S.3 Geometric Sequences and Series

In the previous section, we studied sequences where each term was obtained by adding a constant number to the previous term. In this section, we will take interest in sequences where each term is obtained by multiplying the previous term by a constant number. Such sequences are called **geometric**. For example, the sequence $1, 2, 4, 8, \ldots$ is geometric because each term is multiplied by 2 to obtain the next term. Equivallently, the ratios between consecutive terms of this sequence are always 2.

Definition 2.1 ► A sequence $\{a_n\}$ is called **geometric** if the quotient $r = \frac{a_{n+1}}{a_n}$ of any consecutive terms of the sequence is constantly the same.

The **general term** of a **geometric sequence** is given by the formula

$$a_n = a_1 r^{n-1}$$

The quotient r is referred to as **the common ratio** of the sequence.

Similarly as in the previous section, we can be visualize geometric sequences by plotting their values in a system of coordinates. For instance, *Figure 1* presents the graph of the sequence $1, 2, 4, 8, \ldots$. The common ratio of 2 causes each cosecutive point of the graph to be plotted twice as high as the previous one, and the slope between the n-th and $(n+1)$-st point to be exactly equal to the value of a_n. Generally, the **slope** between the n-th and $(n+1)$-st point of any geometric sequence **is proportional** to the value of a_n. This property characterises exponential functions. Hence, geometric sequences are **exponential** in nature. To develop the formula for the general term, we observe the pattern

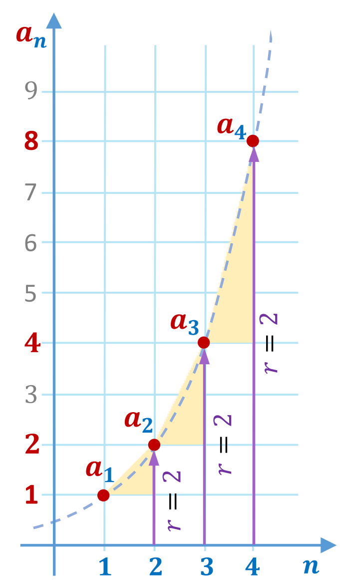


Figure 1

$$\begin{aligned} a_1 &= a_1 \\ a_2 &= a_1 r \\ a_3 &= a_1 r^2 \\ a_4 &= a_1 r^3 \\ &\vdots \end{aligned}$$

so

$$\boxed{a_n = a_1 r^{n-1}}$$

This is the **general term** of a geometric sequence

Particularly, the general term of the sequence $1, 2, 4, 8, \ldots$ is equal to $a_n = 2^{n-1}$, because $a_1 = 1$ and $r = 2$ (the ratios of consecutive terms are constantly equal to 2).

Note: *To find the common ratio of a geometric sequence, divide any of its terms by the preceeding term.*

Example 1 ► **Identifying Geometric Sequences and Writing Their General Terms**

Determine whether the given sequence $\{a_n\}$ is geometric. If it is, then write a formula for the general term of the sequence.

a. $\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \ldots$

b. $\frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \frac{1}{12}, \ldots$

Solution ► **a.** After calculating ratios of terms by their peceding terms, we notice that they are always equal to $-\frac{1}{2}$. Indeed, $\frac{a_2}{a_1} = \frac{-\frac{1}{4}}{\frac{1}{3}} = -\frac{1}{2}$, $\frac{a_3}{a_2} = \frac{\frac{1}{8}}{-\frac{1}{4}} = -\frac{1}{2}$, and so on. Therefore, the given sequence is geometric with $a_1 = \frac{1}{2}$ and the common ratio $r = \frac{1}{2}$.
To find its general term, we follow the formula $a_n = a_1 r^{n-1}$. This gives us

$$\boldsymbol{a_n} = \frac{1}{2}\left(-\frac{1}{2}\right)^{n-1} = \boldsymbol{\frac{(-1)^{n-1}}{2^n}}.$$

b. Here, the ratios of terms by their peceding terms, $\frac{a_2}{a_1} = \frac{\frac{1}{6}}{\frac{1}{3}} = \frac{1}{2}$ and $\frac{a_3}{a_2} = \frac{\frac{1}{9}}{\frac{1}{6}} = \frac{2}{3}$, are not the same. So the sequence is not geometric.

Example 2 ► **Finding Terms of a Gometric Sequence**

Given the information, write out the first five terms of the geometric sequence $\{a_n\}$. Then, find the 8-th term a_8.

a. $a_n = 5(-2)^{n-1}$ **b.** $a_1 = 3,\ r = \frac{2}{3}$

Solution ► **a.** To find the first five terms of this sequence, we evaluate a_n for $n = 1,2,3,4,5$.

$$\begin{aligned} a_1 &= 5(-2)^0 = 5 \\ a_2 &= 5(-2)^1 = -10 \\ a_3 &= 5(-2)^2 = 20 \\ a_4 &= 5(-2)^3 = -40 \\ a_5 &= 5(-2)^4 = 80 \end{aligned}$$

So, the first five terms are $\mathbf{5}, \mathbf{-10}, \mathbf{20}, \mathbf{-40}$, and $\mathbf{80}$.

The 8-th term equals $a_8 = 5(-2)^7 = \mathbf{-640}$.

b. To find the first five terms of a geometric sequence with $\boldsymbol{a_1} = \mathbf{3}$, $\boldsymbol{r} = \frac{\mathbf{2}}{\mathbf{3}}$, we substitute these values into the general term formula

$$\boldsymbol{a_n} = \boldsymbol{a_1 r^{n-1}} = \mathbf{3}\left(\frac{\mathbf{2}}{\mathbf{3}}\right)^{\boldsymbol{n-1}},$$

and then evaluate it for $n = 1,2,3,4,5$.

This gives us

$$\begin{aligned} a_1 &= 3\left(\frac{2}{3}\right)^0 = 3 \\ a_2 &= 3\left(\frac{2}{3}\right)^1 = 2 \\ a_3 &= 3\left(\frac{2}{3}\right)^2 = \frac{4}{3} \\ a_4 &= 3\left(\frac{2}{3}\right)^3 = \frac{8}{9} \\ a_5 &= 3\left(\frac{2}{3}\right)^4 = \frac{16}{27} \end{aligned}$$

So, the first five terms are $\mathbf{3}, \mathbf{2}, \frac{\mathbf{4}}{\mathbf{3}}, \frac{\mathbf{8}}{\mathbf{9}}$, and $\frac{\mathbf{16}}{\mathbf{27}}$.

The 8-th term equals $a_8 = 3\left(\frac{2}{3}\right)^7 = \frac{\mathbf{128}}{\mathbf{6561}}$.

Example 3 ► **Finding the Number of Terms in a Finite Geometric Sequence**

Determine the number of terms in the geometric sequence $1, -3, 9, -27, \dots, 729$.

Solution ► Since the common ratio r of this sequence is -3 and the first term $a_1 = 1$, then the n-th term $a_n = (-3)^{n-1}$. Since the last term is 729, we can set up the equation

$$a_n = (-3)^{n-1} = 729,$$

which can be written as

$$(-3)^{n-1} = (-3)^6$$

This equation holds if $\quad n - 1 = 6,$

which gives us $\quad n = 7.$

So, there are 7 terms in the given sequence.

Example 4 ► **Finding Missing Terms of a Geometric Sequence**

Given the information, determine the values of the indicated terms of a geometric sequence.

a. $a_3 = 5$ and $a_6 = -135$; find a_1 and a_8

b. $a_3 = 200$ and $a_5 = 50$; find a_4 if $a_4 > 0$

Solution ► **a.** As before, let r be the common ratio of the given sequence. Using the general term formula $a_n = a_1 r^{n-1}$ for $n = 6$ and $n = 3$, we can set up a system of two equations in two variables, r and a_1:

$$\begin{cases} -135 = a_1 r^5 \\ 5 = a_1 r^2 \end{cases}$$

To solve this system, let's divide the two equations side by side, obtaining

$$-27 = r^3,$$

which gives us

$$r = -3.$$

Substituting this value to equation (2), we have $5 = a_1 \cdot (-3)^2$, which gives us

$$\boldsymbol{a_1 = \frac{5}{9}}.$$

To find value a_8, we substitute $a_1 = -\frac{5}{2}, r = -3$, and $n = 8$ to the formula $a_n = a_1 r^{n-1}$. This gives us

$$\boldsymbol{a_8} = \frac{5}{9}(-3)^7 = \boldsymbol{-1215}.$$

b. Let r be the common ratio of the given sequence. Since $a_5 = a_4 r$ and $a_4 = a_3 r$, then $a_5 = a_3 r^2$. Hence, $r^2 = \frac{a_5}{a_3}$. Therefore,

$$r = \pm\sqrt{\frac{a_5}{a_3}} = \pm\sqrt{\frac{50}{200}} = \pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2}. \qquad \begin{array}{r}(1)\\(2)\end{array}$$

Since $a_3, a_4 > 0$ and $a_4 = a_3 r$, we choose the positive r-value. So we have

$$\boldsymbol{a_4} = a_3 r = 200\left(\frac{1}{2}\right) = \mathbf{100}.$$

Remark: *A **geometric mean** of two quantities $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined as $\sqrt{\boldsymbol{ab}}$. Notice that $\boldsymbol{a_4} = 100 = \sqrt{50 \cdot 200} = \sqrt{\boldsymbol{a_3 \cdot a_5}}$, so $\boldsymbol{a_4}$ is indeed the geometric mean of $\boldsymbol{a_3}$ and $\boldsymbol{a_5}$. Generally, for any $n > 1$, we have*

$$\boldsymbol{a_n} = a_{n-1}r = \sqrt{a_{n-1}^2 r^2} = \sqrt{a_{n-1} \cdot a_{n-1} r^2} = \sqrt{\boldsymbol{a_{n-1} \cdot a_{n+1}}},$$

so every term (except for the first one) of a geometric sequence is the geometric mean of its adjacent terms.

Partial Sums

Similarly as with arithmetic sequences, we might be interested in evaluating the sum of the first n terms of a geometric sequence.

To find partial sum $\boldsymbol{S_n}$ of the first n terms of a **geometric sequence**, we line up formulas for $\boldsymbol{S_n}$ and $\boldsymbol{-rS_n}$ as shown below and then add the resulting equations side by side.

$$\begin{array}{rcl} \boldsymbol{S_n} & = & \boldsymbol{a_1} \boxed{+\, \boldsymbol{a_1 r}} \boxed{+\, \boldsymbol{a_1 r^2}} + \cdots \boxed{+\, \boldsymbol{a_1 r^{n-1}}} \\ \boldsymbol{-rS_n} & = & \qquad \boxed{-\boldsymbol{a_1 r}} \boxed{-\, \boldsymbol{a_1 r^2}} - \cdots \boxed{-\, \boldsymbol{a_1 r^{n-1}}} - \boldsymbol{a_1 r^n} \\ \hline \end{array}$$

the terms of inside columns add to zero, so they cancel each other out

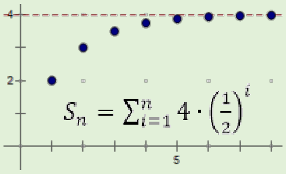
So, we obtain $\quad \boldsymbol{(1 - r)S_n} = \boldsymbol{a_1 - a_1 r^n},$

which gives us

$$\boxed{\boldsymbol{S_n} = \frac{\boldsymbol{a_1(1 - r^n)}}{\boldsymbol{1 - r}},} \qquad (3)$$

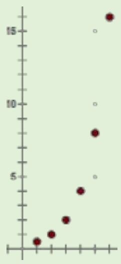
as long as $r \neq 1$.

Observe that



If $|\boldsymbol{r}| < \mathbf{1}$, then the value of r^n gets closer and closer to zero for larger and larger n (we write: $r^n \to 0$ for $n \to \infty$). This means that the sum of all infinitely many terms of such a sequence **exists** and is equal to

$$\boldsymbol{S_\infty} = \frac{\boldsymbol{a_1}}{\mathbf{1} - \boldsymbol{r}} \qquad (4)$$



$S_n = \sum_{i=1}^{n} \frac{1}{2} \cdot (2)^i$

If $|r| > 1$, then the value of $|r^n|$ grows without bound for larger and larger n. Therefore, the sum S_∞ of all terms of such a sequence does not have a finite value. We say that such a sum does not exist.

If $|r| = 1$, then the sum S_∞ becomes $a_1 + a_1 + a_1 + \cdots$, or $a_1 - a_1 + a_1 - \cdots$. Neither of these sums has a finite value, unless $a_1 = 0$.

Hence overall, if $|\boldsymbol{r}| \geq \mathbf{1}$, then the sum $\boldsymbol{S_\infty}$ of a nonzero geometric sequence **does not exist**.

Example 5 ► **Finding a Partial Sum of a Geometric Sequence**

a. Find S_6, for the geometric sequence with $a_1 = 0.5$ and $r = 0.1$.

b. Evaluate the sum $1 - \left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)^2 - \cdots - \left(\frac{3}{4}\right)^9$.

Solution ►

a. Using formula (3) for $n = 6$, $a_1 = 0.5$ and $r = 0.1$, we calculate

$$\boldsymbol{S_6} = \frac{0.5(1 - 0.1^6)}{1 - 0.1} = \frac{0.5 \cdot 0.999999}{0.9} = \mathbf{0.555555}.$$

b. First, we observe that the given series is geometric with $a_1 = 1$ and $r = -\frac{3}{4}$. Equating the formula for the general term to the last term of the sum

$$a_1 r^{n-1} = \left(-\frac{3}{4}\right)^{n-1} = -\left(\frac{3}{4}\right)^9 = \left(-\frac{3}{4}\right)^9$$

and comparing the exponents,

$$n - 1 = 9$$

allows us to find the number of terms $n = \mathbf{10}$.

Now, we are ready to calculate the sum of the given series

$$\boldsymbol{S_{10}} = \frac{1\left(1 - \left(-\frac{3}{4}\right)^{10}\right)}{1 - \left(-\frac{3}{4}\right)} \cong \mathbf{0.539249}$$

Example 6 ► **Evaluating Infinite Geometric Series**

Decide wether or not the overall sum S_∞ of each geometric series exists and if it does, evaluate it.

a. $3 - \frac{9}{2} + \frac{27}{4} - \frac{81}{8} + \cdots$

b. $\sum_{i=0}^{\infty} 3 \cdot \left(\frac{2}{3}\right)^i$

Solution ►

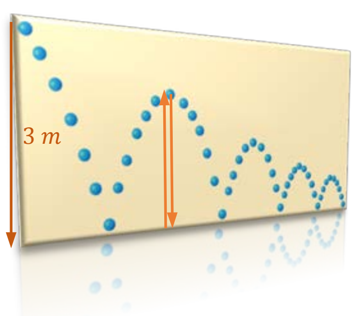
a. Since the common ratio of this series is $|r| = \left|\frac{-\frac{9}{2}}{3}\right| = \left|-\frac{3}{2}\right| = \frac{3}{2} > 1$, then the sum S_∞ does not exist.

b. This time, $|r| = \frac{2}{3} < 1$, so the sum S_∞ exists and can be calculated by following the formula (4). Using $a_1 = 3$ and $r = \frac{2}{3}$, we have

$$S_\infty = \frac{3}{1 - \frac{2}{3}} = \frac{3}{\frac{1}{3}} = \mathbf{9}$$

So, $\sum_{i=0}^{\infty} 3 \cdot \left(\frac{2}{3}\right)^i = \mathbf{9}$.

Example 4 ► **Using Geometric Sequences and Series in Application Problems**



When dropped from a certain height, a ball rebounds $\frac{3}{4}$ of the original height.

a. How high will the ball rebound after the fourth bounce if it was dropped from a height of 3 meters? *Round the answer to the nearest centimeter.*

b. Find a formula for the rebound height of the ball after its n-th bounce.

c. Assuming that the ball bounce forever, what is the total vertical distance traveled by the ball?

Solution ►

a. Let h_n represents the ball's rebound height after the n-th bounce, where $n \in \mathbb{N}$. Since the ball rebounds $\frac{3}{4}$ of the previous height, we have

$$h_1 = 3 \cdot \left(\frac{3}{4}\right)$$

$$h_2 = h_1 \cdot \left(\frac{3}{4}\right) = 3 \cdot \left(\frac{3}{4}\right)^2$$

$$h_3 = h_2 \cdot \left(\frac{3}{4}\right) = 3 \cdot \left(\frac{3}{4}\right)^3$$

$$h_4 = h_3 \cdot \left(\frac{3}{4}\right) = 3 \cdot \left(\frac{3}{4}\right)^4 \simeq .949\, m \simeq \mathbf{95\ cm}$$

After the fourth bounce, the ball will rebound approximately 95 centimeters.

b. Notice that the formulas developed in solution to *Example 4a* follow the pattern

$$h_n = 3 \cdot \left(\frac{3}{4}\right)^n$$

So this is the formula for the rebound height of the ball after its n-th bounce.

c. Let $h_0 = 3$ represents the vertical distance before the first bounce. To find the total verical distance D traveled by the ball, we add the vertical distance h_0 before the first bounce, and twice the vertical distances h_n after each bounce. So we have

$$D = h_0 + \sum_{n=1}^{\infty} h_n = 3 + \sum_{n=1}^{\infty} 3 \cdot \left(\frac{3}{4}\right)^n$$

Applying the formula $\frac{\boldsymbol{a_1}}{\boldsymbol{1-r}}$ for the infinite sum of a geometric series, we calculate

$$D = 3 + \frac{\frac{9}{4}}{1 - \frac{3}{4}} = 3 + \frac{\frac{9}{4}}{\frac{1}{4}} = 3 + \frac{9}{\cancel{4}} \cdot \frac{\cancel{4}}{1} = 3 + 9 = \boldsymbol{12\ m}$$

Thus, the total vertical distance traveled by the ball is 12 meters.

S.3 Exercises

Vocabulary Check *Fill in each blank with appropriate formula, one of the suggested words, or with the most appropriate term or phrase from the given list:* ***consecutive, geometric mean, ratio.***

1. A geometric sequence is a sequence in which there is a constant ___________ between consecutive terms.

2. The sequence $1, 2, 6, 24, \ldots$ $\underset{is\ /\ is\ not}{\underline{\qquad\qquad}}$ geometric because the ratio between _______________terms is not the same.

3. The general term of a geometric sequence is given by the formula $a_n =$ ___________.

4. If the absolute value of the common ratio of a geometric sequence is $\underset{less\ /\ more}{\underline{\qquad\qquad}}$ than ____, then the sum of the associated geometric series exists and it is equal to $S_\infty =$ ___________ .

5. The expression $\sqrt{ab}$ is called the ________________ ____________ of a and b.

Concept Check *True or False?*

6. The sequence $3, -1, \frac{1}{3}, -\frac{1}{9}, \ldots$ is a geometric sequence.

7. The common ratio for $0.05, 0.0505, 0.050505, \ldots$ is 0.05 .

8. The series $\sum_{i=1}^{7}(3 \cdot 2^i)$ is a geometric series.

9. The n-th partial sum S_n of any finite geometric series exists and it can be evaluated by using the formula $S_n = \frac{a_1(1-r^n)}{1-r}$.

Concept Check *Identify whether or not the given sequence is geometric. If it is, write a formula for its n-th term.*

10. $0, 3, 9, 27, \ldots$

11. $1, 5, 25, 125, \ldots$

12. $-9, 3, -1, \frac{1}{3}, \ldots$

13. $\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}, \ldots$

14. $1, -1, 1, -1, \ldots$

15. $0.9, 0.09, 0.009, 0.0009, \ldots$

16. $81, -27, 9, -3, \ldots$

17. $\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{8}{3}, \ldots$

18. $-\frac{1}{4}, -\frac{1}{5}, -\frac{4}{25}, -\frac{16}{125}, \ldots$

Concept Check *Given the information, write out the first four terms of the geometyric sequence $\{a_n\}$. Then, find the 8-th term a_8.*

19. $a_n = 3 \cdot 2^{n-1}$

20. $a_n = (-2)^{-n}$

21. $a_1 = 6,\ r = \frac{1}{3}$

22. $a_1 = 5,\ r = -1$

23. $a_1 = \frac{1}{3},\ a_2 = -\frac{1}{6}$

24. $a_1 = 100,\ a_2 = 10$

Concept Check *Find the number of terms in each geometric sequence.*

25. $1, 2, 4, \ldots, 1024$

26. $20, 10, 5, \ldots, \frac{5}{128}$

27. $-4, 2, -1, \ldots, \frac{1}{32}$

28. $3, -1, \frac{1}{3}, \ldots, \frac{1}{243}$

29. $6, -2, \frac{2}{3}, \ldots, -\frac{2}{81}$

30. $-24, 12, -6, \ldots, -\frac{3}{32}$

Given the information, find the indicated term of each geometric sequence.

31. $a_2 = 40,\ r = 0.1;\quad a_5$

32. $a_3 = 4,\ a_4 = -8;\quad a_{10}$

33. $2, -2, 2, -2, \ldots;\quad a_{50}$

34. $-4, 2, -1, \ldots;\quad a_{12}$

35. $a_1 = 6,\ a_4 = -\frac{2}{9};\quad a_8$

36. $a_1 = \frac{1}{9},\ a_6 = 27;\quad a_9$

37. $a_3 = \frac{1}{2},\ a_7 = \frac{1}{32};\quad a_4$ if $a_4 > 0$

38. $a_5 = 48,\ a_8 = -384;\quad a_{10}$

Given the geometric sequence, evaluate the indicated partial sum. Round your answer to three decimal places, if needed.

39. $a_n = 5\left(\frac{2}{3}\right)^{n-1};\quad S_6$

40. $a_n = -2\left(\frac{1}{4}\right)^{n-1};\quad S_{10}$

41. $2, 6, 18, \ldots;\quad S_8$

42. $6, 3, \frac{3}{2}, \ldots;\quad S_{12}$

43. $1 + \left(\frac{1}{5}\right) + \left(\frac{1}{5}\right)^2 + \cdots + \left(\frac{1}{5}\right)^5$

44. $1 - 3 + 3^2 - \cdots - 3^9$

45. $\sum_{i=1}^{7} 2(1.05)^{i-1}$

46. $\sum_{i=1}^{10} 3(2)^{i-1}$

Decide wether or not the infinite sum S_∞ of each geometric series exists and if it does, evaluate it.

47. $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots$

48. $1-\frac{5}{4}+\frac{25}{16}-\frac{125}{64}+\cdots$

49. $1+1.02+1.02^2+1.02^3+\cdots$

50. $1+0.8+0.8^2+0.8^3+\cdots$

51. $\sum_{i=1}^{\infty}(0.6)^{i-1}$

52. $\sum_{i=1}^{\infty}\frac{2}{5}(1.1)^{i-1}$

53. $\sum_{i=1}^{\infty}2\left(\frac{4}{3}\right)^{i}$

54. $\sum_{i=1}^{\infty}2\left(-\frac{3}{4}\right)^{i}$

Analytic Skills *Solve each problem.*

55. A company is offering a job with a salary of \$30,000 for the first year and a 5% raise each year after that. If that 5% raise continues every year, find the amount of money you would earn in the 10th year of your career.

56. Suppose you go to work for a company that pays one penny on the first day, 2 cents on the second day, 4 cents on the third day and so on. If the daily wage keeps doubling, what will your wage be on the 30th day? What will your total income be for working 30 days?

57. Suppose a deposit of \$2000 is made at the beginning of each year for 45 years into an account paying 12% compounded annually. What is the amount in the account at the end of the forty-fifth year?

58. A father opened a savings account for his daughter on her first birthday, depositing \$1000. Each year on her birthday he deposits another \$1000, making the last deposit on her 21st birthday. If the account pays 4.4% interest compounded annually, how much is in the account at the end of the day on the daughter's 21st birthday?

59. A ball is dropped from a height of 2 m and bounces 90% of its original height on each bounce.

a. How high off the floor is the ball at the top of the eighth bounce?

b. Asuming that the ball moves only vertically, how far has it traveled when it hits the ground for the eighth time?

60. Suppose that a ball always rebounds $\frac{2}{3}$ of the distance from which it falls. If this ball is dropped from a height of 9 ft, then approximately how far does it travel before coming to rest? Assime that the ball moves only vertically.

61. Suppose the midpoints of a **unit square** s_1 (*with the length of each side equal to one*) are connected to form another square, s_2, as in the accompanying figure. Suppose we continue indefinitely the process of creating a new square, s_{n+1}, by connecting the midpoints of the previous square, s_n. Calculate the sum of the areas of the infinite sequence of squares $\{s_n\}$.