S.2 Arithmetic Sequences and Series

Some sequences consist of random values while other sequences follow a certain pattern that is used to arrive at the sequence's terms. In this section, we will take interest in sequences that follow the pattern of having a constant difference between their consecutive terms. Such sequences are called **arithmetic**. For example, the sequence $3, 5, 7, 9, \ldots$ is arithmetic because its consecutive terms always differ by 2.

Definition 2.1 ► A sequence $\{a_n\}$ is called **arithmetic** if the difference $d = a_{n+1} - a_n$ between any consecutive terms of the sequence is constantly the same.

The **general term** of an **arithmetic sequence** is given by the formula

$$a_n = a_1 + (n-1)d$$

The difference d is referred to as **the common difference** of the sequence.

Similarly to functions, sequences can be visualized by plotting their values in a system of coordinates. For instance, *Figure 1* presents the graph of the sequence $3, 5, 7, 9, \ldots$. Notice that the common difference of 2 makes the graph linear in nature. This is because the slope between the consecutive points of the graph is always the same. Generally, any arithmetic sequence follows a linear pattern with the **slope** being the **common difference** and the ***y*-intercept** being the **first term diminished by the common difference**, as illustrated by *Figure 1*. This means that we should be able to write the general term of an arithmetic sequence by following the slope-intercept equation of a line. Using this strategy, we obtain

$$a_n = \textit{slope} \cdot n + (y\textit{-intercept}) = dn + (a_1 - d) = a_1 + dn - d$$

$$= a_1 + (n-1)d$$

Particularly, the general term of the sequence $3, 5, 7, 9, \ldots$ is equal to $a_n = 3 + (n-1)2$, or equivalently to $a_n = 2n + 1$.

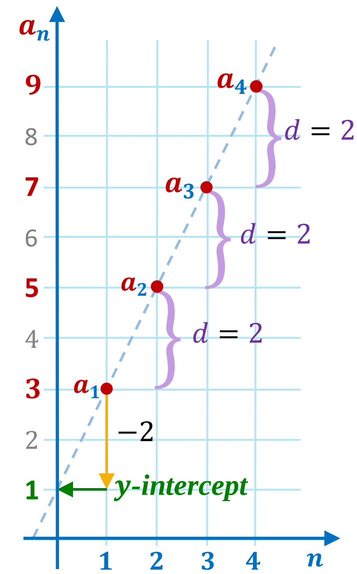

Figure 1

Example 1 ► **Identifying Arithmetic Sequences and Writing its General Term**

Determine whether the given sequence $\{a_n\}$ is arithmetic. If it is, then write a formula for the general term of the sequence.

a. $2, 4, 8, 16, \ldots$

b. $3, 1, -1, -3, \ldots$

Solution ►

a. Since the differences between consecutive terms, $a_2 - a_1 = 4 - 2 = 2$ and $a_3 - a_2 = 8 - 4 = 4$, are not the same, the sequence is not arithmetic.

b. Here, the differences between consecutive terms are constantly equal to -2, so the sequence is arithmetic with $a_1 = 3$, and the common difference $d = -2$. Therefore, using the formula for the general term $a_n = a_1 + (n-1)d$, we have

$$a_n = 3 + (n-1)(-2) = 3 - 2n + 2 = \mathbf{-2n + 5}.$$

Example 2 ► **Finding Terms of an Arithmetic Sequence**

Given the information, write out the first five terms of the arithmetic sequence $\{a_n\}$. Then, find the 10-th term a_{10}.

a. $a_n = 12 - 3n$ **b.** $a_1 = 3,\ d = 5$

Solution ► **a.** To find the first five terms of this sequence, we evaluate a_n for $n = 1,2,3,4,5$.

$$a_1 = 12 - 3 \cdot 1 = 9$$
$$a_2 = 12 - 3 \cdot 2 = 6$$
$$a_3 = 12 - 3 \cdot 3 = 3$$
$$a_4 = 12 - 3 \cdot 4 = 0$$
$$a_5 = 12 - 3 \cdot 5 = -3$$

So, the first five terms are $\mathbf{9}, \mathbf{6}, \mathbf{3}, \mathbf{0}$, and $\mathbf{-3}$.

The 10-th term equals $a_{10} = 12 - 3 \cdot 10 = \mathbf{-18}$.

b. To find the first five terms of an arithmetic sequence with $\boldsymbol{a_1} = \mathbf{3}$, $\boldsymbol{d} = \mathbf{5}$, we substitute these values into the general term formula

$$\boldsymbol{a_n} = \boldsymbol{a_1} + (\boldsymbol{n} - \mathbf{1})\boldsymbol{d} = \mathbf{3} + (\boldsymbol{n} - \mathbf{1})\mathbf{5},$$

and then evaluate it for $n = 1,2,3,4,5$.

This gives us

$$a_1 = 3 + 0 \cdot 5 = 3$$
$$a_2 = 3 + 1 \cdot 5 = 8$$
$$a_3 = 3 + 2 \cdot 5 = 13$$
$$a_4 = 3 + 3 \cdot 5 = 18$$
$$a_5 = 3 + 4 \cdot 5 = 23$$

So, the first five terms are $\mathbf{3}, \mathbf{8}, \mathbf{13}, \mathbf{18}$, and $\mathbf{23}$.

The 10-th term equals $a_{10} = 3 + 9 \cdot 5 = \mathbf{48}$.

Example 3 ► **Finding the Number of Terms in a Finite Arithmetic Sequence**

Determine the number of terms in the arithmetic sequence $1,5,9,13, \dots ,45$.

Solution ► Notice that the common difference d of this sequence is $5 - 1 = 4$ and the first term $a_1 = 1$. Therefore the n-th term $a_n = 1 + (n-1)4 = 4n - 3$. Since the last term is 45, we can set up the equation

$$a_n = 4n - 3 = 45, \quad \text{and solve it for } n.$$

This gives us

$$4n = 48,$$

and finally

$$n = 12.$$

So, there are 12 terms in the given sequence.

Example 4 ► **Finding Missing Terms of an Arithmetic Sequence**

Given the information, determine the values of the indicated terms of an arithmetic sequence.

a. $a_5 = 2$ and $a_7 = 8$; find a_6

b. $a_3 = 5$ and $a_{10} = -9$; find a_1 and a_{15}

Solution ►

a. Let d be the common difference of the given sequence. Since $a_7 = a_6 + d$ and $a_6 = a_5 + d$, then $a_7 = a_5 + 2d$. Hence, $2d = a_7 - a_5$, which gives

$$d = \frac{a_7 - a_5}{2} = \frac{8-2}{2} = 3.$$

Therefore,

$$\boldsymbol{a_6} = a_5 + d = 2 + 3 = \mathbf{5}.$$

Remark: *An **arithmetic mean** of two quantities **a** and **b** is defined as* $\frac{\boldsymbol{a}+\boldsymbol{b}}{\mathbf{2}}$.
Notice that $\boldsymbol{a_6} = 5 = \frac{2+8}{2} = \frac{\boldsymbol{a_5}+\boldsymbol{a_7}}{\mathbf{2}}$, *so* $\boldsymbol{a_6}$ *is indeed the arithmetic mean of* $\boldsymbol{a_5}$ *and* $\boldsymbol{a_7}$.
Generally, for any $n > 1$, *we have*

$$\boldsymbol{a_n} = a_{n-1} + d = \frac{2a_{n-1} + 2d}{2} = \frac{a_{n-1} + (a_{n-1} + 2d)}{2} = \frac{\boldsymbol{a_{n-1}} + \boldsymbol{a_{n+1}}}{\mathbf{2}},$$

so every term (except for the first one) of an arithmetic sequence is the arithmetic mean of its adjacent terms.

b. As before, let d be the common difference of the given sequence. Using the general term formula $a_n = a_1 + (n-1)d$ for $n = 10$ and $n = 3$, we can set up a system of two equations in two variables, d and a_1:

$$\begin{cases} -9 = a_1 + 9d & (1) \\ 5 = a_1 + 2d & (2) \end{cases}$$

To solve this system, we can subtract the two equations side by side, obtaining

$$-14 = 7d,$$

which gives

$$d = -2.$$

After substitution to equation (2), we have $5 = a_1 + 2 \cdot 2$, which allows us to find the value a_1:

$$\boldsymbol{a_1} = 5 - 4 = \mathbf{1}.$$

To find the value of a_{15}, we substitute $a_1 = 1, d = -2$, and $n = 15$ to the formula $a_n = a_1 + (n-1)d$ to obtain

$$\boldsymbol{a_{15}} = 1 + (15-1)(-2) = 2 - 28 = \mathbf{-26}.$$

Partial Sums

Sometimes, we are interested in evaluating the sum of the first n terms of a sequence. For example, we might be interested in finding a formula for the sum $S_n = 1 + 2 + \cdots n$ of the first n consecutive natural numbers. To do this, we can write this sum in increasing and decreasing order, as below.

$$\begin{array}{rcl} S_n &=& 1 + \quad 2 \quad + \cdots + (n-1) + n \\ S_n &=& n + (n-1) + \cdots + \quad 2 \quad + 1 \end{array}$$

Now, observe that the sum of terms in each column is always $(n+1)$, and there are n columns. Therefore, after adding the two equations side by side, we obtain:

$$2S_n = n(n+1),$$

which in turn gives us a very useful formula

$$S_n = \frac{\boldsymbol{n(n+1)}}{\mathbf{2}} \qquad (1)$$

for the sum of the first n consecutive natural numbers.

Figure 2 shows us a geometrical interpretation of this formula, for $n = 6$. For example, to find the area of the shape composed of blocks of heights from 1 to 6, we cut the shape at half the height and rearrange it to obtain a rectangle of length $6 + 1 = 7$ and height $\frac{6}{2} = 3$. This way, the area of the original shape equals to the area of the 7 by 3 rectangle, which according to equation (1), is calculated as $\frac{6(6+1)}{2} = \frac{6}{2} \cdot (6+1) = 3 \cdot 7 = 21$.

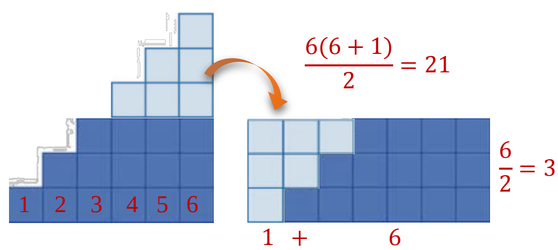

Figure 2

Formally, a partial sum of any sequence is defined as follows:

Definition 1.2 ► Let $\{a_n\}$ be a sequence and $a_1 + a_2 + \cdots + a_n + \cdots$ be its associated series.
The ***n*-th partial sum**, denoted $\boldsymbol{S_n}$, of the sequence (or the series) is the sum

$$\boldsymbol{a_1 + a_2 + \cdots + a_n}.$$

The overall sum of the entire series can be denoted by $\boldsymbol{S_\infty}$.
The **partial sums** on its own create a **sequence** $\{\boldsymbol{S_n}\}$.

Observation:

$$\boldsymbol{S_1 = a_1}$$

$$\boldsymbol{a_n = (a_1 + a_2 + \cdots + a_{n-1} + a_n) - (a_1 + a_2 + \cdots + a_{n-1}) = S_n - S_{n-1}}$$

To find the partial sum $\boldsymbol{S_n}$ of the first n terms of an **arithmetic sequence**, as before, we write it in increasing and decreasing order of terms and then add the resulting equations side by side.

$$\begin{array}{rcl} \boldsymbol{S_n} &=& \boldsymbol{a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + (a_1 + (n-1)d)} \\ \boldsymbol{S_n} &=& \boldsymbol{a_n + (a_n - d) + (a_n - 2d) + \cdots + (a_n - (n-1)d)} \\ \hline \end{array}$$

each column adds to $\boldsymbol{a_1 + a_n}$ *and there are* $\boldsymbol{n}$ *columns*

So, we obtain $\quad \boldsymbol{2S_n = n(a_1 + a_n)},$

which gives us

$$\boldsymbol{S_n = \frac{n(a_1 + a_n)}{2}} \qquad (2)$$

Notice that by substituting of the general term $a_n = a_1 + (n-1)d$ into the above formula, we can express the partial sum S_n in terms of the first term a_1 and the common difference d, as follows:

$$\boldsymbol{S_n = \frac{n(2a_1 + (n-1)d)}{2} \overset{\text{or equivalently}}{=} \frac{n}{2}(2a_1 + (n-1)d)} \qquad (3)$$

Example 5 ► **Finding a Partial Sum of an Arithmetic Sequence**

a. Find the sum of the first 100 consecutive natural numbers.
b. Find S_{20}, for the sequence $-10, -5, 0, 5, \dots$.
c. Evaluate the sum $2 + (-1) + (-4) + \cdots + (-25)$.

Solution ► **a.** Using the formula (1) for $n = 100$, we have

$$\boldsymbol{S_{100}} = \frac{100 \cdot (100+1)}{2} = 50 \cdot 101 = \mathbf{5050}.$$

So the sum of the first 100 consecutive natural numbers is 5050.

b. To find S_{20}, we can use either formula (2) or formula (3). We are given $n = 20$ and $a_1 = -10$. To use formula (2) it is enough to calculate a_{20}. Since $d = 5$, we have

$$\boldsymbol{a_{20}} = a_1 + 19d = -10 + 19 \cdot 5 = \mathbf{85},$$

which gives us

$$\boldsymbol{S_{20}} = \frac{20(-10+85)}{2} = 10 \cdot 75 = \mathbf{750}.$$

Alternatively, using formula (3), we also have

$$\boldsymbol{S_{20}} = \frac{20}{2}(2(-10) + 19 \cdot 5) = 10(-20 + 95) = 10 \cdot 75 = \mathbf{750}.$$

c. This time, we are given $a_1 = 2$ and $a_n = -25$, but we need to figure out the number of terms n. To do this, we can use the n-th term formula $a_1 + (n-1)d$ and equal it to -25. Since $d = -1 - 2 = -3$, then we have

$$2 + (n-1)(-3) = -25,$$

which becomes

$$(n-1) = \frac{-27}{-3}$$

and finally

$$n = 10.$$

Now, using formula (2), we evaluate the requested sum to be

$$\boldsymbol{S_{10}} = \frac{10\big(2 + (-25)\big)}{2} = 5 \cdot (-23) = \mathbf{-115}.$$

As we saw in the beginning of this section, an **arithmetic sequence** is **linear** in nature and, as such, it can be identified by the formula $a_n = dn + b$, where $n \in \mathbb{N}$, $d, b \in \mathbb{R}$, and $b = a_1 - d$. This means that the n-th partial sum $S_n = a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_n$ of the associated arithmetic series can be written as

$$\sum_{i=1}^{n}(di + b),$$

and otherwise; each such sum represents the n-th partial sum S_n of an arithmetic series with the first term $d + b$ and the common difference d. Therefore, the above sum can be evaluated with the aid of formula (2), as shown in the next example.

Example 6 ► **Evaluating Finite Arithmetic Series Given in Sigma Notation**

Evaluate the sum $\sum_{i=1}^{16}(2i - 1)$.

Solution ► First, notice that the sum $\sum_{i=1}^{16}(2i - 1)$ represents S_{16} of an arithmetic series with the general term $a_n = 2n - 1$. Since $a_1 = 2 \cdot 1 - 1 = 1$ and $a_{16} = 2 \cdot 16 - 1 = 31$, then applying formula (2), we have

$$\sum_{i=1}^{16}(2i - 1) = \frac{16(1 + 31)}{2} = 8 \cdot 32 = \mathbf{256}.$$

Example 7 ► **Using Arithmetic Sequences and Series in Application Problems**

A worker is stacking wooden logs in layers. Each layer contains three logs less than the layer below it. There are two logs in the topmost layer, five logs in the next layer, and so on. There are 7 layers in the stack.

a. How many logs are in the bottom layer?
b. How many logs are in the entire stack?

Solution ►

a. First, we observe that the number of logs in consecutive layers, starting from the top, can be expressed by an arithmetic sequence with $a_1 = 2$ and $d = 3$. Since we look for the number of logs in the seventh layer, we use $n = 7$ and the formula

$$a_n = 2 + (n - 1)3 = 3n - 1.$$

This gives us $a_7 = 3 \cdot 7 - 1 = \mathbf{20}$.
Therefore, there are 20 wooden logs in the bottom layer.

b. To find the total number of logs in the stack, we can evaluate the 7-th partial sum $\sum_{i=1}^{7}(3i - 1)$. Using formula (2), we have

$$\sum_{i=1}^{7}(3i - 1) = \frac{7(2 + 20)}{2} = 7 \cdot 11 = \mathbf{77}.$$

So, the entire stack consists of 77 wooden logs.

S.1 Exercises

Vocabulary Check *Fill in each blank with one of the suggested words, or the most appropriate term or phrase from the given list:* ***arithmetic, consecutive natural, difference, general, linear, partial sum, sigma.***

1. A sequence with a common difference between consecutive terms is called an ________________ sequence.

2. The sequence $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ $\underset{is\ /is\ not}{\underline{\hspace{2cm}}}$ arithmetic because the _______________ between consecutive terms is not the same.

3. The _______________ term of an arithmetic sequence is given by the formula $a_n = a_1 + (n-1)d$.

4. A graph of an arithmetic sequence follows a _____________ pattern, therefore the general term of this sequence can be written in the fom $a_n = dn + b$.

5. The n-th ______________ _______ of a sequence is the sum of its first n terms. Partial sums can be written using _____________ notation.

6. The formula $\frac{n(n+1)}{2}$ allows for calculation of the sum of the first n ____________ __________ numbers.

Concept Check *True or False?*

7. The sequence $3, 1, -1, -3, \dots$ is an arithmetic sequence.

8. The common difference for $2, 4, 2, 4, 2, 4, \dots$ is 2.

9. The series $\sum_{i=1}^{12}(3 + 2i)$ is an arithmetic series.

10. The n-th partial sum S_n of any series can be calculated according to the formula $S_n = \frac{n(a_1+a_n)}{2}$.

Concept Check *Write a formula for the n-th term of each arithmetic sequence.*

11. $1, 3, 5, 7, 9, \dots$
12. $0, 6, 12, 18, 24, \dots$
13. $-4, -2, 0, 2, 4, \dots$
14. $5, 1, -3, -7, -11, \dots$
15. $-2, -\frac{3}{2}, -1, -\frac{1}{2}, 0, \dots$
16. $1, \frac{5}{3}, \frac{7}{3}, 3, \frac{11}{3}, \dots$

Concept Check *Given the information, write out the first five terms of the arithmetic sequence $\{a_n\}$. Then, find the 12-th term a_{12}.*

17. $a_n = 3 + (n-1)(-2)$
18. $a_n = 3 + 5n$
19. $a_1 = -8,\ d = 4$
20. $a_1 = 5,\ d = -2$
21. $a_1 = 10,\ a_2 = 8$
22. $a_1 = -7,\ a_2 = 3$

Concept Check *Find the number of terms in each arithmetic sequence.*

23. $3, 5, 7, 9, \dots, 31$

24. $0, 5, 10, 15, \dots, 55$

25. $4, 1, -2, \dots, -32$

26. $-3, -7, -11, \dots, -39$

27. $-2, -\frac{3}{2}, -1, -\frac{1}{2}, \dots, 5$

28. $\frac{3}{4}, 3, \frac{21}{4}, \dots, 12$

Given the information, find the indicated term of each arithmetic sequence.

29. $a_2 = 5,\ d = 3;\quad a_8$

30. $a_3 = -4,\ a_4 = -6;\quad a_{20}$

31. $1, 5, 9, 13, \dots;\quad a_{50}$

32. $6, 3, 0, -3, \dots;\quad a_{25}$

33. $a_1 = -8,\ a_9 = -64;\quad a_{10}$

34. $a_1 = 6,\ a_{18} = 74;\quad a_{20}$

35. $a_8 = 28,\ a_{12} = 40;\quad a_1$

36. $a_{10} = -37,\ a_{12} = -45;\quad a_2$

Given the arithmetic sequence, evaluate the indicated partial sum.

37. $a_n = 3n - 8;\quad S_{12}$

38. $a_n = 2 - 3n;\quad S_{16}$

39. $6, 3, 0, -3, \dots;\quad S_9$

40. $1, 6, 11, 16, \dots;\quad S_{15}$

41. $a_1 = 4,\ d = 3;\quad S_{10}$

42. $a_1 = 6,\ a_4 = -2;\quad S_{19}$

Use a formula for S_n to evaluate each series.

43. $1 + 2 + 3 + \cdots + 25$

44. $2 + 4 + 6 + \cdots + 50$

45. $\sum_{i=1}^{17} 3i$

46. $\sum_{i=1}^{22} (5i + 4)$

47. $\sum_{i=1}^{15} \left(\frac{1}{2}i + 1\right)$

48. $\sum_{i=1}^{20} (4i - 7)$

49. $\sum_{i=1}^{25} (-3 - 2i)$

50. $\sum_{i=1}^{13} \left(\frac{1}{4} + \frac{3}{4}i\right)$

Analytic Skills *Solve each problem.*

51. The sum of the interior angles of a triangle is 180º, of a quadrilateral is 360º and of a pentagon is 540º. Assuming this pattern continues, find the sum of the interior angles of a dodecagon (*12-sided figure*).

52. Deanna's aunt has promised to deposit \$1 in her account on the first day of her birthday month, \$2 on the second day, \$3 on the third day, and so on for 30 days. How much will this amount to over the entire month?

53. Ben is learning to drive. His first lesson is 26 minutes long, and each subsequent lesson is 4 minutes longer than the lesson before.

a. How long will his 15-th lesson be?

b. Overall, how long will Ben's training be after his 15-th lesson?

54. Suppose you visit the Grand Canyon and drop a penny off the edge of a cliff. The distance the penny will fall is 16 feet the first second, 48 feet the next second, 80 feet the third second, and so on in an arithmetic progression. What is the total distance the object will fall in 6 seconds?

55. If a contractor does not complete a multimillion-dollar construction project on time, he must pay a penalty of \$500 for the first day that he is late, \$700 for the second day, \$900 for the third day, and so on. Each day the penalty is \$200 larger than the previous day.

a. Write a formula for the penalty on the n-th day.

b. What is the penalty for the 10-th day?

c. If the contractor completes the project 14 days late, then what is the total amount of the penalties that the contractor must pay?

56. On the first day of October, an English teacher suggests to his students that they read five pages of a novel and every day thereafter increase their daily reading by two pages. If his students follow this suggestion, then how many pages will they read during October?