Review of Operations on the Set of Real Numbers

Before we start our journey through algebra, let us review the structure of the real number system, properties of four operations, order of operations, the concept of absolute value, and set-builder and interval notation.

R1 Structure of the Set of Real Numbers

It is in human nature to group and classify objects with the same properties. For instance, items found in one's home can be classified as furniture, clothing, appliances, dinnerware, books, lighting, art pieces, plants, etc., depending on what each item is used for, what it is made of, how it works, etc. Furthermore, each of these groups could be subdivided into more specific categories (groups). For example, furniture includes tables, chairs, bookshelves, desks, etc. Sometimes an item can belong to more than one group. For example, a piece of furniture can also be a piece of art. Sometimes the groups do not have any common items (e.g. plants and appliances). Similarly to everyday life, we like to classify numbers with respect to their properties. For example, even or odd numbers, prime or composite numbers, common fractions, finite or infinite decimals, infinite repeating decimals, negative numbers, etc. In this section, we will take a closer look at commonly used groups (sets) of real numbers and the relations between those groups.

Set Notation and Frequently Used Sets of Numbers

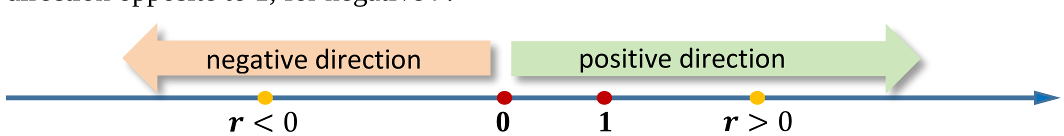
We start with terminology and notation related to sets.

Definition 1.1 ► A **set** is a collection of objects, called **elements** (or **members**) of this set.

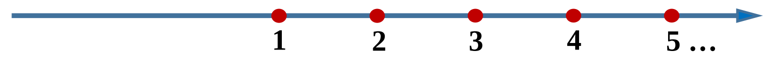
Roster Notation: A set can be given by listing its elements within the **set brackets { }** (braces).
The elements of the set are separated by commas.
To indicate that a pattern continues, we use three dots … .
Examples:
If set A consists of the numbers 1, 2, and 3, we write $A = \{1,2,3\}$.
If set B consists of all consecutive numbers, starting from 5, we write $B = \{5,6,7,8, \dots\}$.

More on Notation: To indicate that the number 2 **is an element** of set A, we write $\mathbf{2 \in A}$.
To indicate that the number 2 **is <u>not</u> an element** of set B, we write $\mathbf{2 \notin A}$.
A set with no elements, called **empty** set, is denoted by the symbol $\emptyset$ or **{ }**.

In this course we will be working with the set of **real numbers**, denoted by $\mathbb{R}$. To visualise this set, we construct a line and choose two distinct points on it, 0 and 1, to establish direction and scale. This makes it a **number line**. Each real number r can be identified with exactly one point on such a number line by choosing the endpoint of the segment of length $|r|$ that starts from 0 and follows the line in the direction of 1, for positive r, or in the direction opposite to 1, for negative r.

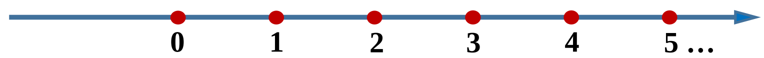


The set of real numbers contains several important subgroups (**subsets**) of numbers. The very first set of numbers that we began our mathematics education with is the set of counting numbers $\{\mathbf{1}, \mathbf{2}, \mathbf{3}, \dots\}$, called **natural numbers** and denoted by $\mathbb{N}$.



Natural numbers $\mathbb{N}$

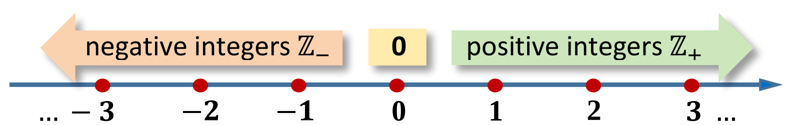
The set of natural numbers together with the number **0** creates the set of **whole numbers** $\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \dots\}$, denoted by $\mathbb{W}$.



Whole numbers $\mathbb{W}$

Notice that if we perform addition or multiplication of numbers from either of the above sets, $\mathbb{N}$ and $\mathbb{W}$, the result will still be an element of the same set. We say that the set of **natural numbers** $\mathbb{N}$ and the set of **whole numbers** $\mathbb{W}$ are both **closed** under **addition** and **multiplication**.

However, if we wish to perform subtraction of natural or whole numbers, the result may become a negative number. For example, $2 - 5 = -3 \notin \mathbb{W}$, so neither the set of whole numbers nor natural numbers is not closed under subtraction. To be able to perform subtraction within the same set, it is convenient to extend the set of whole numbers to include negative counting numbers. This creates the set of **integers** $\{\dots, -\mathbf{3}, -\mathbf{2}, -\mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \dots\}$, denoted by $\mathbb{Z}$.

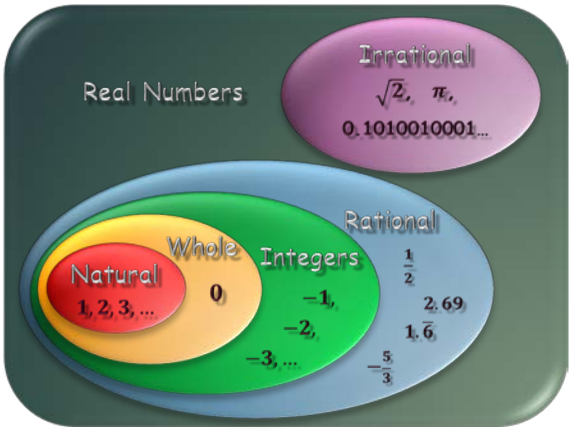


Integers $\mathbb{Z}$

Alternatively, the set of integers can be recorded using the $\pm$ sign: $\{\mathbf{0}, \pm\mathbf{1}, \pm\mathbf{2}, \pm\mathbf{3}, \dots\}$. The $\pm$ sign represents two numbers at once, the positive and the negative.

So the set of **integers** $\mathbb{Z}$ is **closed** under **addition**, **subtraction** and **multiplication**. What about division? To create a set that would be closed under division, we extend the set of integers by including all quotients of integers (all common fractions). This new set is called the set of **rational numbers** and denoted by $\mathbb{Q}$. Here are some examples of rational numbers: $\frac{3}{1} = \mathbf{3}$, $\frac{1}{2} = \mathbf{0.5}$, $-\frac{7}{4}$, or $\frac{4}{3} = \mathbf{1}.\overline{\mathbf{3}}$.

Thus, the set of **rational numbers** $\mathbb{Q}$ is **closed** under **all four operations**. It is quite difficult to visualize this set on the number line as its elements are nearly everywhere. Between any two rational numbers, one can always find another rational number, simply by taking an average of the two. However, all the points corresponding to rational numbers still do not fulfill the whole number line. Actually, the number line contains a lot more unassigned points than points that are assigned to rational numbers. The remaining points correspond to numbers called **irrational** and denoted by $\mathbb{IQ}$. Here are some examples of irrational numbers: $\sqrt{\mathbf{2}}, \boldsymbol{\pi}, \boldsymbol{e}$, or $\mathbf{0.1010010001}\dots$.



By the definition, the two sets, $\mathbb{Q}$ and $\mathbb{IQ}$ fulfill the entire number line, which represents the set of **real numbers**, $\mathbb{R}$.

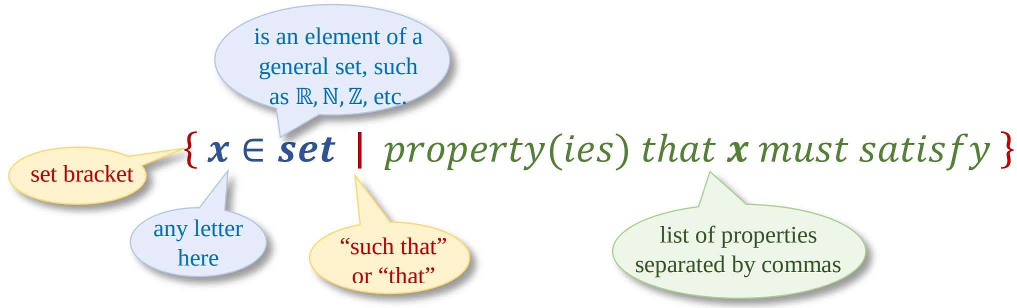
The sets $\mathbb{N}, \mathbb{W}, \mathbb{Z}, \mathbb{Q}, \mathbb{IQ}$, and $\mathbb{R}$ are related to each other as in the accompanying diagram. One can make the following observations:

$\mathbb{N} \subset \mathbb{W} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$, where $\subset$ (*read* is a **subset**) represents the operator of **inclusion of sets**;

$\mathbb{Q}$ and $\mathbb{IQ}$ are **disjoint** (they have **no common element**);

$\mathbb{Q}$ together with $\mathbb{IQ}$ create $\mathbb{R}$.

So far, we introduced six double-stroke letter signs to denote the main sets of numbers. However, there are many more sets that one might be interested in describing. Sometimes it is enough to use a subindex with the existing letter-name. For instance, the set of all positive real numbers can be denoted as $\mathbb{R}_+$ while the set of negative integers can be denoted by $\mathbb{Z}_-$. But how would one represent, for example, the set of even or odd numbers or the set of numbers divisible by 3, 4, 5, and so on? To describe numbers with a particular property, we use the **set-builder notation**. Here is the structure of set-builder notation:



For example, to describe the set of even numbers, first, we think of a property that distinguishes even numbers from other integers. This is divisibility by 2. So each even number n can be expressed as $2k$, for some integer k. Therefore, the set of even numbers could be stated as $\{n \in \mathbb{Z} \mid n = 2k, k \in \mathbb{Z}\}$ (*read*: The set of all integers n such that each n is of the form $2k$, for some integral k.)

To describe the set of rational numbers, we use the fact that any rational number can be written as a common fraction. Therefore, the set of rational numbers $\mathbb{Q}$ can be described as $\left\{x \,\middle|\, x = \frac{p}{q},\ p, q \in \mathbb{Z},\ q \neq 0\right\}$ (*read*: The set of all real numbers x that can be expressed as a fraction $\frac{p}{q}$, for integral p and q, with $q \neq 0$.)

Convention: *If the description of a set refers to the set of real numbers, there is no need to state $x \in \mathbb{R}$ in the first part of set-builder notation. For example, we can write $\{\boldsymbol{x} \in \mathbb{R} \mid \boldsymbol{x} > \boldsymbol{0}\}$ or $\{\boldsymbol{x} \mid \boldsymbol{x} > \boldsymbol{0}\}$. Both sets represent the set of all positive real numbers, which could also be recorded as simply $\mathbb{R}_+$. However, if we work with any other major set, this set must be stated. For example, to describe all positive integers $\mathbb{Z}_+$ using set-builder notation, we write $\{\boldsymbol{x} \in \mathbb{Z} \mid \boldsymbol{x} > \boldsymbol{0}\}$ and $\mathbb{Z}$ is essential there.*

Example 1 ► **Listing Elements of Sets Given in Set-builder Notation**

List the elements of each set.

a. $\{n \in \mathbb{Z} \mid -2 \leq n < 5\}$ **b.** $\{n \in \mathbb{N} \mid n = 5k, k \in \mathbb{N}\}$

Solution ►

a. This is the set of integers that are at least -2 but smaller than 5. So this is $\{-2, -1, 0, 1, 2, 3, 4\}$.

b. This is the set of natural numbers that are multiples of 5. Therefore, this is the infinite set $\{5, 10, 15, 20, \dots\}$.

Example 2 ► **Writing Sets with the Aid of Set-builder Notation**

Use set-builder notation to describe each set.

a. $\{1, 4, 9, 16, 25, \dots\}$ **b.** $\{-2, 0, 2, 4, 6\}$

Solution ►

a. First, we observe that the given set is composed of consecutive perfect square numbers, starting from 1. Since all the elements are natural numbers, we can describe this set using the set-builder notation as follows: $\{n \in \mathbb{N} \mid n = k^2, \text{for } k \in \mathbb{N}\}$.

b. This time, the given set is finite and lists all even numbers starting from -2 up to 6. Since the general set we work with is the set of integers, the corresponding set in set-builder notation can be written as $\{n \in \mathbb{Z} \mid n \text{ is even}, -2 \leq n \leq 6\}$, or $\{n \in \mathbb{Z} \mid -2 \leq n \leq 6, n = 2k, \text{for } k \in \mathbb{Z}\}$.

Observations:

* *There are many equivalent ways to describe a set using the set-builder notation.*

* *The commas used between the conditions (properties) stated after the "such that" bar play the same role as the connecting word "and".*

Rational Decimals

How can we recognize if a number in decimal notation is rational or irrational?

A **terminating** decimal (with a **finite** number of nonzero digits after the decimal dot, like 1.25 or 0.1206) can be converted to a common fraction by replacing the decimal dot with the division by the corresponding power of 10 and then simplifying the resulting fraction. For example,

$$1.25 = \frac{125}{100} = \frac{5}{4}, \quad \text{or} \quad 0.1206 = \frac{1206}{10000} = \frac{603}{5000}.$$

2 places → 2 zeros; simplify by 25

4 places → 4 zeros; simplify by 2

Therefore, any **terminating decimal is a rational number**.

One can also convert a **nonterminating** (**infinite**) decimal to a common fraction, as long as there is a **recurring** (**repeating**) sequence of digits in the decimal expansion. This can be done using the method shown in *Example 3a*. Hence, any **infinite repeating decimal is a rational number**.

Also, notice that any fraction $\frac{m}{n}$ can be converted to either a finite or infinite repeating decimal. This is because since there are only finitely many numbers occurring as remainders in the long division process when dividing by n, eventually, either a remainder becomes zero, or the sequence of remainders starts repeating.

So **a number is rational if and only if it can be represented by a finite or infinite repeating decimal**. Since the irrational numbers are defined as those that are not rational, we can conclude that **a number is irrational if and only if it can be represented as an infinite non-repeating decimal**.

Example 3 ► **Proving that an Infinite Repeating Decimal is a Rational Number**

Show that the given decimal is a rational number.

a. $0.333\ldots$ **b.** $2.3\overline{45}$

Solution ►

a. Let $a = 0.333\ldots$. After multiplying this equation by 10, we obtain $10a = 3.333\ldots$. Since in both equations, the number after the decimal dot is exactly the same, after subtracting the equations side by side, we obtain

$$-\begin{cases} 10a = 3.\cancel{333}\ldots \\ \ \ a = 0.\cancel{333}\ldots \end{cases}$$
$$9a = 3$$

which solves to $a = \frac{3}{9} = \frac{1}{3}$. So $0.333\ldots = \frac{1}{3}$ is a rational number.

b. Let $a = 2.3\overline{45}$. The bar above 45 tells us that the sequence 45 repeats forever. To use the subtraction method as in solution to *Example 3a*, we need to create two equations involving the given number with the decimal dot moved after the repeating sequence and before the repeating sequence. This can be obtained by multiplying the equation $a = 2.3\overline{45}$ first by 1000 and then by 10, as below.

$$-\begin{cases} 1000a = 2345.\cancel{\overline{45}} \\ \ \ \ 10a = 23.\cancel{\overline{45}} \end{cases}$$
$$990a = 2322$$

Therefore, $a = \frac{2322}{990} = \frac{129}{55} = 2\,\frac{19}{55}$, which proves that $2.3\overline{45}$ is rational.

Example 4 ► **Identifying the Main Types of Numbers**

List all numbers of the set

$\left\{-10,\ -5.34,\ 0,\ 1,\ \frac{12}{3},\ 3.\overline{16},\ \frac{4}{7},\ \sqrt{2},\ -\sqrt{36},\ \sqrt{-4},\ \pi,\ 9.010010001\ldots\right\}$ that are

a. natural **b.** whole **c.** integral **d.** rational **e.** irrational

Solution

a. The only natural numbers in the given set are 1 and $\frac{12}{3} = 4$.

b. The whole numbers include the natural numbers and the number 0, so we list 0, 1 and $\frac{12}{3}$.

c. The integral numbers in the given set include the previously listed $0, 1, \frac{12}{3}$, and the negative integers -10 and $-\sqrt{36} = -6$.

d. The rational numbers in the given set include the previously listed integers $0, 1, \frac{12}{3}, -10, -\sqrt{36}$, the common fraction $\frac{4}{7}$, and the decimals -5.34 and $3.\overline{16}$.

e. The only irrational numbers in the given set are the constant π and the infinite decimal $9.010010001\ldots$.

Note: $\sqrt{-4}$ is not a real number.

R.1 Exercises

True or False? If it is false, explain why.

1. Every natural number is an integer.

2. Some rational numbers are irrational.

3. Some real numbers are integers.

4. Every integer is a rational number.

5. Every infinite decimal is irrational.

6. Every square root of an odd number is irrational.

*Use **roster notation** to list all elements of each set.*

7. The set of all positive integers less than 9

8. The set of all odd whole numbers less than 11

9. The set of all even natural numbers

10. The set of all negative integers greater than -5

11. The set of natural numbers between 3 and 9

12. The set of whole numbers divisible by 4

*Use **set-builder notation** to describe each set.*

13. $\{0, 1, 2, 3, 4, 5\}$

14. $\{4, 6, 8, 10, 12, 14\}$

15. The set of all real numbers greater than -3

16. The set of all real numbers less than 21

17. The set of all multiples of 3

18. The set of perfect square numbers up to 100

Fill in each box with one of the signs $\in, \notin, \subset, \not\subset$ or $=$ to make the statement true.

19. $-3 \ \square \ \mathbb{Z}$

20. $\{0\} \ \square \ \mathbb{W}$

21. $\mathbb{Q} \ \square \ \mathbb{Z}$

22. $0.3555\ldots \ \square \ \mathbb{IQ}$

23. $\sqrt{3} \ \square \ \mathbb{Q}$

24. $\mathbb{Z}_- \ \square \ \mathbb{Z}$

25. $\pi \ \square \ \mathbb{R}$ **26.** $\mathbb{N} \ \square \ \mathbb{Q}$ **27.** $\mathbb{Z}_+ \ \square \ \mathbb{N}$

For the given set, state the subset of ***(a)*** *natural numbers,* ***(b)*** *whole numbers,* ***(c)*** *integers,* ***(d)*** *rational numbers,* ***(e)*** *irrational numbers,* ***(f)*** *real numbers.*

28. $\left\{-1,\ 2.16,\ -\sqrt{25},\ \frac{12}{2}, -\frac{12}{5},\ 3.\overline{25},\ \sqrt{5},\ \pi,\ 3.565665666\ldots\right\}$

29. $\left\{0.999\ldots,\ -5.001,\ 0,\ 5\frac{3}{4},\ 1.4\overline{05},\ \frac{7}{8},\ \sqrt{2},\ \sqrt{16},\ \sqrt{-9},\ 9.010010001\ldots\right\}$

Show that the given decimal is a rational number.

30. $0.555\ldots$ **31.** $1.\overline{02}$ **32.** $0.1\overline{34}$

33. $2.0\overline{125}$ **34.** $0.25\overline{7}$ **35.** $5.22\overline{54}$

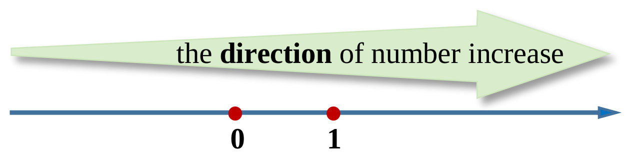
R2 Number Line and Interval Notation

As mentioned in the previous section, it is convenient to visualise the set of real numbers by identifying each number with a unique point on a number line.

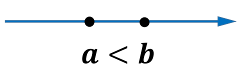
Order on the Number Line and Inequalities

Definition 2.1 ► A **number line** is a line with two distinct points chosen on it. One of these points is designated as **0** and the other point is designated as **1**.

The length of the segment from 0 to 1 represents one **unit** and provides the scale that allows to locate the rest of the numbers on the line. The **direction** from 0 to 1, marked by an **arrow** at the end of the line, indicates the **increasing order** on the number line. The numbers corresponding to the points on the line are called the **coordinates** of the points.



Note: *For simplicity, the <u>coordinates</u> of points on a number line are often identified with the points themselves.*



To compare numbers, we use **inequality signs** such as $<, \leq, >, \geq$, or $\neq$. For example, if a is **smaller than** b we write $a < b$. This tells us that the location of point a on the number line is to the left of point b. Equivalently, we could say that b is **larger than** a and write $b > a$. This means that the location of b is to the right of a.

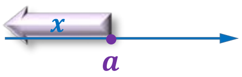
Example 1 ► **Identifying Numbers with Points on a Number Line**

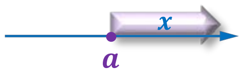
Match the numbers $-2,\ 3.5,\ \pi,\ -1.5,\ \frac{5}{2}$ with the letters on the number line:

Solution ► To match the given numbers with the letters shown on the number line, it is enough to order the numbers from the smallest to the largest. First, observe that negative numbers are smaller than positive numbers and $-2 < -1.5$. Then, observe that $\pi \approx 3.14$ is larger than $\frac{5}{2}$ but smaller than 3.5. Therefore, the numbers are ordered as follows:

$$-2 < -1.5 < \frac{5}{2} < \pi < 3.5$$

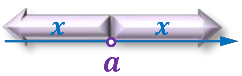
Thus, $\boldsymbol{A = -2, B = -1.5, C = \frac{5}{2}, D = \pi}$, and $\boldsymbol{E = 3.5}$.





To indicate that a number x is **smaller or equal** a, we write $x \leq a$. This tells us that the location of point x on the number line is to the left of point a or exactly at point a. Similarly, if x is **larger or equal** a, we write $x \geq a$, and we locate x to the right of point a or exactly at point a.

To indicate that a number x is **between** a and b, we write $a < x < b$. This means that the location of point x on the number line is somewhere on the segment joining points a and b, but not at a nor at b. Such stream of two inequalities is referred to as a **three-part inequality**.



Finally, to state that a number x is **different than** a, we write $x \neq a$. This means that the point x can lie anywhere on the entire number line, except at the point a.

Here is a list of some English phrases that indicate the use of particular inequality signs.

English Phrases	Inequality Sign(s)
less than; smaller than	$<$
less or equal; smaller or equal; at most; no more than	$\leq$
more than; larger than; greater than;	$>$
more or equal; larger or equal; greater or equal; at least; no less than	$\geq$
different than	$\neq$
between	$<\quad<$

Example 2 ► **Using Inequality Symbols**

Write each statement as a single or a three-part inequality.

a. -7 is **less than** 5
b. $2x$ is **greater or equal** 6
c. $3x + 1$ is **between** -1 and 7
d. x is **between** 1 and 8, **including** 1 and **excluding** 8
e. $5x - 2$ is **different than** 0
f. x is **negative**

Solution ►

a. Write $-7 < 5$. *Notice:* The inequality "points" to the smaller number. This is an example of a **strong** inequality. One side is "strongly" smaller than the other side.

b. Write $2x \geq 6$. This is an example of a **weak** inequality, as it allows for equation.

c. Enclose $3x + 1$ within two strong inequalities to obtain $-1 < 3x + 1 < 7$. *Notice:* The word "between" indicates that the endpoints are not included.

d. Since 1 is included, the statement is $1 \le x < 8$.

e. Write $5x - 2 \ne 0$.

f. Negative x means that x is smaller than zero, so the statement is $x < 0$.

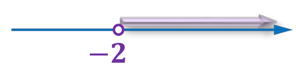
Example 3 ► **Graphing Solutions to Inequalities in One Variable**

Using a number line, graph all x-values that satisfy (are **solutions** of) the given inequality or inequalities:

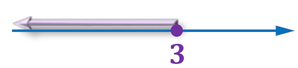
a. $x > -2$ **b.** $x \le 3$ **c.** $1 \le x < 4$

Solution ►

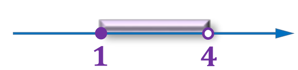
a. The x-values that satisfy the inequality $x > -2$ are larger than -2, so we shade the part of the number line that corresponds to numbers greater than -2. Those are all points to the right of -2, but not including -2. To indicate that the -2 is not a solution to the given inequality, we draw a hollow circle at -2.



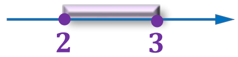
b. The x-values that satisfy the inequality $x \le 3$ are smaller than or equal to 3, so we shade the part of the number line that corresponds to the number 3 or numbers smaller than 3. Those are all points to the right of 3, including the point 3. To indicate that the 3 is a solution to the given inequality, we draw a filled-in circle at 3.



c. The x-values that satisfy the inequalities $1 \le x < 4$ are larger than or equal to 1 and at the same time smaller than 4. Thus, we shade the part of the number line that corresponds to numbers between 1 and 4, including the 1 but excluding the 4. Those are all the points that lie between 1 and 4, including the point 1 but excluding the point 4. So, we draw a segment connecting 1 with 4, with a filled-in circle at 1 and a hollow circle at 4.



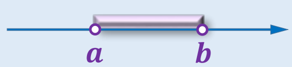
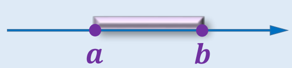
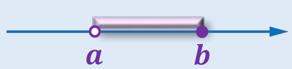
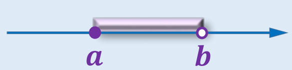
Interval Notation

As shown in the solution to *Example 3*, the graphical solutions of inequalities in one variable result in a segment of a number line (if we extend the definition of a segment to include the endpoint at infinity). To record such a solution segment algebraically, it is convenient to write it by stating its left endpoint (corresponding to the lower number) and then the right endpoint (corresponding to the higher number), using appropriate brackets that would indicate the inclusion or exclusion of the endpoint. For example, to record algebraically the segment that starts from 2 and ends on 3, including both endpoints, we write $[2, 3]$. Such notation very closely depicts the graphical representation of the segment,  , and is called **interval notation**.

Interval Notation: A set of numbers satisfying a single inequality of the type $<, \leq, >$, or $\geq$ can be recorded in interval notation, as stated in the table below.

inequality	set-builder notation	graph	interval notation	comments
$x > a$	$\{x \mid x > a\}$		$(\boldsymbol{a}, \infty)$	- list the endvalues from **left to right** - to exclude the endpoint use a **round bracket (** or **)**
$x \geq a$	$\{x \mid x \geq a\}$		$[\boldsymbol{a}, \infty)$	- **infinity** sign is used with a **round bracket**, as there is no last point to include - to include the endpoint use a **square bracket [** or **]**
$x < a$	$\{x \mid x < a\}$		$(-\infty, \boldsymbol{a})$	- to indicate **negative infinity**, use the **negative sign** in front of ∞ - to indicate **positive infinity**, there is **no need** to write **a positive sign** in front of the infinity sign
$x \leq a$	$\{x \mid x \leq a\}$		$(-\infty, \boldsymbol{a}]$	- remember to list the endvalues from **left to right**; this also refers to infinity signs

Similarly, a set of numbers satisfying two inequalities resulting in a segment of solutions can be recorded in interval notation, as stated below.

inequality	set-builder notation	graph	interval notation	comments
$a < x < b$	$\{x \mid a < x < b\}$		$(\boldsymbol{a}, \boldsymbol{b})$	- we read: an **open interval** from a to b
$a \leq x \leq b$	$\{x \mid a \leq x \leq b\}$		$[\boldsymbol{a}, \boldsymbol{b}]$	- we read: a **closed interval** from a to b
$a < x \leq b$	$\{x \mid a < x \leq b\}$		$(\boldsymbol{a}, \boldsymbol{b}]$	- we read: an interval from a to b, without a but with b This is called **half-open** or **half-closed** interval.
$a \leq x < b$	$\{x \mid a \leq x < b\}$		$[\boldsymbol{a}, \boldsymbol{b})$	- we read: an interval from a to b, with a but without b This is called **half-open** or **half-closed** interval.

In addition, the set of all real numbers $\mathbb{R}$ is represented in the interval notation as $(-\infty, \infty)$.

Example 4 ► **Writing Solutions to One Variable Inequalities in Interval Notation**

Write solutions to the inequalities from *Example 3* in set-builder and interval notation.

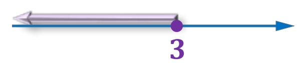
a. $x > -2$ **b.** $x \leq 3$ **c.** $1 \leq x < 4$

Solution ►

a. The solutions to the inequality $x > -2$ can be stated in set-builder notation as $\{\boldsymbol{x}|\boldsymbol{x} > -\mathbf{2}\}$. Reading the graph of this set

from **left to right**, we start from -2, without -2, and go towards infinity. So, the interval of solutions is written as $(-\mathbf{2}, \infty)$. We use the round bracket to indicate that the endpoint is not included. The infinity sign is always written with the round bracket, as infinity is a concept, not a number. So, there is no last number to include.

b. The solutions to the inequality $x \leq 3$ can be stated in set-builder notation as $\{\boldsymbol{x}|\boldsymbol{x} \leq \mathbf{3}\}$. Again, reading the graph of this set



from **left to right**, we start from $-\infty$ and go up to 3, including 3. So, the interval of solutions is written as $(-\infty, \mathbf{3}]$. We use the square bracket to indicate that the endpoint is included. As before, the infinity sign takes the round bracket. Also, we use "$-$" in front of the infinity sign to indicate negative infinity.

c. The solutions to the three-part inequality $1 \leq x < 4$ can be stated in set-builder notation as $\{\boldsymbol{x}|\mathbf{1} \leq \boldsymbol{x} < \mathbf{4}\}$. Reading the graph of this set

from **left to right**, we start from 1, including 1, and go up to 4, excluding 4. So, the interval of solutions is written as $[\mathbf{1}, \mathbf{4})$. We use the square bracket to indicate 1 and the round bracket, to exclude 4.

Absolute Value, and Distance

The **absolute value** of a number x, denoted $|x|$, can be thought of as the distance from x to 0 on a number line. Based on this interpretation, we have $|\boldsymbol{x}| = |-\boldsymbol{x}|$. This is because both numbers x and $-x$ are at the same distance from 0. For example, since both 3 and -3 are exactly three units apart from the number 0, then $|3| = |-3| = 3$.

Since distance can not be negative, we have $|\boldsymbol{x}| \geq \mathbf{0}$.

Here is a formal definition of the absolute value operator.

Definition 2.1 ► For any real number x,

$$|x| \overset{def}{=} \begin{cases} \boldsymbol{x}, & \boldsymbol{if} \quad \boldsymbol{x} \geq \mathbf{0} \\ -\boldsymbol{x}, & \boldsymbol{if} \quad \boldsymbol{x} < \mathbf{0} \end{cases}.$$

The above definition of absolute value indicates that for $x \geq 0$ we use the equation $|x| = x$, and for $x < 0$ we use the equation $|x| = -x$ (the absolute value of x is the opposite of x, which is a positive number).

Example 5 ► **Evaluating Absolute Value Expressions**

Evaluate.

a. $-|-4|$ **b.** $|-5| - |2|$ **c.** $|-5 - (-2)|$

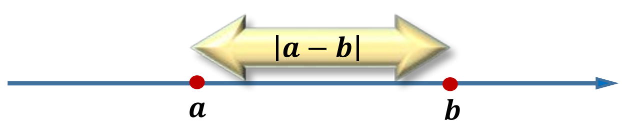
Solution ►

a. Since $|-4| = 4$ then $-|-4| = \mathbf{-4}$.

b. Since $|-5| = 5$ and $|2| = 2$ then $|-5| - |2| = 5 - 2 = \mathbf{3}$.

c. Before applying the absolute value operator, we first simplify the expression inside the absolute value sign. So we have $|-5 - (-2)| = |-5 + 2| = |-3| = \mathbf{3}$.

On a number line, the **distance** between two points with coordinates a and b is calculated by taking the difference between the two coordinates. So, if $b > a$, the distance is $b - a$. However, if $a > b$, the distance is $a - b$. What if we don't know which value is larger, a or b? Since the distance must be positive, we can choose to calculate any of the differences and apply the absolute value on the result.



Definition 2.2 ► The **distance** $d(a, b)$ between points a and b on a number line is given by the expression $|\boldsymbol{a} - \boldsymbol{b}|$, or equivalently $|\boldsymbol{b} - \boldsymbol{a}|$.

Notice that $d(x, 0) = |x - 0| = |x|$, which is consistent with the intuitive definition of absolute value of x as the distance from x to 0 on the number line.

Example 6 ► **Finding Distance Between Two Points on a Number Line**

Find the distance between the two given points on the number line.

a. -3 and 5 **b.** x and 2

Solution ►

a. Using the distance formula for two points on a number line, we have $d(-3,5) = |-3 - 5| = |-8| = \mathbf{8}$. Notice that we could also calculate $|5 - (-3)| = |8| = \mathbf{8}$.

b. Following the formula, we obtain $d(x, 2) = |x - 2|$. Since a is unknown, the distance between a and 2 is stated as an expression $|\boldsymbol{x} - \mathbf{2}|$ rather than a specific number.

R.2 Exercises

Write each statement with the use of an ***inequality*** *symbol.*

1. -6 is less than -3

2. 0 is more than -1

3. 17 is greater or equal to x

4. x is smaller or equal to 8

5. $2x + 3$ is different than zero

6. $2 - 5x$ is negative

7. x is between 2 and 5

8. $3x$ is between -5 and 7

9. $2x$ is between -2 and 6, including -2 and excluding 6

10. $x + 1$ is between -5 and 11, excluding -5 and including 11

Graph each set of numbers on a number line and write it in ***interval notation****.*

11. $\{x \mid x \geq -4\}$

12. $\{x \mid x \leq -3\}$

13. $\left\{x \,\middle|\, x < \frac{5}{2}\right\}$

14. $\left\{x \,\middle|\, x > -\frac{2}{5}\right\}$

15. $\{x \mid 0 < x < 6\}$

16. $\{x \mid -1 \leq x \leq 4\}$

17. $\{x \mid -5 \leq x < 16\}$

18. $\{x \mid -12 < x \leq 4.5\}$

Evaluate.

19. $-|-7|$

20. $|5| - |-13|$

21. $|11 - 19|$

22. $|-5 - (-9)|$

23. $-|9| - |-3|$

24. $-|-13 + 7|$

Replace each $\square$ *with one of the signs* $<, >, \leq, \geq, =$ *to make the statement true.*

25. $-7 \square -5$

26. $|-16| \square -|16|$

27. $-3 \square -|3|$

28. $x^2 \square 0$

29. $x \square |x|$

30. $|x| \square |-x|$

Find the distance between the given points.

31. $-7, -32$

32. $46, -13$

33. $-\frac{2}{3}, \frac{5}{6}$

34. $x, 0$

35. $5, y$

36. x, y

Find numbers that are 5 units apart from the given point.

37. 0

38. 3

39. a

R3 Properties and Order of Operations on Real Numbers

In algebra, we are often in need of changing an expression to a different but equivalent form. This can be observed when simplifying expressions or solving equations. To change an expression equivalently from one form to another, we use appropriate properties of operations and follow the order of operations.

Properties of Operations on Real Numbers

The four basic operations performed on real numbers are addition $(+)$, subtraction $(-)$, multiplication $(\cdot)$, and division $(\div)$. Here are the main properties of these operations:

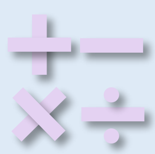
Closure: The result of an operation on real numbers is also a real number. We can say that the **set of real numbers** is **closed** under **addition**, **subtraction** and **multiplication**.

We cannot say this about division, as **division by zero is not allowed**.

Neutral Element: A real number that leaves other real numbers unchanged under a particular operation.

For example, **zero** is the **neutral element** (also called **additive identity**) of **addition**, since $\boldsymbol{a} + \mathbf{0} = \boldsymbol{a}$, and $\mathbf{0} + \boldsymbol{a} = \boldsymbol{a}$, for any real number a.
Similarly, **one** is the **neutral element** (also called **multiplicative identity**) of **multiplication**, since $\boldsymbol{a} \cdot \mathbf{1} = \boldsymbol{a}$, and $\mathbf{1} \cdot \boldsymbol{a} = \boldsymbol{a}$, for any real number a.

Inverse Operations: Operations that reverse the effect of each other. For example, **addition and subtraction** are **inverse operations**, as $\boldsymbol{a} + \boldsymbol{b} - \boldsymbol{b} = \boldsymbol{a}$, and $\boldsymbol{a} - \boldsymbol{b} + \boldsymbol{b} = \boldsymbol{a}$, for any real a and b.



Similarly, **multiplication and division** are **inverse operations**, as $\boldsymbol{a} \cdot \boldsymbol{b} \div \boldsymbol{b} = \boldsymbol{a}$, and $\boldsymbol{a} \cdot \boldsymbol{b} \div \boldsymbol{b} = \boldsymbol{a}$ for any real a and $b \neq 0$.

Opposites: Two quantities are **opposite** to each other if they **add to zero**. Particularly, $\boldsymbol{a}$ and $-\boldsymbol{a}$ are **opposites** (also referred to as **additive inverses**), as $\boldsymbol{a} + (-\boldsymbol{a}) = \mathbf{0}$. For example, the opposite of 3 is -3, the opposite of $-\frac{3}{4}$ is $\frac{3}{4}$, the opposite of $x + 1$ is $-(x + 1) = -x - 1$.

Reciprocals: Two quantities are **reciprocals** of each other if they **multiply to one**. Particularly, $\boldsymbol{a}$ and $\frac{\mathbf{1}}{\boldsymbol{a}}$ are **reciprocals** (also referred to as **multiplicative inverses**), since $\boldsymbol{a} \cdot \frac{\mathbf{1}}{\boldsymbol{a}} = \mathbf{1}$. For example, the reciprocal of 3 is $\frac{1}{3}$, the reciprocal of $-\frac{3}{4}$ is $-\frac{4}{3}$, the reciprocal of $x + 1$ is $\frac{1}{x+1}$.

Multiplication by 0: Any real quantity **multiplied by zero** becomes **zero**. Particularly, $\boldsymbol{a} \cdot \mathbf{0} = \mathbf{0}$, for any real number a.

Zero Product: If a product of two real numbers is zero, then at least one of these numbers must be zero. Particularly, for any real $\boldsymbol{a}$ and $\boldsymbol{b}$, if $\boldsymbol{a} \cdot \boldsymbol{b} = \mathbf{0}$, then $\boldsymbol{a} = \mathbf{0}$ or $\boldsymbol{b} = \mathbf{0}$.

For example, if $x(x - 1) = 0$, then either $x = 0$ or $x - 1 = 0$.

Commutativity: **The order of numbers does not change the value** of a particular operation. In particular, **addition** and **multiplication** is **commutative**, since

$$a + b = b + a \text{ and } a \cdot b = b \cdot a,$$

for any real a and b. For example, $5 + 3 = 3 + 5$ and $5 \cdot 3 = 3 \cdot 5$.

Note: Neither subtraction nor division is commutative. See a counterexample: $5 - 3 = 2$ but $3 - 5 = -2$, so $5 - 3 \neq 3 - 5$. Similarly, $5 \div 3 \neq 3 \div 5$.

Associativity: **Association (grouping) of numbers does not change the value** of an expression involving only one type of operation. In particular, **addition** and **multiplication** is **associative**, since

$$(a + b) + c = a + (b + c) \text{ and } (a \cdot b) \cdot c = a \cdot (b \cdot c),$$

for any real a and b. For example, $(5 + 3) + 2 = 5 + (3 + 2)$ and $(5 \cdot 3) \cdot 2 = 5 \cdot (3 \cdot 2)$.

Note: Neither subtraction nor division is associative. See a counterexample:
$(8 - 4) - 2 = 2$ but $8 - (4 - 2) = 6$, so $(8 - 4) - 2 \neq 8 - (4 - 2)$.
Similarly, $(8 \div 4) \div 2 = 1$ but $8 \div (4 \div 2) = 4$, so $(8 \div 4) \div 2 \neq 8 \div (4 \div 2)$.

Distributivity: Multiplication can be **distributed** over addition or subtraction by following the rule:

$$a(b \pm c) = ab \pm ac,$$

for any real a, b and c. For example, $2(3 \pm 5) = 2 \cdot 3 \pm 2 \cdot 5$, or $2(x \pm y) = 2x \pm 2y$.

Note: The reverse process of distribution is known as **factoring a common factor** out. For example, $2ax + 2ay = 2a(x + y)$.

Example 1 ► **Showing Properties of Operations on Real Numbers**

Complete each statement to illustrate the indicated property.

a. $mn =$ _______ (commutativity of multiplication)
b. $5x + (7x + 8) =$ __________________ (associativity of addition)
c. $5x(2 - x) =$ __________________ (distributivity of multiplication)
d. $-y +$ _____ $= 0$ (additive inverse)
e. $-6 \cdot$ _____ $= 1$ (multiplicative inverse)
f. If $7x = 0$, then ____ $= 0$ (zero product)

Solution ►

a. To show that multiplication is commutative, we change the order of letters, so $mn = nm$.

b. To show that addition is associative, we change the position of the bracket, so $5x + (7x + 8) = (5x + 7x) + 8$.

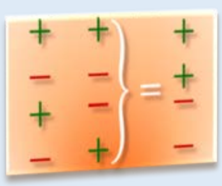
c. To show the distribution of multiplication over subtraction, we multiply $5x$ by each term of the bracket. So we have $5x(2 - x) = 5x \cdot 2 - 5x \cdot x$.

d. Additive inverse to $-y$ is its opposite, which equals to $-(-y) = y$.
So we write $-y + y = 0$.

e. Multiplicative inverse of -6 is its reciprocal, which equals to $-\frac{1}{6}$.
So we write $-6 \cdot \left(-\frac{1}{6}\right) = 1$.

f. By the zero product property, one of the factors, 7 or x, must equal to zero.
Since $7 \neq 0$, then x must equal to zero. So, we write: If $7x = 0$, then $x = 0$.

Sign Rule:



When multiplying or dividing two numbers of the **same sign**, the result is **positive**.

When multiplying or dividing two numbers of **different signs**, the result is **negative**.

This rule also applies to double signs. If the two **signs** are the **same**, they can be replaced by a **positive** sign. For example $+(+3) = 3$ and $-(-3) = 3$.
If the two **signs** are **different**, they can be replaced by a **negative** sign. For example $-(+3) = -3$ and $+(-3) = -3$.

Observation:

*Since a double negative $(- -)$ can be replaced by a positive sign $(+)$, the **opposite of an opposite** leaves the original quantity unchanged. For example, $-(-2) = 2$, and generally $-(-a) = a$.*
*Similarly, taking the **reciprocal of a reciprocal** leaves the original quantity unchanged. For example, $\frac{1}{\frac{1}{2}} = 1 \cdot \frac{2}{1} = 2$, and generally $\frac{1}{\frac{1}{a}} = 1 \cdot \frac{a}{1} = a$.*

Example 2 ► **Using Properties of Operations on Real Numbers**

Use applicable properties of real numbers to simplify each expression.

a. $-\frac{-2}{-3}$

b. $3 + (-2) - (-7) - 11$

c. $2x(-3y)$

d. $3a - 2 - 5a + 4$

e. $-(2x - 5)$

f. $2(x^2 + 1) - 2(x - 3x^2)$

g. $-\frac{100ab}{25a}$

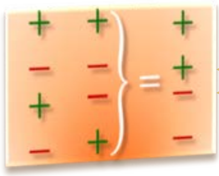
h. $\frac{2x-6}{2}$

Solution ►

a. The quotient of two negative numbers is positive, so $-\frac{-2}{-3} = -\frac{\mathbf{2}}{\mathbf{3}}$.

Note: *To determine the overall sign of an expression involving only multiplication and division of signed numbers, it is enough to count how many of the negative signs appear in the expression. An **even number of negatives** results in a **positive** value; an **odd number of negatives** leaves the answer **negative**.*

b. First, according to the sign rule, replace each double sign by a single sign. Therefore,



$$3+(-2)-(-7)-11=3-2+7-11.$$

It is convenient to treat this expression as a **sum** of signed numbers. So, it really means

$$3+-2+7+-11$$

but, for shorter notation, we tend not to write the plus signs.

Then, using the commutative property of addition, we collect all positive numbers, and all negative numbers to obtain

$$3\underbrace{-2+7}_{\substack{switch\\addends}}-11=\underbrace{3+7}_{\substack{collect\\positive}}\ \underbrace{-2-11}_{\substack{collect\\negative}}=\underbrace{10-13}_{subtract}=\boldsymbol{-3}.$$

c. Since associativity of multiplication tells us that the order of performing multiplication does not change the outcome, there is no need to use any brackets in expressions involving only multiplication. So, the expression $2x(-3y)$ can be written as $2\cdot x\cdot(-3)\cdot y$. Here, the bracket is used only to isolate the negative number, not to prioritize any of the multiplications. Then, applying commutativity of multiplication to the middle two factors, we have

$$2\cdot\underbrace{x\cdot(-3)}_{\substack{switch\\factors}}\cdot y=\underbrace{2\cdot(-3)}_{\substack{perform\\multiplication}}\cdot x\cdot y=\boldsymbol{-6xy}$$

d. First, use commutativity of addition to switch the two middle addends, then factor out the a, and finally perform additions where possible.

$$3a\underbrace{-2-5a}_{\substack{switch\\addends}}+4=\underbrace{3a-5a}_{factor\ \boldsymbol{a}\ out}-2+4=\underbrace{(3-5)}_{combine}a\ \underbrace{-2+4}_{combine}=\boldsymbol{-2a+2}$$

Note: *In practice, to combine terms with the same variable, add their coefficients.*

e. The expression $-(2x-5)$ represents the **opposite** to $2x-5$, which is $\boldsymbol{-2x+5}$. This expression is indeed the opposite because

$$-2x+5+2x-5=-2x\ \underbrace{+2x+5}_{\substack{commutativity\\of\ addition}}-5=\underbrace{2x-2x}_{opposites}\ \underbrace{-5+5}_{opposites}=0+0=0.$$

Notice that the negative sign in front of the bracket in the expression $-(2x-5)$ can be treated as multiplication by -1. Indeed, using the distributive property of multiplication over subtraction and the sign rule, we achieve the same result

$$-1(2x-5)=-1\cdot 2x+(-1)(-5)=\boldsymbol{-2x+5}.$$

Note: *In practice, to release a bracket with a negative sign (or a negative factor) in front of it, change all the addends into opposites. For example*

$$-(2x-y+1)=-2x+y-1$$
$$\textit{and}\quad -3(2x-y+1)=-6x+3y-3$$

f. To simplify $2(x^2+1)-2(x-3x^2)$, first, we apply the distributive property of multiplication and the sign rule.

$$2(x^2+1)-2(x-3x^2)=2x^2+2-2x+6x^2$$

Then, using the commutative property of addition, we group the terms with the same powers of x. So, the equivalent expression is

$$2x^2+6x^2-2x+2$$

Finally, by factoring x^2 out of the first two terms, we can add them to obtain

$$(2+6)x^2-2x+2=\boldsymbol{8x^2-2x+2}.$$

Note: *In practice, to combine terms with the same powers of a variable (or variables), add their coefficients. For example*

$$\underline{\underline{2x^2}}\ \underline{\underline{-5x^2}}\ \underline{+3xy}\ \underline{-xy}\ -3+2=\underline{\underline{-3x^2}}\ \underline{+2xy}-1.$$

g. To simplify $-\frac{100ab}{25a}$, we reduce the common factors of the numerator and denominator by following the property of the neutral element of multiplication, which is one. So,

$$-\frac{100ab}{25a}=-\frac{25\cdot 4ab}{25a}=-\frac{25a\cdot 4b}{25a\cdot 1}=-\frac{25a}{25a}\cdot\frac{4b}{1}=-1\cdot\frac{4b}{1}=\boldsymbol{-4b}.$$

This process is called **canceling** and can be recorded in short as

$$-\frac{\overset{4}{\cancel{100}}\cancel{a}b}{\cancel{25}\cancel{a}}=\boldsymbol{-4b}.$$

h. To simplify $\frac{2x-6}{2}$, factor the numerator and then remove from the fraction the factor of one by canceling the common factor of 2 in the numerator and the denominator. So, we have

$$\frac{2x-6}{2}=\frac{\cancel{2}\cdot(2x-6)}{\cancel{2}}=\boldsymbol{2x-6}.$$

In the solution to *Example 2d* and *2f*, we used an intuitive understanding of what a "term" is. We have also shown how to combine terms with a common variable part (like terms). Here is a more formal definition of a term and of like terms.

Definition 3.1 ► A **term** is a **product** of constants (numbers), variables, or expressions. Here are examples of single terms:

$$1,\ x,\ \frac{1}{2}x^2,\ -3xy^2,\ 2(x+1),\ \frac{x+2}{x(x+1)},\ \pi\sqrt{x}.$$

Observe that the expression $2x+2$ consists of two terms connected by addition, while the equivalent expression $2(x+1)$ represents just one term, as it is a product of the number 2 and the expression $(x+1)$.

Like terms are the terms that have exactly the same variable part (the same variables or expressions raised to the same exponents). Like terms can be **combined** by adding their **coefficients** (numerical part of the term).

For example, $5x^2$ and $-2x^2$ are like, so they can be combined (added) to $3x^2$,
$(x+1)$ and $3(x+1)$ are like, so they can be combined to $4(x+1)$,
but $5x$ and $2y$ are unlike, so they cannot be combined.

Example 3 ► Combining Like Terms

Simplify each expression by combining like terms.

a. $-x^2+3y^2+x-6+2y^2-x+1$

b. $\frac{2}{x+1}-\frac{5}{x+1}+\sqrt{x}-\frac{\sqrt{x}}{2}$

Solution ►

a. Before adding like terms, it is convenient to underline the groups of like terms by the same type of underlining. So, we have

add to zero

$$-x^2+3y^2+x-6+2y^2-x+1=\mathbf{-x^2+5y^2-5}$$

b. Notice that the numerical coefficients of the first two like terms in the expression

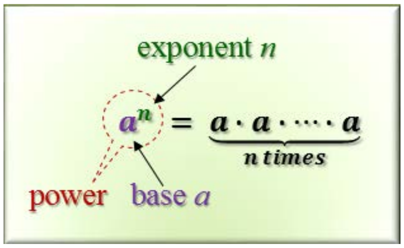
$$\frac{2}{x+1}-\frac{5}{x+1}+\sqrt{x}-\frac{\sqrt{x}}{2}$$

are 2 and -5, and of the last two like terms are 1 and $-\frac{1}{2}$. So, by adding these coefficients, we obtain

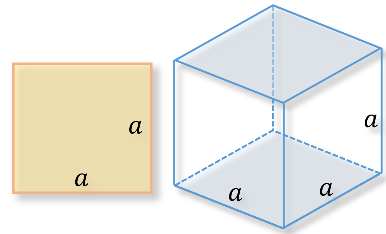
$$\mathbf{-\frac{3}{x+1}+\frac{1}{2}\sqrt{x}}$$

Observe that $\frac{1}{2}\sqrt{x}$ can also be written as $\frac{\sqrt{x}}{2}$. Similarly, $-\frac{3}{x+1}$, $\frac{-3}{x+1}$, or $-3\cdot\frac{1}{x+1}$ are equivalent forms of the same expression.

Exponents and Roots



Exponents are used as a shorter way of recording repeated multiplication by the same quantity. For example, to record the product $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$, we write 2^5. The **exponent** 5 tells us how many times to multiply the **base** 2 by itself to evaluate the product, which is 32. The expression 2^5 is referred to as the 5th **power** of 2, or "2 to the 5th". In the case of exponents 2 or 3, terms "squared" or "cubed" are often used. This is because of the connection to geometric figures, a square and a cube.



The area of a square with sides of length a is expressed by a^2 (*read:* "a squared" or "the square of a") while the volume of a cube with sides of length a is expressed by a^3 (*read:* "a cubed" or "the cube of a").

If a negative number is raised to a certain exponent, a bracket must be used around the base number. For example, if we wish to multiply -3 by itself two times, we write $(-3)^2$, which equals $(-3)(-3) = 9$. The notation -3^2 would indicate that only 3 is squared, so $-3^2 = -3 \cdot 3 = -9$. This is because an **exponent refers only to the number immediately below the exponent**. Unless we use a bracket, a negative sign in front of a number is not under the influence of the exponent.

Example 4 ► **Evaluating Exponential Expressions**

Evaluate each exponential expression.

a. -3^4 **b.** $(-2)^6$

c. $(-2)^5$ **d.** $-(-2)^3$

e. $\left(-\frac{2}{3}\right)^2$ **f.** $-\left(-\frac{2}{3}\right)^5$

Solution ►

a. $-3^4 = (-1) \cdot 3 \cdot 3 \cdot 3 \cdot 3 = \mathbf{-81}$

b. $(-2)^6 = (-2)(-2)(-2)(-2)(-2)(-2) = \mathbf{64}$

c. $(-2)^5 = (-2)(-2)(-2)(-2)(-2) = \mathbf{-32}$

Observe: *Negative sign in front of a power works like multiplication by* -1.

A ***negative*** *base raised to an* ***even exponent*** *results in a* ***positive*** *value.*
A ***negative*** *base raised to an* ***odd exponent*** *results in a* ***negative*** *value.*

d. $-(-2x)^3 = -(-2x)(-2x)(-2x)$

$= -(-2)(-2)(-2)xxx = -(-2)^3x^3 = -(-8)x^3 = \mathbf{8x^3}$

e. $\left(-\frac{2}{3}\right)^2 = \left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right) = \frac{(-2)^2}{3^2} = \mathbf{\frac{4}{9}}$

f. $-\left(-\frac{2}{3}\right)^5 = -\left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right) = -\frac{(-2)^5}{3^5} = -\frac{-32}{243} = \mathbf{\frac{32}{243}}$

Observe: *Exponents apply to every factor of the numerator and denominator of the base. This exponential property can be stated as*

$$\boldsymbol{(ab)^n = a^n b^n} \quad \textit{and} \quad \boldsymbol{\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}}$$

To reverse the process of squaring, we apply a **square root**, denoted by the **radical sign** $\sqrt{}$. For example, since $5 \cdot 5 = 25$, then $\sqrt{25} = 5$. Notice that $(-5)(-5) = 25$ as well, so we could also claim that $\sqrt{25} = -5$. However, we wish to define the operation of taking square root in a unique way. We choose to take the **positive** number (called **principal square root**) as the value of the square root. Therefore $\sqrt{25} = 5$, and generally

$$\boldsymbol{\sqrt{x^2} = |x|.}$$

Since the square of any nonzero real number is positive, the square root of a negative number is not a real number. For example, we can say that $\boldsymbol{\sqrt{-16}}$ **does not exist** (in the set of real numbers), as there is no real number a that would satisfy the equation $a^2 = -16$.

Example 5 ► **Evaluating Radical Expressions**

Evaluate each radical expression.

a. $\sqrt{0}$ **b.** $\sqrt{64}$

c. $-\sqrt{121}$ **d.** $\sqrt{-100}$

e. $\sqrt{\frac{1}{9}}$ **f.** $\sqrt{0.49}$

Solution ►

a. $\sqrt{0} = \mathbf{0}$, as $0 \cdot 0 = 0$

b. $\sqrt{64} = \mathbf{8}$, as $8 \cdot 8 = 64$

c. $-\sqrt{121} = \mathbf{-11}$, as we copy the negative sign and $11 \cdot 11 = 121$

d. $\sqrt{-100} = \boldsymbol{DNE}$ (*read:* doesn't exist), as no real number squared equals -100

e. $\sqrt{\frac{1}{9}} = \mathbf{\frac{1}{3}}$, as $\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$.

Notice that $\frac{\sqrt{1}}{\sqrt{9}}$ also results in $\frac{1}{3}$. So, $\sqrt{\frac{1}{9}} = \frac{\sqrt{1}}{\sqrt{9}}$ and generally $\boldsymbol{\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}}$ for any nonnegative real numbers a and $b \neq 0$.

f. $\sqrt{0.49} = \mathbf{0.7}$, as $0.7 \cdot 0.7 = 0.49$

Order of Operations

In algebra, similarly as in arithmetic, we like to perform various operations on numbers or on variables. To record in what order these operations should be performed, we use grouping signs, mostly brackets, but also division bars, absolute value symbols, radical symbols, etc. In an expression with many grouping signs, we perform operations in the **innermost grouping sign first**. For example, the innermost grouping sign in the expression

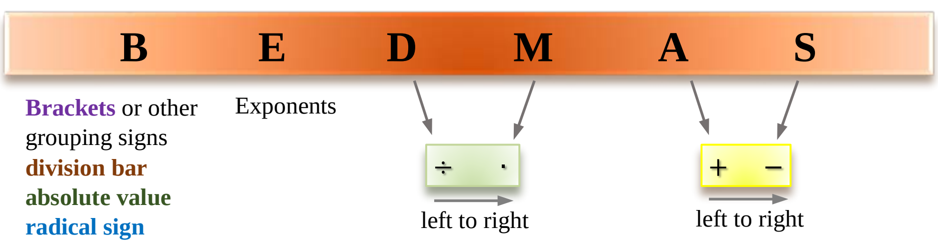
$$[4 + (3 \cdot |2 - 4|)] \div 2$$

is the absolute value sign, then the round bracket, and finally, the square bracket. So first, perform subtraction, then apply the absolute value, then multiplication, addition, and finally the division. Here are the calculations:

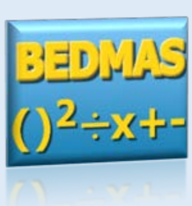
$$\begin{aligned} &[4 + (3 \cdot |2 - 4|)] \div 2 \\ &= [4 + (3 \cdot |-2|)] \div 2 \\ &= [4 + (3 \cdot 2)] \div 2 \\ &= [4 + 6] \div 2 \\ &= 10 \div 2 \\ &= \mathbf{5} \end{aligned}$$

Observe that the more operations there are to perform, the more grouping signs would need to be used. To simplify the notation, additional rules of order of operations have been created. These rules, known as BEDMAS, allow for omitting some of the grouping signs, especially brackets. For example, knowing that multiplication is performed before addition, the expression $[4 + (3 \cdot |2 - 4|)] \div 2$ can be written as $[4 + 3 \cdot |2 - 4|] \div 2$ or $\frac{4+3\cdot|2-4|}{2}$.

Let's review the BEDMAS rule.



BEDMAS Rule:



1. Perform operations in the innermost **B**racket (or other grouping sign) first.
2. Then work out **E**xponents.
3. Then perform **D**ivision and **M**ultiplication in order of their occurrence (left to right). *Notice that there is no priority between division and multiplication. However, both division and multiplication have priority before any addition or subtraction.*
4. Finally, perform **A**ddition and **S**ubtraction in order of their occurrence (left to right). *Again, there is no priority between addition and subtraction.*

Example 6 ► **Simplifying Arithmetic Expressions According to the Order of Operations**

Use the order of operations to simplify each expression.

a. $3 + 2 \cdot 6$

b. $4 \cdot 6 \div 3 - 2$

c. $12 \div 4 + 2|3 - 4|$

d. $2 \cdot 3^2 - 3(-2 + 6)$

e. $\sqrt{30 - 5} - 2\big(3 + 4 \cdot (-2)\big)^2$

f. $\frac{3-2(-3^2)}{3\cdot\sqrt{4}-6\cdot 2}$

Solution ► **a.** Out of the two operations, + and $\cdot$, multiplication is performed first. So, we have

$3 + 2 \cdot 6$
$= 3 + 12$
$= \mathbf{15}$

b. There is no priority between multiplication and division, so we perform these operations in the order in which they appear, from left to right. Then we subtract. Therefore,

$4 \cdot 6 \div 3 - 2$
$= 24 \div 3 - 2$
$= 8 - 2$
$= \mathbf{6}$

c. In this expression, we have a grouping sign (the absolute value bars), so we perform the subtraction inside the absolute value first. Then, we apply the absolute value and work out the division and multiplication before the final addition. So, we obtain

$12 \div 4 + 2|3 - 4|$
$= 12 \div 4 + 2|-1|$
$= 12 \div 4 + 2 \cdot 1$
$= 3 + 2$
$= \mathbf{5}$

d. In this expression, work out the bracket first, then perform the exponent, then both multiplications, and finally the subtraction. Thus,

$2 \cdot 3^2 - 3(-2 + 6)$
$= 2 \cdot 3^2 - 3(4)$
$= 2 \cdot 9 - 3(4)$
$= 18 - 12$
$= \mathbf{6}$

e. The expression $\sqrt{30 - 5} - 2\big(3 + 4 \cdot (-2)\big)^2$ contains two grouping signs, the bracket and the radical sign. Since these grouping signs are located at separate places (they are not nested), they can be worked out simultaneously. As usual, out of the operations inside the bracket, multiplication is done before subtraction. So, we calculate

$\sqrt{30 - 5} - 2\big(3 + 4 \cdot (-2)\big)^2$
$= \sqrt{25} - 2\big(3 + (-8)\big)^2$
$= 5 - 2(-5)^2$

Work out the power first, then multiply, and finally subtract.

$= 5 - 2 \cdot 25$
$= 5 - 50$
$= \mathbf{-45}$

f. To simplify the expression $\frac{3-2(-3^2)}{3\cdot\sqrt{4}-6\cdot 2}$, work on the numerator and the denominator before performing the division. Therefore,

$$\frac{3-2(-3^2)}{3\cdot\sqrt{4}-6\cdot 2}$$

$-3^2 = -9$

$$= \frac{3-(-18)}{3\cdot 2-6\cdot 2}$$

$$= \frac{3+18}{6-12}$$

$$= \frac{21}{-6}$$

reduce the common factor of 3

$$= -\frac{\mathbf{7}}{\mathbf{2}}$$

Example 7 ► **Simplifying Expressions with Nested Brackets**

Simplify the expression $2\{1-5[3x+2(4x-1)]\}$.

Solution ► The expression $2\{1-5[3x+2(4x-1)]\}$ contains three types of brackets: the innermost parenthesis ( ), the middle brackets [ ], and the outermost braces { }. We start with working out the innermost parenthesis first, and then after collecting like terms, we proceed with working out consecutive brackets. So, we simplify

$2\{1-5[3x+2(4x-1)]\}$ — distribute 2 over the ( ) bracket
$= 2\{1-5[3x+8x-2]\}$ — collect like terms before working out the [ ] bracket
$= 2\{1-5[11x-2]\}$ — distribute -5 over the [ ] bracket
$= 2\{1-55x+10\}$ — collect like terms before working out the { } bracket
$= 2\{-55x+11\}$ — distribute 2 over the { } bracket
$= \mathbf{-110x+22}$

Evaluation of Algebraic Expressions

An **algebraic expression** consists of letters, numbers, operation signs, and grouping symbols. Here are some examples of algebraic expressions:

$$6ab,\quad x^2-y^2,\quad 3(2a+5b),\quad \frac{x-3}{3-x},\quad 2\pi r,\quad \frac{d}{t},\quad Prt,\quad \sqrt{x^2+y^2}$$

When a letter is used to stand for various numerical values, it is called a **variable**. For example, if t represents the number of hours needed to drive between particular towns, then t changes depending on the average speed used during the trip. So, t is a variable. Notice however, that the distance d between the two towns represents a constant number. So, even though letters in algebraic expressions usually represent variables, sometimes they may represent a **constant** value. One such constant is the letter π, which represents approximately 3.14.

Notice that algebraic expressions do not contain any comparison signs (equality or inequality, such as $=$, $\neq$, $<$, $\leq$, $>$, $\geq$), therefore, they are **not to be solved** for any variable. Algebraic expressions can only be **simplified** by implementing properties of operations (see *Example 2* and *3*) or **evaluated** for particular values of the variables. The evaluation process involves substituting given values for the variables and evaluating the resulting arithmetic expression by following the order of operations.

Advice: *To evaluate an algebraic expression for given variables, first rewrite the expression replacing each variable with* ***empty brackets*** *and then write appropriate values inside these brackets. This will help to avoid possible errors of using incorrect signs or operations.*

Example 8 ► **Evaluating Algebraic Expressions**

Evaluate each expression for $a = -2$, $b = 3$, and $c = 6$.

a. $b^2 - 4ac$ **b.** $2c \div 3a$ **c.** $\frac{|a^2 - b^2|}{-a^2 + \sqrt{a+c}}$

Solution ►

a. First, we replace each letter in the expression $b^2 - 4ac$ with an empty bracket. So, we write

$$(\quad)^2 - 4(\quad)(\quad).$$

Now, we fill in the brackets with the corresponding values and evaluate the resulting expression. So, we have

$$(3)^2 - 4(-2)(6) = 9 - (-48) = 9 + 48 = \mathbf{57}.$$

b. As above, we replace the letters with their corresponding values to obtain

$$2c \div 3a = 2(6) \div 3(-2).$$

Since we work only with multiplication and division here, they are to be performed in order from left to right. Therefore,

$$2(6) \div 3(-2) = 12 \div 3(-2) = 4(-2) = \mathbf{-8}.$$

c. As above, we replace the letters with their corresponding values to obtain

$$\frac{|a^2 - b^2|}{-a^2 + \sqrt{b + c}} = \frac{|(-2)^2 - (3)^2|}{-(-2)^2 + \sqrt{(3) + (6)}} = \frac{|4 - 9|}{-4 + \sqrt{9}} = \frac{|-5|}{-4 + 3} = \frac{5}{-1} = \mathbf{-5}.$$

Equivalent Expressions

Algebraic expressions that produce the same value for all allowable values of the variables are referred to as **equivalent expressions**. Notice that properties of operations allow us to rewrite algebraic expressions in a simpler but equivalent form. For example,

$$\frac{\boldsymbol{x-3}}{\boldsymbol{3-x}} = \frac{\cancel{x-3}}{-(\cancel{x-3})} = \boldsymbol{-1}$$

or

$$(\boldsymbol{x}+\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y}) = (x+y)x - (x+y)y = x^2 + \cancel{yx} - \cancel{xy} - y^2 = \boldsymbol{x^2 - y^2}.$$

To show that two expressions are **<u>not</u> equivalent**, it is enough to find a particular set of variable values for which the two expressions evaluate to a different value. For example,

$$\sqrt{x^2+y^2} \neq x + y$$

because if $x = 1$ and $y = 1$ then $\sqrt{x^2+y^2} = \sqrt{1^2+1^2} = \sqrt{2}$ while $x + y = 1 + 1 = 2$. Since $\sqrt{2} \neq 2$ the two expressions $\sqrt{x^2+y^2}$ and $x + y$ are not equivalent.

Example 9 ► **Determining Whether a Pair of Expressions is Equivalent**

Determine whether the given expressions are equivalent.

a. $(a+b)^2$ and a^2+b^2 **b.** $\frac{x^8}{x^4}$ and x^4

Solution ► **a.** Suppose $a = 1$ and $b = 1$. Then

$$(a+b)^2 = (1+1)^2 = 2^2 = 4$$

but

$$a^2 + b^2 = 1^2 + 1^2 = 2.$$

So the expressions $(a+b)^2$ and a^2+b^2 are not equivalent.

Using the distributive property and commutativity of multiplication, check on your own that

$$(a+b)^2 = a^2 + 2ab + b^2.$$

b. Using properties of exponents and then removing a factor of one, we show that

$$\frac{x^8}{x^4} = \frac{x^4 \cdot x^4}{x^4} = x^4.$$

So the two expressions are indeed equivalent.

Review of Operations on Fractions

A large part of algebra deals with performing operations on algebraic expressions by generalising the ways that these operations are performed on real numbers, particularly, on common fractions. Since operations on fractions are considered to be one of the most challenging topics in arithmetic, it is a good idea to review the rules to follow when performing these operations before we move on to other topics of algebra.

Operations on Fractions:

Simplifying

To simplify a fraction to its lowest terms, **remove** the **greatest common factor** (**GCF**) of the numerator and denominator. For example, $\frac{48}{64} = \frac{3 \cdot \cancel{16}}{4 \cdot \cancel{16}} = \frac{3}{4}$, and generally $\frac{a\cancel{k}}{b\cancel{k}} = \frac{a}{b}$.

This process is called ***reducing*** or ***canceling***.

Note that the reduction can be performed several times, if needed. In the above example, if we didn't notice that 16 is the greatest common factor for 48 and 64, we could reduce the fraction by dividing the numerator and denominator by any common factor (2, or 4, or 8) first, and then repeat the reduction process until there is no common factors (other than 1) anymore. For example,

$$\frac{\cancel{48}}{\cancel{64}} = \frac{\cancel{24}}{\cancel{32}} = \frac{\cancel{6}}{\cancel{8}} = \frac{\mathbf{3}}{\mathbf{4}}.$$

÷ by 2 ÷ by 4 ÷ by 2

Multiplying

To multiply fractions, we multiply their numerators and denominators. So generally,

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

However, before performing multiplication of numerators and denominators, it is a good idea to reduce first. This way, we work with smaller numbers, which makes the calculations easier. For example,

$$\frac{18}{15} \cdot \frac{25}{14} = \frac{18 \cdot \cancel{25}}{\cancel{15} \cdot 14} = \frac{\cancel{18} \cdot 5}{\cancel{3} \cdot 14} = \frac{\cancel{6} \cdot 5}{1 \cdot \cancel{14}} = \frac{3 \cdot 5}{1 \cdot 7} = \frac{\mathbf{15}}{\mathbf{7}}.$$

÷ by 5 ÷ by 3 ÷ by 2

Dividing

To divide fractions, we **multiply** the dividend (the first fraction) **by the reciprocal** of the **divisor** (the second fraction). So generally,

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}.$$

· by reciprocal

For example,

÷ by 4

$$\frac{8}{15} \div \frac{4}{5} = \frac{\cancel{8}}{\cancel{15}} \cdot \frac{\cancel{5}}{\cancel{4}} = \frac{2 \cdot 1}{3 \cdot 1} = \frac{\mathbf{2}}{\mathbf{3}}.$$

÷ by 5

Adding or Subtracting

To add or subtract fractions, follow the steps:

1. Find the **Lowest Common Denominator** (**LCD**).
2. Extend each fraction to higher terms to obtain the desired common denominator.
3. Add or subtract the numerators, keeping the common denominator.
4. Simplify the resulting fraction, if possible.

For example, to evaluate $\frac{5}{6}+\frac{3}{4}-\frac{4}{15}$, first we find the LCD for denominators 6, 4, and 15. We can either guess that 60 is the least common multiple of 6, 4, and 15, or we can use the following method of finding LCD:

2	**6**	**4**	**15**
· 3	3	2	15
·	1	· 2	· 5

$= 60$

- divide by a common factor of at least two numbers; for example, by 2
- write the quotients in the line below; 15 is not divisible by 2, so just copy it down
- keep dividing by common factors till all numbers become relatively prime
- the LCD is the product of all numbers listed in the letter L, so it is 60

Then, we extend the fractions so that they share the same denominator of 60, and finally perform the operations in the numerator. Therefore,

$$\frac{5}{6}+\frac{3}{4}-\frac{4}{5}=\underbrace{\frac{5\cdot 10}{6\cdot 10}+\frac{3\cdot 15}{4\cdot 15}-\frac{4\cdot 12}{5\cdot 12}}_{\text{in practice, this step doesn't have to be written}}=\frac{5\cdot 10+3\cdot 15-4\cdot 12}{60}=\frac{50+45-48}{60}=\mathbf{\frac{47}{60}}.$$

Example 10 ► **Evaluating Fractional Expressions**

Simplify each expression.

a. $-\frac{2}{3}-\left(-\frac{5}{12}\right)$ **b.** $-3\left[\frac{3}{2}+\frac{5}{6}\div\left(-\frac{3}{8}\right)\right]$

Solution ►

a. After replacing the double negative by a positive sign, we add the two fractions, using 12 as the lowest common denominator. So, we obtain

$$-\frac{2}{3}-\left(-\frac{5}{12}\right)=-\frac{2}{3}+\frac{5}{12}=\frac{-2\cdot 4+5}{12}=\frac{-\cancel{3}}{\cancel{12}}=-\mathbf{\frac{1}{4}}.$$

b. Following the order of operations, we calculate

$$-3\left[\frac{3}{2}+\frac{5}{6}\div\left(-\frac{3}{8}\right)\right]$$

First, perform the division in the bracket by converting it to a multiplication by the reciprocal. The quotient becomes negative.

$$=-3\left[\frac{3}{2}-\frac{5}{\cancel{6}_{3}}\cdot\frac{\cancel{8}^{4}}{3}\right]$$

Reduce, before multiplying.

$$=-3\left(\frac{3}{2}-\frac{20}{9}\right)$$

Extend both fractions to higher terms using the common denominator of 18.

$$=-3\left(\frac{27-40}{18}\right)$$

Perform subtraction.

$$=-\cancel{3}\left(\frac{-13}{\cancel{18}}\right)$$

Reduce before multiplying. The product becomes positive.

$$=\mathbf{\frac{13}{6}}\text{ or equivalently }\mathbf{2\frac{1}{6}}.$$

R.3 Exercises

True or False?

1. The set of integers is closed under multiplication.

2. The set of natural numbers is closed under subtraction.

3. The set of real numbers different than zero is closed under division.

4. According to the BEDMAS rule, division should be performed before multiplication.

5. For any real number $\sqrt{x^2} = x$.

6. Square root of a negative number is not a real number.

7. If the value of a square root exists, it is positive.

8. $-x^3 = (-x)^3$

9. $-x^2 = (-x)^2$

Complete each statement to illustrate the indicated property.

10. $x + (-y) =$ _________ , commutative property of addition

11. $(7 \cdot 5) \cdot 2 =$ ___________ , associative property of multiplication

12. $(3 + 8x) \cdot 2 =$ ______________ , distributive property of multiplication over addition

13. $a +$ _____ $= 0$, additive inverse

14. $-\frac{a}{b} \cdot$ _____ $= 1$, multiplicative inverse

15. $\frac{3x}{4y} \cdot$ _____ $= \frac{3x}{4y}$, multiplicative identity

16. _____ $+ (-a) = -a$, additive identity

17. $(2x - 7) \cdot$ ____ $= 0$, multiplication by zero

18. If $(x + 5)(x - 1) = 0$, then ________ $= 0$ or ________ $= 0$, zero product property

Perform operations.

19. $-\frac{2}{5} + \frac{3}{4}$

20. $\frac{5}{6} - \frac{2}{9}$

21. $\frac{5}{8} \cdot \left(-\frac{2}{3}\right) \cdot \frac{18}{15}$

22. $-3\left(-\frac{5}{9}\right)$

23. $-\frac{3}{4}(8x)$

24. $\frac{15}{16} \div \left(-\frac{9}{12}\right)$

Use order of operations to evaluate each expression.

25. $64 \div (-4) \div 2$

26. $3 + 3 \cdot 5$

27. $8 - 6(5 - 2)$

28. $20 + 4^3 \div (-8)$

29. $6\left(9 - 3\sqrt{9 - 5}\right)$

30. $-2^5 - 8 \div 4 - (-2)$

31. $-\frac{5}{6}+\left(-\frac{7}{4}\right)\div 2$ **32.** $\left(-\frac{3}{2}\right)\cdot\frac{1}{6}-\frac{2}{5}$ **33.** $-\frac{3}{2}\div\left(-\frac{4}{9}\right)-\frac{5}{4}\cdot\frac{2}{3}$

34. $-3\left(-\frac{4}{9}\right)-\frac{1}{4}\div\frac{3}{5}$ **35.** $2-3|3-4\cdot 6|$ **36.** $\frac{3|5-7|-6\cdot 4}{5\cdot 6-2|4-1|}$

Simplify each expression.

37. $-(x-y)$ **38.** $-2(3a-5b)$ **39.** $\frac{2}{3}(24x+12y-15)$

40. $\frac{3}{4}(16a-28b+12)$ **41.** $5x-8x+2x$ **42.** $3a+4b-5a+7b$

43. $5x-4x^2+7x-9x^2$ **44.** $8\sqrt{2}-5\sqrt{2}+\frac{1}{x}+\frac{3}{x}$ **45.** $2+3\sqrt{x}-6-\sqrt{x}$

46. $\frac{a-b}{b-a}$ **47.** $\frac{2(x-3)}{3-x}$ **48.** $-\frac{100ab}{75a}$

49. $-(5x)^2$ **50.** $\left(-\frac{2}{3}a\right)^2$ **51.** $5a-(4a-7)$

52. $6x+4-3(9-2x)$ **53.** $5x-4(2x-3)-7$

54. $8x-(-4y+7)+(9x-1)$ **55.** $6a-[4-3(9a-2)]$

56. $5\{x+3[4-5(2x-3)-7]\}$ **57.** $-2\{2+3[4x-3(5x+1)]\}$

58. $4\{[5(x-3)+5^2]-3[2(x+5)-7^2]\}$ **59.** $3\{[6(x+4)-3^3]-2[5(x-8)-8^2]\}$

Evaluate each algebraic expression for $a=-2$, $b=3$, and $c=2$.

60. b^2-a^2 **61.** $6c\div 3a$ **62.** $\frac{c-a}{c-b}$

63. $b^2-3(a-b)$ **64.** $\frac{-b+\sqrt{b^2-4ac}}{2a}$ **65.** $c\left(\frac{a}{b}\right)^{|a|}$

Determine whether each pair of expressions is equivalent.

66. $x^3\cdot x^2$ and x^5 **67.** a^2-b^2 and $(a-b)^2$

68. $\sqrt{x^2}$ and x **69.** $(x^3)^2$ and x^5

Use the distributive property to calculate each value mentally.

70. $96\cdot 18+4\cdot 18$ **71.** $29\cdot 70+29\cdot 30$

72. $57\cdot\frac{3}{5}-7\cdot\frac{3}{5}$ **73.** $\frac{8}{5}\cdot 17+\frac{8}{5}\cdot 13$

Insert one pair of parentheses to make the statement true.

74. $2\cdot 3+6\div 5-3=9$ **75.** $9\cdot 5+2-8\cdot 3+1=22$

Attributions

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