 90 km/h.

Solution  a. The total distance, d , for the whole trip is $x + x = 2x$. The total time, t , for the whole trip is the sum of the times for the two parts of the trip, t_1 and t_2 . From the relation $rate \cdot time = distance$, we have

$$t_1 = \frac{x}{v_1} \quad \text{and} \quad t_2 = \frac{x}{v_2}.$$

Therefore,

$$t = \frac{x}{v_1} + \frac{x}{v_2},$$

which after substituting to the formula for the average speed, $v = \frac{d}{t}$, gives us

$$v = \frac{2x}{\frac{x}{v_1} + \frac{x}{v_2}}.$$

Since the formula involves a complex fraction, it should be simplified. We can do this by multiplying the numerator and denominator by the $LCD = v_1 v_2$. So, we have

$$v = \frac{2x}{\frac{x}{v_1} + \frac{x}{v_2}} \cdot \frac{v_1 v_2}{v_1 v_2}$$

$$v = \frac{2x v_1 v_2}{\cancel{x v_1 v_2} / v_1 + \cancel{x v_1 v_2} / v_2}$$

$$v = \frac{2x v_1 v_2}{x v_2 + x v_1}$$

factor the x

$$v = \frac{2xv_1v_2}{x(v_2 + v_1)}$$

$$v = \frac{2v_1v_2}{v_2 + v_1}$$

Note 1: The average speed in this formula does not depend on the distance travelled.

Note 2: The average speed for the total trip is not the average (arithmetic mean) of the speeds for each part of the trip. In fact, this formula represents the **harmonic mean** of the two speeds.

- b. Since $v_1 = 60$ km/h and $v_2 = 90$ km/h, using the formula developed in *Example 2a*, we calculate

$$v = \frac{2 \cdot 60 \cdot 90}{60 + 90} = \frac{10800}{150} = 72 \text{ km/h}$$

Observation: Notice that the average speed for the whole trip is lower than the average of the speeds for each part of the trip, which is $\frac{60+90}{2} = 75$ km/h.

Applied Problems

Many types of application problems were already introduced in *sections L3* and *E2*. Some of these types, for example motion problems, may involve solving rational equations. Below we show examples of proportion and motion problems as well as introduce another type of problems, work problems.

Proportion Problems

When forming a proportion,

$$\frac{\text{category I before}}{\text{category II before}} = \frac{\text{category I after}}{\text{category II after}}$$

it is essential that the same type of data are placed in the same row or the same column.

Recall: To solve a proportion

$$\frac{a}{b} = \frac{c}{d},$$

for example, for a , it is enough to multiply the equation by b . This gives us

$$a = \frac{bc}{d}.$$

Similarly, to solve

$$\frac{a}{b} = \frac{c}{d}$$

for b , we can use the cross-multiplication method, which eventually (we encourage the reader to check this) leads us to

$$a = \frac{ad}{c}.$$

Notice that in both cases the desired variable equals the **product** of the blue variables lying **across** each other, **divided by the remaining** purple variable. This is often referred to as the ‘cross multiply and divide’ approach to solving a proportion.

In statistics, proportions are often used to estimate the population by analysing its sample in situations where the exact count of the population is too costly or not possible to obtain.

Example 3 ▶ Estimating Numbers of Wild Animals



To estimate the number of wild horses in Utah, a forest ranger catches 620 wild horses, tags them, and releases them. Later, 122 horses are caught and it is found that 31 of them are tagged. Assuming that the horses mix freely when they are released, estimate how many wild horses there are in Utah.

Solution ▶ Suppose there are x wild horses in Utah. 620 of them were tagged, so the ratio of the tagged horses in the whole population of the wild horses in Utah is

$$\frac{620}{x}$$

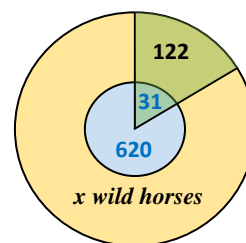
The ratio of the tagged horses found in the sample of 122 horses caught in the later time is

$$\frac{31}{122}$$

So, we form the proportion:

$$\frac{620}{x} = \frac{31}{122}$$

population
sample
tagged horses
all horses



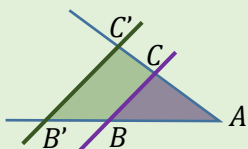
After solving for x , we have

$$x = \frac{620 \cdot 122}{31} = 2440$$

So, we can estimate that over 2400 wild horses live in Utah.

In geometry, proportions are the defining properties of similar figures. One frequently used theorem that involves proportions is the theorem about similar triangles, attributed to the Greek mathematician Thales.

Thales' Theorem ▶ Two triangles are **similar** iff the ratios of the corresponding sides are the same.



$$\triangle ABC \sim \triangle AB'C' \Leftrightarrow \frac{AB}{AB'} = \frac{AC}{AC'} = \frac{BC}{B'C'}$$

Example 4 ▶ **Using Similar Triangles in an Application Problem**

A cross-section of a small storage room is in the shape of a right triangle with a height of 2 meters and a base of 1.2 meters, as shown in *Figure 6.1*. What is the largest cubic box that can fit in this room? Assume that the base of the box is positioned on the floor of the storage room.

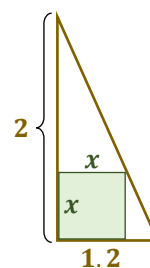


Figure 6.1a

Solution ▶ Suppose that the height of the box is x meters. Since the height of the storage room is 2 meters, the expression $2 - x$ represents the height of the wall above the box, as shown in *Figure 6.1b*.

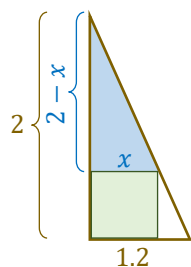


Figure 6.1b

Since the blue and brown triangles are similar, we can use the Thales' Theorem to form the proportion

$$\frac{2 - x}{2} = \frac{x}{1.2}$$

Employing cross-multiplication, we obtain

$$2.4 - 1.2x = 2x$$

$$2.4 = 3.2x$$

$$x = \frac{2.4}{3.2} = \mathbf{0.75}$$

So, the dimensions of the largest cubic box fitting in this storage room are 75 cm by 75 cm by 75 cm.

Motion Problems

Motion problems in which we compare times usually involve solving rational equations. This is because when solving the motion formula $\text{rate } R \cdot \text{time } T = \text{distance } D$ for time, we create a fraction

$$\text{time } T = \frac{\text{distance } D}{\text{rate } R}$$

Example 5 ▶ **Solving a Motion Problem Where Times are the Same**

The speed of one mountain biker is 3 km/h faster than the speed of another biker. The first biker travels 40 km in the same amount of time that it takes the second to travel 30 km. Find the speed of each biker.

Solution ▶ Let r represent the speed of the slower biker. Then $r + 3$ represents the speed of the faster biker. The slower biker travels 30 km, while the faster biker travels 40 km. Now, we can complete the table

	R	T	$= D$
slower biker	r	$\frac{30}{r}$	30
faster biker	$r + 3$	$\frac{40}{r + 3}$	40

To complete the *Time* column, we divide the *Distance* by the *Rate*.

Since the time of travel is the same for both bikers, we form and then solve the equation:

$$\begin{aligned} \frac{30}{r} &= \frac{40}{r + 3} && / \div 10 \\ &&& \text{and cross-multiply} \\ 3(r + 3) &= 4r \\ 3r + 9 &= 4r && / -3r \\ r &= 9 \end{aligned}$$

Thus, the speed of the slower biker is $r = 9$ km/h and the speed of the faster biker is $r + 3 = 12$ km/h.

Example 6 ▶ **Solving a Motion Problem Where the Total Time is Given**

Kris and Pat are driving from Vancouver to Princeton, a distance of 297 km. Kris, whose average rate is 6 km/h faster than Pat's, will drive the first 153 km of the trip, and then Pat will drive the rest of the way to their destination. If the total driving time is 3 hours, determine the average rate of each driver.

Solution ▶ Let r represent Pat's average rate. Then $r + 6$ represents Kris' average rate. Kris travelled 153 km, while Pat travelled $297 - 153 = 144$ km. Now, we can complete the table:

	R	T	$= D$
Kris	$r + 6$	$\frac{153}{r + 6}$	153
Pat	r	$\frac{144}{r}$	144
total		3	297

Note: In motion problems we may add *times* or *distances* but we usually **do not add rates!**

The equation to solve comes from the **Time** column.

$$\begin{aligned} \frac{153}{r+6} + \frac{144}{r} &= 3 && / \cdot r(r+6) \\ 153r + 144(r+6) &= 3r(r+6) && / \div 3 \\ 51r + 48r + 288 &= r^2 + 6r && \text{and distribute; then} \\ 0 &= r^2 - 93r - 288 && \text{collect like terms on} \\ &&& \text{one side} \\ (r-96)(r+3) &= 0 && \text{factor} \\ r = 96 \text{ or } r = -3 &&& \end{aligned}$$

Since a rate cannot be negative, we discard the solution $r = -3$. Therefore, Pat's average rate was $r = 96$ km/h and Kris' average rate was $r + 6 = 102$ km/h.

Work Problems

Notice the similarity to the formula $R \cdot T = D$ used in motion problems.

When solving work problems, refer to the formula

$$\text{Rate of work} \cdot \text{Time} = \text{amount of Job completed}$$

and organize data in a table like this:

	R	\cdot	T	$=$	J
worker I					
worker II					
together					

Note: In work problems we usually add rates but **do not add times!**

Example 7 ▶ Solving a Work Problem Involving Addition of Rates

Alex can trim the shrubs at Beecher Community College in 6 hr. Bruce can do the same job in 4 hr. How long would it take them to complete the same trimming job if they work together?



Solution ▶ Let t be the time needed to trim the shrubs when Alex and Bruce work together. Since trimming the shrubs at Beecher Community College is considered to be the whole one job to complete, then the rate R in which this work is done equals

$$R = \frac{\text{Job}}{\text{Time}} = \frac{1}{\text{Time}}$$

To organize the information, we can complete the table below.

	R	T	$= J$
Alex	$\frac{1}{6}$	6	1
Bruce	$\frac{1}{4}$	4	1
together	$\frac{1}{t}$	t	1

The job column is often equal to **1**, although sometimes other values might need to be used.

To complete the *Rate* column, we divide the *Job* by the *Time*.

Since the rate of work when both Alex and Bruce trim the shrubs is the sum of rates of individual workers, we form and solve the equation

$$\begin{aligned} \frac{1}{6} + \frac{1}{4} &= \frac{1}{t} && / \cdot 12t \\ 2t + 3t &= 12 \\ 5t &= 12 && / \div 5 \\ t &= \frac{12}{5} = 2.4 \end{aligned}$$

So, if both Alex and Bruce work together, the amount of time needed to complete the job is 2.4 hours = **2 hours 24 minutes**.

Note: The time needed for both workers is **shorter** than either of the individual times.

Example 8 ▶ Solving a Work Problem Involving Subtraction of Rates



One pipe can fill a swimming pool in 10 hours, while another pipe can empty the pool in 15 hours. How long would it take to fill the pool if both pipes were left open?

Solution ▶ Suppose t is the time needed to fill the pool when both pipes are left open. If filling the pool is the whole one job to complete, then emptying the pool corresponds to -1 job. This is because when emptying the pool, we reverse the filling job.

To organize the information given in the problem, we complete the following table.

	R	T	$= J$
pipe I	$\frac{1}{10}$	10	1
pipe II	$-\frac{1}{15}$	15	-1
both pipes	$\frac{1}{t}$	t	1

The equation to solve comes from the **Rate** column.

$$\begin{aligned}\frac{1}{10} - \frac{1}{15} &= \frac{1}{t} && / \cdot 30t \\ 3t - 2t &= 30 \\ t &= 30\end{aligned}$$

So, it will take **30 hours** to fill the pool when both pipes are left open.

RT.6 Exercises

Vocabulary Check Complete each blank with the most appropriate term or phrase from the given list: **column, fraction, motion, numerator, proportions, row, similar, work.**

- When solving a formula for a specified variable, we want to keep the variable in the _____.
- The strategy of taking reciprocal of each side of an equation is applicable only in solving _____. This means that each side of such equation must be in the form of a single _____.
- When forming a proportion, it is essential that the same type of data are placed in the same _____ or in the same _____.
- The Thales' Theorem states that for any two _____ triangles the ratios of the corresponding sides are the same.
- In _____ problems, we usually add times or distances but not the rates.
- In _____ problems, we usually add rates but not times.

Concept Check

- Using the formula $\frac{1}{a} = \frac{1}{b} + \frac{1}{c}$, find b if $a = 8$ and $c = 12$.
- The gravitational force between two masses is given by the formula $F = \frac{GMm}{d^2}$. Find M if $F = 10$, $G = 6.67 \cdot 10^{-11}$, $m = 1$, and $d = 3 \cdot 10^{-6}$. Round your answer to two decimal places.

Concept Check

- What is the first step in solving the formula $rp - rq = p + q$ for r ?
- What is the first step in solving the formula $m = \frac{ab}{a-b}$ for a ?

Solve each formula for the specified variable.

11. $m = \frac{F}{a}$ for a

12. $I = \frac{E}{R}$ for R

13. $\frac{W_1}{W_2} = \frac{d_1}{d_2}$ for d_1

14. $F = \frac{GMm}{d^2}$ for m

15. $s = \frac{(v_1+v_2)t}{2}$ for t

16. $s = \frac{(v_1+v_2)t}{2}$ for v_1

17. $\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2}$ for R

18. $\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2}$ for r_1

19. $\frac{1}{p} + \frac{1}{q} = \frac{1}{f}$ for q

20. $\frac{t}{a} + \frac{t}{b} = 1$ for a

21. $\frac{PV}{T} = \frac{pv}{t}$ for v

22. $\frac{PV}{T} = \frac{pv}{t}$ for T

23. $A = \frac{h(a+b)}{2}$ for b

24. $a = \frac{V-v}{t}$ for V

25. $R = \frac{gs}{g+s}$ for s

26. $I = \frac{2V}{V+2r}$ for V

27. $I = \frac{nE}{E+nr}$ for n

28. $\frac{E}{e} = \frac{R+r}{r}$ for e

29. $\frac{E}{e} = \frac{R+r}{r}$ for r

30. $S = \frac{H}{m(t_1-t_2)}$ for t_1

31. $V = \frac{\pi h^2(3R-h)}{3}$ for R

32. $P = \frac{A}{1+r}$ for r

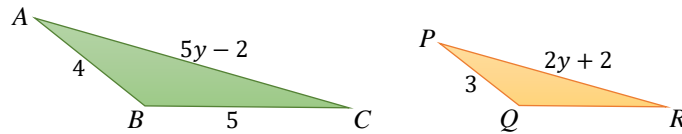
33. $\frac{V^2}{R^2} = \frac{2g}{R+h}$ for h

34. $v = \frac{d_2-d_1}{t_2-t_1}$ for t_2

Analytic Skills Solve each problem

35. The ratio of the weight of an object on the moon to the weight of an object on Earth is 0.16 to 1. How much will an 80-kg astronaut weigh on the moon?
36. A rope is 24 meters long. How can the rope be cut in such a way that the ratio of the resulting two segments is 3 to 5?
37. Walking 4 miles in 2 hours will use up 650 calories. Walking at the same rate, how many miles would a person need to walk to lose 1 lb? (Burning 3500 calories is equivalent to losing 1 pound.) *Round to the nearest hundredth.*
38. On a map of Canada, the linear distance between Vancouver and Calgary is 1.8 cm. The airline distance between the two cities is about 675 kilometers. On this same map, what would be the linear distance between Calgary and Montreal if the airline distance between the two cities is approximately 3000 kilometers?
39. To estimate the deer population of a forest preserve, wildlife biologists caught, tagged, and then released 42 deer. A month later, they returned and caught a sample of 75 deer and found that 15 of them were tagged. Based on this experiment, approximately how many deer lived in the forest preserve?
40. Biologists tagged 500 fish in a lake on January 1. On February 1, they returned and collected a random sample of 400 fish, 8 of which had been previously tagged. On the basis of this experiment, approximately how many fish does the lake have?
41. Twenty-two bald eagles are tagged and released into the wilderness. Later, an observed sample of 56 bald eagles contains 7 eagles that are tagged. Estimate the bald eagle population in this wilderness area.
42. A 6-foot tall person casts a 4-foot long shadow. If a nearby tree casts a 44-foot long shadow, estimate the height of the tree.

43. Suppose the following triangles are similar. Find y and the lengths of the unknown sides of each triangle.



44. A rectangle has sides of 9 cm and 14 cm. In a similar rectangle the longer side is 8 cm. What is the length of the shorter side?
45. What is the average speed if Shane runs 18 kilometers per hour for the first half of a race and 22 kilometers per hour for the second half of the race?
46. If you average x kilometers per hour during the first half of a trip, find the speed y in kilometers per hour needed during the second half of the trip to reach 80 kilometers per hour as an overall average.



47. Allen's boat goes 12 mph. Find the rate of the current of the river if he can go 6 mi upstream in the same amount of time he can go 10 mi downstream.

48. A plane averaged 500 mph on a trip going east, but only 350 mph on the return trip. The total flying time in both directions was 8.5 hr. What was the one-way distance?

49. A Boeing 747 flies 2420 mi with the wind. In the same amount of time, it can fly 2140 mi against the wind. The cruising speed (in still air) is 570 mph. Find the speed of the wind.

50. A moving sidewalk moves at a rate of 1.7 ft/sec. Walking on the moving sidewalk, Brenda can travel 120 ft forward in the same time it takes to travel 52 ft in the opposite direction. How fast would Brenda be walking on a non-moving sidewalk?



51. On his drive from Montpelier, Vermont, to Columbia, South Carolina, Arthur averaged 51 mph. If he had been able to average 60 mph, he would have reached his destination 3 hr earlier. What is the driving distance between Montpelier and Columbia?

52. On the first part of a trip to Carmel traveling on the freeway, Margaret averaged 60 mph. On the rest of the trip, which was 10 mi longer than the first part, she averaged 50 mph. Find the total distance to Carmel if the second part of the trip took 30 min more than the first part.

53. Barb is a college professor who lives in an off-campus apartment. On days when she rides her bike to campus, she gets to her first class 36 min faster than when she walks. If her average walking rate is 3 mph and her average biking rate is 12 mph, how far is it from her apartment to her first class?

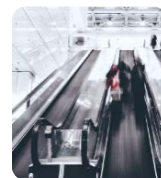
54. Kate can file all of the daily invoices in 4 hr and Ben can do the same job in 6 hr. If they work together, then what portion of the invoices can they file in 1 hr?

55. Stephanie can paint the entire house in x hours and Tim can do the same job in y hours. Write a rational expression that represents the portion of the house that they can paint in 2 hr working together.



56. John and Nessa are asked to paint a house. John can paint the house by himself in 12 hours and Nessa can paint the house by herself in 16 hours. How long would it take to paint the house if they worked together?

57. Brian, Mark, and Jeff are painting a house. Working together they can paint the house in 6 hours. Working alone Brian can paint the house in 15 hours and Jeff can paint the house in 20 hours. How long would it take Mark to paint the house working alone?
58. An experienced carpenter can frame a house twice as fast as an apprentice. Working together, it takes the carpenters 2 days. How long would it take the apprentice working alone?
59. A tank can be filled in 9 hr and drained in 11 hr. How long will it take to fill the tank if the drain is left open?
60. A cold water faucet can fill the bath tub in 12 minutes, and a hot water faucet can fill the bath tub in 18 minutes. The drain can empty the bath tub in 24 minutes. If both faucets are on and the drain is open, how long would it take to fill the bath tub?
61. Together, a 100-cm wide escalator and a 60-cm wide escalator can empty a 1575-person auditorium in 14 min. The wider escalator moves twice as many people as the narrower one does. How many people per hour does the 60-cm wide escalator move?



Discussion Point



62. At what time after 4:00 will the minute hand overlap the hour hand of a clock for the first time?
63. Michelle drives to work at 50 mph and arrives 1 min late. When she drives to work at 60 mph, she arrives 5 min early. How far does Michelle live from work?