Sequences and Series

In everyday life, we can observe sequences or series of events in many contexts. For instance, we line up to enter a store in a sequence, we make a sequence of mortgage payments, or we observe a series of events that lead to a particular outcome. In this section, we will consider mathematical definitions for sequences and series, and explore some applications of these concepts.

S.1 Sequences and Series

Think of a **sequence of numbers** as an **ordered list of numbers**. For example, the waiting time, in minutes, of each person standing in line to Tim Horton's to be served

$$0, 1, 2, 2, 3, 5, 5, 5, 7, 9, 10, 11$$

or the number of bacteria in a colony after each hour, if the colony starts with one bacteria and each bacteria divides into two every hour

$$1, 2, 4, 8, 16, 32, \ldots, 2^{n-1}, \ldots$$

The first example illustrates a finite sequence, while the second example shows an infinite sequence. Notice that numbers listed in a sequence, called **terms**, can repeat, like in the first example, or they can follow a certain pattern, like in the second example. If we can recognize the pattern of the listed terms, it is convenient to state it as a general rule by listing the n-th term. The sequence of numbers in our second example shows consecutive powers of two, starting with 2^0, so the n-th term of this sequence is 2^{n-1}.

Sequences as Functions

Formally, the definition of sequence can be stated by using the terminology of functions.

Definition 1.1 ► An **infinite sequence** is a function whose **domain** is the set of all **natural numbers**.
A **finite sequence** is a function whose **domain** is the first n natural numbers $\{\mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots, \boldsymbol{n}\}$.
The **terms** (or elements) of a sequence are the function values, the entries of the ordered list of numbers.
The **general term** of a sequence is its n-th term.

Notation: Customarily, sequence functions assume names such as a, b, c, rather than f, g, h.
If the name of a sequence function is a, than the function values (the **terms** of the sequence) are denoted $\boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_3}, \ldots$ rather than $a(1), a(2), a(3), \ldots$.
The **index** $\boldsymbol{k}$ in the notation $\boldsymbol{a_k}$ indicates the position of the term in the sequence.
$\boldsymbol{a_n}$ denotes the **general term** of the sequence and $\{\boldsymbol{a_n}\}$ represents the entire sequence.

Example 1 ► **Finding Terms of a Sequence When Given the General Term**

Given the sequence $a_n = \frac{n-1}{n+1}$, find the following

a. the first four terms of $\{a_n\}$ **b.** the 12-th term a_{12}

Solution ► **a.** To find the first four terms of the given sequence, we evaluate a_n for $n = 1,2,3,4$.

We have
$$a_1 = \frac{1-1}{1+1} = 0$$
$$a_2 = \frac{2-1}{2+1} = \frac{1}{3}$$
$$a_3 = \frac{3-1}{3+1} = \frac{2}{4} = \frac{1}{2}$$
$$a_4 = \frac{4-1}{4+1} = \frac{3}{5}$$

so the first four terms are $0, \frac{1}{3}, \frac{1}{2}, \frac{3}{5}$.

b. The twelfth term is $a_{12} = \frac{12-1}{12+1} = \frac{11}{13}$.

Example 2 ► **Finding the General Term of a Sequence**

Determine the expression for the general term a_n of the sequence

a. $3, 9, 27, 81, \ldots$

b. $-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \ldots$

Solution ► **a.** Observe that all terms of the given sequence are powers of 3.

We have
$$a_1 = 3 = 3^1$$
$$a_2 = 9 = 3^2$$
$$a_3 = 27 = 3^3$$
$$a_4 = 81 = 3^4, \text{ and so on.}$$

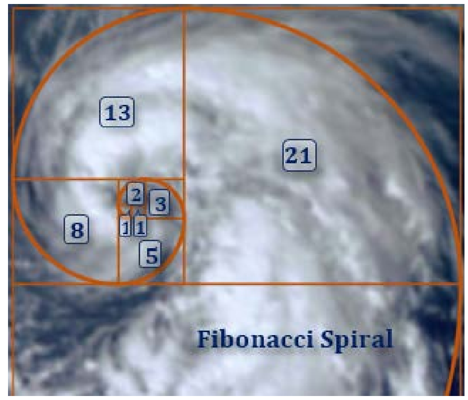
Notice that in each term, the exponent of 3 is the same as the index of the term. The above pattern suggests the candidate $a_n = 3^n$ for the general term of this sequence. To convince oneself that this is indeed the general term, one may want to generate the given terms with the aid of the developed formula $a_n = 3^n$. If the generated terms match the given ones, the formula is correct.

b. Here, observe that the signs of the given terms alter, starting with the negative sign. While building a formula for the general term, to accommodate for this change in signs, we may want to use the factor of $(-1)^n$. This is because
$$(-1)^n = \begin{cases} -1, & \text{for } n = 1, 3, 5, \ldots \\ 1\ , & \text{for } n = 2, 4, 6, \ldots \end{cases}$$
Then we observe that all terms may be seen as fractions with the numerator equal to 1 and denominator matching the index of the term,

$$a_1 = -1 = (-1)^1 \frac{1}{1}$$
$$a_2 = \frac{1}{2} = (-1)^2 \frac{1}{2}$$
$$a_3 = -\frac{1}{3} = (-1)^3 \frac{1}{3}$$
$$a_4 = \frac{1}{4} = (-1)^4 \frac{1}{4}$$
$$a_5 = -\frac{1}{5} = (-1)^5 \frac{1}{5}, \text{ and so on.}$$

The above pattern suggests that the formula $a_n = (-1)^n \frac{1}{n}$ would work for the general term of this sequence. As before, please convince yourself that this is indeed the general term of the sequence by generating the given terms with the aid of the suggested formula.

Sometimes it is difficult to describe a sequence by stating the explicit formula for its general term. For example, in the case of the **Fibonacci sequence** $1, 1, 2, 3, 5, 8, 13, 21, \dots$, one can observe the rule of obtaining the next term by adding the previous two terms (for terms after the second term), but it would be very difficult to come up with an explicit formula for the general term a_n. Yet the Fibonacci sequence can be defined through the following equations $a_1 = a_2 = 1$ and $a_n = a_{n-2} + a_{n-1}$, for $n \geq 3$. Notice that the n-th term is not given explicitly but it can be found as long as the previous terms are known. In such a case we say that the sequence is **defined recursively**.



Definition 1.2 ► A sequence is defined **recursively** if
- the initial term or terms are given, and
- the n-th term is defined by a formula that refers to the *preceding* terms.

Example 3 ► **Finding Terms of a Sequence Given Recursively**

Find the first 5 terms of the sequence given by the conditions
$a_1 = 1$, $a_2 = 2$, and $a_n = 2a_{n-1} + a_{n-2}$, for $n \geq 3$.

Solution ► The first two terms are given, $a_1 = \mathbf{1}$, $a_2 = \mathbf{2}$. To find the third term, we substitute $n = 3$ into the recursive formula, to obtain

$$\begin{aligned} a_3 &= 2a_{3-1} + a_{3-2} \\ &= 2a_2 + a_1 = 2 \cdot \mathbf{2} + \mathbf{1} = \mathbf{5} \end{aligned}$$

Similarly
$$\begin{aligned} a_4 &= 2a_{4-1} - a_{4-2} \\ &= 2a_3 - a_2 = 2 \cdot \mathbf{5} + \mathbf{2} = \mathbf{12} \end{aligned}$$

and
$$\begin{aligned} a_5 &= 2a_{5-1} - a_{5-2} \\ &= 2a_4 - a_3 = 2 \cdot \mathbf{12} + \mathbf{5} = \mathbf{29} \end{aligned}$$

So the first five terms of this sequence are: **1**, **2**, **5**, **12**, and **29**.

Example 4 ► **Using Sequences in Application Problems**

Peter borrowed \$6000. To pay off this debt, the lender requests monthly payments of \$600 and 1% interest of the unpaid balance from the previous month. If his first payment is due one month from the date of borrowing, find

a. the total number of payments needed to pay off the debt,
b. the sequence of his first four payments,
c. the general term of the sequence of payments,
d. the last payment.

Solution ►

a. Since Peter pays \$600 off his \$6000 principal each time, the total number of payments is $\frac{6000}{600} = \mathbf{10}$.

b. Let $a_1, a_2, \dots, a_{10}$ be the sequence of Peter's payments.
After the first month, Peter pays $a_1 = \$600 + 0.01 \cdot \$6000 = \mathbf{\$660}$ and the remaining balance becomes $\$6000 - \$600 = \$5400$.
Then, Peter's second payment is $a_2 = \$600 + 0.01 \cdot \$5400 = \mathbf{\$654}$ and the remaining balance becomes $\$5400 - \$600 = \$4800$.
The third payment is equal to $a_3 = \$600 + 0.01 \cdot \$4800 = \mathbf{\$648}$ and the remaining balance becomes $\$4800 - \$600 = \$4200$.
Finally, the fourth payment is $a_4 = \$600 + 0.01 \cdot \$4200 = \mathbf{\$642}$ with the remaining balance of $\$4200 - \$600 = \$3600$.

So the sequence of Peter's first four payments is $\mathbf{\$660, \$654, \$648, \$642.}$

c. Notice that the terms of the above sequence diminish by 6.
We have
$$a_1 = 660 = 660 - \mathbf{0} \cdot 6$$
$$a_2 = 654 = 660 - \mathbf{1} \cdot 6$$
$$a_3 = 648 = 660 - \mathbf{2} \cdot 6$$
$$a_4 = 642 = 660 - \mathbf{3} \cdot 6, \text{ and so on.}$$

Since the blue coefficient by "6" is one lower than the index of the term, we can write the general term as $a_n = 660 - (\boldsymbol{n - 1}) \cdot 6$, which after simplifying can take the form $a_n = 660 - 6n + 6 = \mathbf{666 - 6n}$.

d. Since there are 10 payments, the last one equals to $a_{10} = 666 - 6 \cdot 10 = \$\mathbf{606}$.

Series and Summation Notation

Often, we take interest in finding sums of terms of a sequence. For instance, in *Example 4*, we might be interested in finding the total amount paid in the first four months $\$550 + \$545 + \$540 + \$535$, or the total cost of borrowing $\$550 + \$545 + \cdots + \$505$. The terms of a sequence connected by the operation of addition create an expression called a **series**.

Note: *The word "**series**" is both singular and plural.*

Definition 1.3 ► A **series** is the sum of terms of a finite or infinite sequence, before evaluation.
The value of a **finite series** can always be determined because addition of a finite number of values can always be performed.
The value of an **infinite series** may not exist. For example, $\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \cdots = 1$ but $1 + 2 + \cdots + n + \cdots = DNE$ (doesn't exist).

Series involve writing sums of many terms, which is often cumbersome. To write such sums in compact form, we use summation notation referred to as **sigma notation**, where the Greek letter $\boldsymbol{\Sigma}$ (sigma) is used to represent the operation of adding all the terms of a sequence. For example, the finite series $1^2 + 2^2 + 3^2 + \cdots + 10^2$ can be recorded in sigma notation as

$$\sum_{i=1}^{10} i^2 \quad \text{or} \quad \sum\nolimits_{i=1}^{10} i^2$$

Here, the letter i is called the **index of summation** and takes integral values from 1 to 10. The expression i^2 (the general term of the corresponding sequence) generates the terms being added. The number 1 is the lower limit of the summation, and the number 10 is the upper limit of the summation. We read "the sum from $i = 1$ to 10 of i^2." To find this sum, we replace the letter i in i^2 with 1, 2, 3, …, 10, and add the resulting terms.

Note: *Any letter can be used for the index of summation; however, the most commonly used letters are i,j,k,m,n.*

A finite series with an unknown number of terms, such as $1 + 2 + \cdots + n$, can be recorded as

$$\sum\nolimits_{i=1}^{n} i$$

Here, since the last term equals to n, the value of the overall sum is an expression in terms of n, rather than a specific number.

An infinite series, such as $0.3 + 0.03 + 0.003 + \cdots$ can be recorded as

$$\sum\nolimits_{i=1}^{\infty} \frac{3}{10^i}$$

In this case, the series can be evaluated and its sum equals to $0.333\ldots = \frac{1}{3}$.

Example 5 ► **Evaluating Finite Series Given in Sigma Notation**

Evaluate the sum.

a. $\sum_{i=1}^{5}(2i + 1)$

b. $\sum_{k=1}^{6}(-1)^k \frac{1}{k}$

Solution ► **a.** $\sum_{i=0}^{5}(2i + 1) = (2 \cdot 0 + 1) + (2 \cdot 1 + 1) + (2 \cdot 2 + 1) + (2 \cdot 3 + 1) + (2 \cdot 4 + 1)$

$= 1 + 3 + 5 + 7 + 9 = \mathbf{25}$

b. $\sum_{k=1}^{6}(-1)^k \frac{1}{k} = (-1)^1 \frac{1}{1} + (-1)^2 \frac{1}{2} + (-1)^3 \frac{1}{3} + (-1)^4 \frac{1}{4} + (-1)^5 \frac{1}{5} + (-1)^6 \frac{1}{6}$

$= -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} = \frac{-60+30-20+15-12+10}{60} = -\frac{\mathbf{37}}{\mathbf{60}}$

Example 6 ► **Writing Series in Sigma Notation**

Write the given series using sigma notation.

a. $5 + 7 + 9 + \cdots + 47 + 49$

b. $\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \cdots$

Solution ► **a.** Observe that the series consists of a sequence of odd integers, from 5 to 49. An odd integer can be represented by the expression $2n + 1$ or $2n - 1$. The first expression assumes a value of 5 for $n = 2$, and a value of 49 for $n = 24$. Therefore, the general term of the series could be written as $a_n = 2n + 1$, for $n = 2,3, \ldots ,24$. Hence, the series might be written in the form

$$\sum_{i=2}^{24}(2i+1)$$

Using the second expression, $2n-1$, the general term $a_n = 2n-1$ would work for $n = 3,4, \dots ,25$. Hence, the series might also be written in the form

$$\sum_{i=3}^{25}(2i-1)$$

The sequence can also be written with the index of summation set to start with 1. Then we would have

$$\sum_{i=1}^{23}(2i+3)$$

Check that all of the above sigma expressions produce the same series.

b. In this infinite series $\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\cdots$ the signs of consecutive terms alter. To accommodate for the change of signs, we may want to use a factor of $(-1)^n$ or $(-1)^{n+1}$, depending on the sign of the first term. Since the first term is positive, we use the factor of $(-1)^{n+1}$ that equals to 1 for $n=1$. In addition, the terms consist of fractions with constant numerators equal to 1 and denominators equal to consecutive even numbers that could be represented by $2n$. Hence, the series might be written in the form

$$\sum_{i=1}^{\infty}(-1)^{i+1}\frac{1}{2i}$$

Notice that by renaming the index of summation to, for example, $k = i-1$, the series takes the form

$$\sum_{k=0}^{\infty}(-1)^{k}\frac{1}{2(k+1)}$$

Check on your own that both of the above sigma expressions produce the same series.

Observation: *Series in sigma notation can be written in many different yet equivalent forms. This is because the starting value of the index of summation is arbitrary. Commonly, we start at 1, or 0, unless other values make the general term formula simpler.*

Example 7 ► **Adjusting the Index of Summation**

In each series, change index j to index k that starts at 1.

a. $\sum_{j=2}^{7}(-1)^{j-1}j^3$ **b.** $\sum_{j=0}^{\infty}3^{2j-1}$

Solution ► **a.** If $k=1$ when $j=2$, then $j-k=1$, or equivalently $j=k+1$. In this relation, the upper limit $j=7$ corresponds to $k=6$. So by substitution, we obtain

$$\sum_{j=2}^{7}(-1)^{j-1}j^3 = \sum_{k=1}^{6}(-1)^{k+1-1}(k+1)^3 = \sum_{k=1}^{6}(-1)^{k}(k+1)^3$$

b. If $k = 1$ when $j = 0$, then $k - j = 1$, or equivalently $j = k - 1$. By substitution, we have

$$\sum_{j=0}^{\infty} 3^{2j-1} = \sum_{k=1}^{\infty} 3^{2(k-1)-1} = \sum_{k=1}^{\infty} \mathbf{3^{2k-3}}$$

Example 8 ► **Using Series in Application Problems**

In reference to Example 4 of this section:

Peter borrowed \$6000. To pay off this debt, the lender requests monthly payments of \$300 and 1% interest of the unpaid balance from the previous month. If his first payment is due one month from the date of borrowing, find

a. the sequence $\{b_n\}$, where b_n represents the remaining balance before the n-th payment,
b. the total interest paid by Peter.

Solution ►

a. As indicated in the solution to *Example 4b*, the sequence of monthly balances before the n-th payment is $6000, 5400, 4800, \ldots, 600$. Since the balance decreases each month by 600, the general term of this sequence is

$$\boldsymbol{b_n} = 6000 - (n-1)600 = \mathbf{6600 - 600}\boldsymbol{n}.$$

b. Since Peter pays 1% on the unpaid balance b_n each month and the number of payments is 20, the total interest paid can be represented by the series

$$\sum_{k=1}^{10} (0.01 \cdot b_k) = \sum_{k=1}^{10} [0.01 \cdot (6600 - 600k)] = \sum_{k=1}^{10} (66 - 6k)$$

$$= 60 + 54 + 48 + 42 + 36 + 30 + 24 + 18 + 12 + 6 = 330$$

Therefore, Peter paid the total interest of **\$330**.

Arithmetic Mean

When calculating the final mark in a course, we often take an average of a sequence of marks we received on assignments, quizzes, or tests. We do this by adding all the marks and dividing the sum into the number of marks used. This average gives us some information about the overall performance on the particular task.

We are often interested in finding averages in many other life situations. For example, we might like to know the average number of calories consumed per day, the average lifetime of particular appliances, the average salary of a particular profession, the average yield from a crop, the average height of a Christmas tree, etc.
In mathematics, the commonly used term "**average**" is called the **arithmetic mean** or just **mean**. If a mean is calculated for a large set of numbers, it is convenient to write it using summation notation.

Definition 1.4 ► The **arithmetic mean (average)** of the numbers $\boldsymbol{x_1, x_2, x_3, \ldots, x_n}$, often denoted $\boldsymbol{\bar{x}}$, is given by the formula

$$\boldsymbol{\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}}$$

Example 9 ► **Calculating the Arithmetic Mean**

a. Find the arithmetic mean of $-12, 3, 0, 4, -2, 10$.

b. Evaluate the mean $\bar{x} = \frac{\sum_{i=1}^{5} i^2}{5}$.

Solution ► **a.** The arithmetic mean equals

$$\frac{-12 + 3 + 0 + 4 + (-2) + 10}{6} = \frac{3}{6} = \boldsymbol{\frac{1}{2}}$$

b. To evaluate this mean by may want to write out all the terms first. We have

$$\bar{x} = \frac{\sum_{i=1}^{5} i^2}{5} = \frac{1^2 + 2^2 + 3^2 + 4^2 + 5^2}{5} = \frac{1 + 4 + 9 + 16 + 25}{5} = \frac{55}{5} = \boldsymbol{11}$$

S.1 Exercises

*Find the **first four terms** and the **10-th** term of each infinite sequence whose n-th term is given.*

1. $a_n = 2n - 3$

2. $a_n = \frac{n+2}{n}$

3. $a_n = (-1)^{n-1}3n$

4. $a_n = 1 - \frac{1}{n}$

5. $a_n = \frac{(-1)^n}{n^2}$

6. $a_n = (n+1)(2n-3)$

7. $a_n = (-1)^n(n-2)^2$

8. $a_n = \frac{1}{n(n+1)}$

9. $a_n = \frac{(-1)^{n+1}}{2n-1}$

Write a formula for the n-th term of each sequence.

10. $2, 5, 8, 11, \ldots$

11. $1, -1, 1, -1, \ldots$

12. $\frac{\pi}{12}, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \ldots$

13. $3, 9, 27, 81, \ldots$

14. $15, 10, 5, 0, \ldots$

15. $6, 9, 12, 15, \ldots$

16. $-1, \frac{1}{4}, -\frac{1}{9}, \frac{1}{25}, \ldots$

17. $1, -\frac{1}{8}, \frac{1}{27}, -\frac{1}{64}, \ldots$

18. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

Find the first five terms of each infinite sequence given by a recursion formula.

19. $a_n = 2a_{n-1} + 5,\ a_1 = -3$

20. $a_n = 1 - \frac{1}{a_{n-1}},\ a_1 = 2$

21. $a_n = 2a_{n-1} + a_{n-2},\ a_1 = 1, a_2 = 2$

22. $a_n = (a_{n-1})^2 - 1,\ a_1 = 2$

Solve each applied problem by writing the first few terms of a sequence.

23. Lucy borrowed \$2600. To pay off her debt, she agreed to make monthly payments of \$200 and 2% interest on the unpaid balance from the previous month. If her first payment is due one month from the date of borrowing, find her first five payments and the remaining balance at the end of that period.

24. This year, Max was hired by a company that offered him a salary of $24{,}800 + 800(n-1)$ dollars per year at the beginning of the n-th year.

a) Write a sequence representing Max's salary for the first 5 years of work.

b) What is his salary increase per year?

c) Write the first five terms of a sequence of the percent increase in his yearly salary. Round the percentages up to two decimal places.

d) How much would he earn during the tenth year of work?

25. Suppose a penalty for not returning a book to the library on time is 50 cents for the first day plus 30 cents for each additional day after the due date. Write a formula for the penalty p_n on returning the book n days after the due day. What is the penalty for returning the book two weeks late?

26. It is estimated that the value of a one-year-old passenger car depreciates at a 10% annual rate. Corina bought a one-year-old Honda Accord in 2018 for \$19,600.

a. Applying the 10% depreciation rule, determine the value of this car in the years 2019 through 2022. *Round these values to the nearest dollar.*

b. Write a formula for the sequence p_n representing the value of this car n years after 2018.

c. Using the formula from part (**b**), approximate the value of this car in 2030.

27. It is advised that the chlorine level in a swimming pool water is kept between 1 and 3 ppm (parts per million). Assume that the chlorine level decreases by approximately 25% per day.

a. If the chlorine was initially at $a_0 = 3$ ppm level and no chlorine was added, construct a sequence a_n that expresses the amount of chlorine present in the pool water after n days.

b. In how many days the chlorine needs to be added to the pool because its level drops below 1 ppm?

Evaluate each sum.

28. $\sum_{i=1}^{5}(i+2)$

29. $\sum_{i=1}^{10} 5$

30. $\sum_{i=1}^{8}(-1)^i i$

31. $\sum_{i=1}^{4}(-1)^i(2i-1)$

32. $\sum_{i=1}^{6}(i^2-1)$

33. $\sum_{i=3}^{7} 2^i$

34. $\sum_{i=0}^{5}(i-1)\,i$

35. $\sum_{i=2}^{5}\frac{(-1)^i}{i}$

36. $\sum_{i=0}^{7}(i-2)\,(i-3)$

Write each series using sigma notation.

37. $2 + 4 + 6 + 8 + 10 + 12$

38. $3 - 6 + 9 - 12 + 15 - 18$

39. $\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\cdots+\frac{1}{50}$

40. $1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\cdots-\frac{1}{100}$

41. $1+8+27+64+\cdots$

42. $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\cdots$

In each series change index ***n*** *to index* ***m*** *that starts at* ***1****.*

43. $\sum_{n=0}^{9}(3n-1)$

44. $\sum_{n=3}^{7} 2^{n-2}$

45. $\sum_{n=2}^{6} \frac{n}{n+2}$

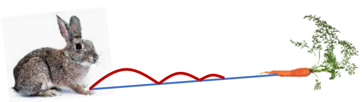
46. $\sum_{n=0}^{\infty}(-1)^{n+1}n$

47. $\sum_{n=2}^{\infty}(-1)^{n}(n-1)$

48. $\sum_{n=2}^{\infty} \frac{1}{n^2}$

Use a series to model the situation in each of the following problems.

49. A rabbit sees a carrot 2 meters away from him. Since he suspects that this could be a trap, he moves slowly towards the carrot by jumping one-third of the way and looking around to determine if it is safe to get closer. He repeats this strategy several times, each time jumping one-third of the remaining distance. For the first three jumps, calculate the length of each jump. Observe the pattern and then, using summation notation, write a series representing the total distance the rabbit has moved after six such jumps.



50. Sonia plans to put \$2000 at the beginning of each year into an account that pays 6% interest compounded annually. Using summation notation, write a series that represents Sonia's predicted savings in this account at the end of the

a. third year;

b. tenth year.

Find the ***arithmetic mean*** *of each sequence, or evaluate the mean written in sigma notation.*

51. $10, 12, 8, 0, 2, 19, 23, 6$

52. $5, -9, 8, 2, -4, 7, 5$

53. $\bar{x} = \frac{\sum_{i=-5}^{5} i}{11}$

54. $\bar{x} = \frac{\sum_{i=1}^{8} 2^i}{8}$

S.2 Arithmetic Sequences and Series

Some sequences consist of random values while other sequences follow a certain pattern that is used to arrive at the sequence's terms. In this section, we will take interest in sequences that follow the pattern of having a constant difference between their consecutive terms. Such sequences are called **arithmetic**. For example, the sequence $3, 5, 7, 9, \ldots$ is arithmetic because its consecutive terms always differ by 2.

Definition 2.1 ► A sequence $\{a_n\}$ is called **arithmetic** if the difference $\boldsymbol{d} = \boldsymbol{a_{n+1}} - \boldsymbol{a_n}$ between any consecutive terms of the sequence is constantly the same.

The **general term** of an **arithmetic sequence** is given by the formula

$$\boldsymbol{a_n = a_1 + (n-1)d}$$

The difference $\boldsymbol{d}$ is referred to as **the common difference** of the sequence.

Similarly to functions, sequences can be visualized by plotting their values in a system of coordinates. For instance, *Figure 1* presents the graph of the sequence $3, 5, 7, 9, \ldots$. Notice that the common difference of 2 makes the graph linear in nature. This is because the slope between the consecutive points of the graph is always the same. Generally, any arithmetic sequence follows a linear pattern with the **slope** being the **common difference** and the $\boldsymbol{y}$**-intercept** being the **first term diminished by the common difference**, as illustrated by *Figure 1*. This means that we should be able to write the general term of an arithmetic sequence by following the slope-intercept equation of a line. Using this strategy, we obtain

$$\boldsymbol{a_n} = \textbf{\textit{slope}} \cdot \boldsymbol{n} + (\boldsymbol{y}\textbf{\textit{-intercept}}) = dn + (a_1 - d) = a_1 + dn - d$$
$$= \boldsymbol{a_1 + (n-1)d}$$

Particularly, the general term of the sequence $3, 5, 7, 9, \ldots$ is equal to $a_n = 3 + (n-1)2$, or equivalently to $\boldsymbol{a_n = 2n + 1}$.

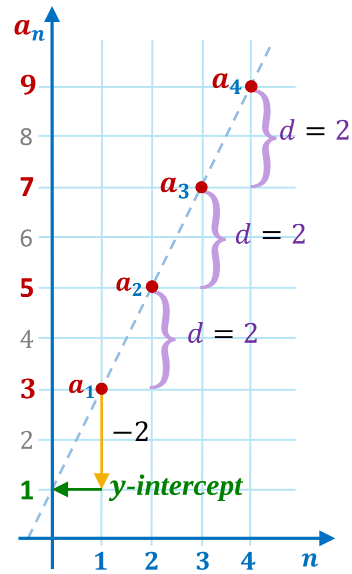


Figure 1

Example 1 ► **Identifying Arithmetic Sequences and Writing its General Term**

Determine whether the given sequence $\{a_n\}$ is arithmetic. If it is, then write a formula for the general term of the sequence.

a. $2, 4, 8, 16, \ldots$ **b.** $3, 1, -1, -3, \ldots$

Solution ►

a. Since the differences between consecutive terms, $a_2 - a_1 = 4 - 2 = 2$ and $a_3 - a_2 = 8 - 4 = 4$, are not the same, the sequence is not arithmetic.

b. Here, the differences between consecutive terms are constantly equal to -2, so the sequence is arithmetic with $a_1 = 3$, and the common difference $d = -2$. Therefore, using the formula for the general term $a_n = a_1 + (n-1)d$, we have

$$\boldsymbol{a_n} = 3 + (n-1)(-2) = 3 - 2n + 2 = \boldsymbol{-2n + 5}.$$

Example 2 ► **Finding Terms of an Arithmetic Sequence**

Given the information, write out the first five terms of the arithmetic sequence $\{a_n\}$. Then, find the 10-th term a_{10}.

a. $a_n = 12 - 3n$ **b.** $a_1 = 3,\ d = 5$

Solution ► **a.** To find the first five terms of this sequence, we evaluate a_n for $n = 1,2,3,4,5$.

$$a_{\mathbf{1}} = 12 - 3 \cdot \mathbf{1} = 9$$
$$a_{\mathbf{2}} = 12 - 3 \cdot \mathbf{2} = 6$$
$$a_{\mathbf{3}} = 12 - 3 \cdot \mathbf{3} = 3$$
$$a_{\mathbf{4}} = 12 - 3 \cdot \mathbf{4} = 0$$
$$a_{\mathbf{5}} = 12 - 3 \cdot \mathbf{5} = -3$$

So, the first five terms are $\mathbf{9}, \mathbf{6}, \mathbf{3}, \mathbf{0}$, and $-\mathbf{3}$.

The 10-th term equals $a_{\mathbf{10}} = 12 - 3 \cdot \mathbf{10} = -\mathbf{18}$.

b. To find the first five terms of an arithmetic sequence with $\boldsymbol{a_1} = \mathbf{3}$, $\boldsymbol{d} = \mathbf{5}$, we substitute these values into the general term formula

$$\boldsymbol{a_n} = \boldsymbol{a_1} + (\boldsymbol{n} - \mathbf{1})\boldsymbol{d} = \mathbf{3} + (\boldsymbol{n} - \mathbf{1})\mathbf{5},$$

and then evaluate it for $n = 1,2,3,4,5$.

This gives us
$$a_{\mathbf{1}} = 3 + \mathbf{0} \cdot 5 = 3$$
$$a_{\mathbf{2}} = 3 + \mathbf{1} \cdot 5 = 8$$
$$a_{\mathbf{3}} = 3 + \mathbf{2} \cdot 5 = 13$$
$$a_{\mathbf{4}} = 3 + \mathbf{3} \cdot 5 = 18$$
$$a_{\mathbf{5}} = 3 + \mathbf{4} \cdot 5 = 23$$

So, the first five terms are $\mathbf{3}, \mathbf{8}, \mathbf{13}, \mathbf{18}$, and $\mathbf{23}$.

The 10-th term equals $a_{\mathbf{10}} = 3 + 9 \cdot 5 = \mathbf{48}$.

Example 3 ► **Finding the Number of Terms in a Finite Arithmetic Sequence**

Determine the number of terms in the arithmetic sequence $1,5,9,13, \dots ,45$.

Solution ► Notice that the common difference d of this sequence is $5 - 1 = 4$ and the first term $a_1 = 1$. Therefore the n-th term $a_n = 1 + (n-1)4 = 4n - 3$. Since the last term is 45, we can set up the equation

$$a_n = 4n - 3 = 45, \quad \text{and solve it for } n.$$

This gives us
$$4n = 48,$$

and finally
$$n = 12.$$

So, there are 12 terms in the given sequence.

Example 4 ► **Finding Missing Terms of an Arithmetic Sequence**

Given the information, determine the values of the indicated terms of an arithmetic sequence.

a. $a_5 = 2$ and $a_7 = 8$; find a_6

b. $a_3 = 5$ and $a_{10} = -9$; find a_1 and a_{15}

Solution ►

a. Let d be the common difference of the given sequence. Since $a_7 = a_6 + d$ and $a_6 = a_5 + d$, then $a_7 = a_5 + 2d$. Hence, $2d = a_7 - a_5$, which gives

$$d = \frac{a_7 - a_5}{2} = \frac{8-2}{2} = 3.$$

Therefore,

$$\boldsymbol{a_6} = a_5 + d = 2 + 3 = \mathbf{5}.$$

Remark: *An **arithmetic mean** of two quantities $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined as $\boldsymbol{\frac{a+b}{2}}$.*

Notice that $\boldsymbol{a_6} = 5 = \frac{2+8}{2} = \boldsymbol{\frac{a_5+a_7}{2}}$, so $\boldsymbol{a_6}$ is indeed the arithmetic mean of $\boldsymbol{a_5}$ and $\boldsymbol{a_7}$. Generally, for any $n > 1$, we have

$$\boldsymbol{a_n} = a_{n-1} + d = \frac{2a_{n-1} + 2d}{2} = \frac{a_{n-1} + (a_{n-1} + 2d)}{2} = \boldsymbol{\frac{a_{n-1} + a_{n+1}}{2}},$$

so every term (except for the first one) of an arithmetic sequence is the arithmetic mean of its adjacent terms.

b. As before, let d be the common difference of the given sequence. Using the general term formula $a_n = a_1 + (n-1)d$ for $n = 10$ and $n = 3$, we can set up a system of two equations in two variables, d and a_1:

$$\begin{cases} -9 = a_1 + 9d & (1) \\ 5 = a_1 + 2d & (2) \end{cases}$$

To solve this system, we can subtract the two equations side by side, obtaining

$$-14 = 7d,$$

which gives

$$d = -2.$$

After substitution to equation (2), we have $5 = a_1 + 2 \cdot 2$, which allows us to find the value a_1:

$$\boldsymbol{a_1} = 5 - 4 = \mathbf{1}.$$

To find the value of a_{15}, we substitute $a_1 = 1, d = -2$, and $n = 15$ to the formula $a_n = a_1 + (n-1)d$ to obtain

$$\boldsymbol{a_{15}} = 1 + (15-1)(-2) = 2 - 28 = \mathbf{-26}.$$

Partial Sums

Sometimes, we are interested in evaluating the sum of the first n terms of a sequence. For example, we might be interested in finding a formula for the sum $S_n = 1 + 2 + \cdots n$ of the first n consecutive natural numbers. To do this, we can write this sum in increasing and decreasing order, as below.

$$\begin{array}{rcccccccc} S_n = & 1 & + & 2 & + \cdots + & (n-1) & + & n \\ S_n = & n & + & (n-1) & + \cdots + & 2 & + & 1 \end{array}$$

Now, observe that the sum of terms in each column is always $(n+1)$, and there are n columns. Therefore, after adding the two equations side by side, we obtain:

$$2S_n = n(n+1),$$

which in turn gives us a very useful formula

$$\boldsymbol{S_n = \frac{n(n+1)}{2}} \tag{1}$$

for the sum of the first n consecutive natural numbers.

Figure 2 shows us a geometrical interpretation of this formula, for $n = 6$. For example, to find the area of the shape composed of blocks of heights from 1 to 6, we cut the shape at half the height and rearrange it to obtain a rectangle of length $6 + 1 = 7$ and height $\frac{6}{2} = 3$. This way, the area of the original shape equals to the area of the 7 by 3 rectangle, which according to equation (1), is calculated as $\frac{6(6+1)}{2} = \frac{6}{2} \cdot (6+1) = 3 \cdot 7 = 21$.

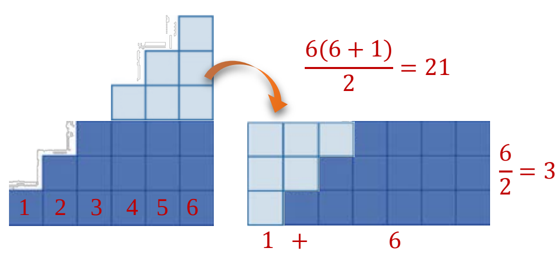


Figure 2

Formally, a partial sum of any sequence is defined as follows:

Definition 1.2 ► Let $\{a_n\}$ be a sequence and $a_1 + a_2 + \cdots + a_n + \cdots$ be its associated series.
The ***n*-th partial sum**, denoted $\boldsymbol{S_n}$, of the sequence (or the series) is the sum

$$\boldsymbol{a_1 + a_2 + \cdots + a_n}.$$

The overall sum of the entire series can be denoted by $\boldsymbol{S_\infty}$.
The **partial sums** on its own create a **sequence** $\{\boldsymbol{S_n}\}$.

Observation:

$$\boldsymbol{S_1 = a_1}$$

$$\boldsymbol{a_n = (a_1 + a_2 + \cdots + a_{n-1} + a_n) - (a_1 + a_2 + \cdots + a_{n-1}) = S_n - S_{n-1}}$$

To find the partial sum $\boldsymbol{S_n}$ of the first n terms of an **arithmetic sequence**, as before, we write it in increasing and decreasing order of terms and then add the resulting equations side by side.

$$\begin{array}{rcccccccc} \boldsymbol{S_n} = & \boldsymbol{a_1} & + & \boldsymbol{(a_1 + d)} & + & \boldsymbol{(a_1 + 2d)} & + \cdots + & \boldsymbol{(a_1 + (n-1)d)} \\ \boldsymbol{S_n} = & \boldsymbol{a_n} & + & \boldsymbol{(a_n - d)} & + & \boldsymbol{(a_n - 2d)} & + \cdots + & \boldsymbol{(a_n - (n-1)d)} \\ \hline \end{array}$$

each column adds to $\boldsymbol{a_1 + a_n}$ *and there are* $\boldsymbol{n}$ *columns*

So, we obtain $\qquad \boldsymbol{2S_n = n(a_1 + a_n)},$

which gives us

$$\boldsymbol{S_n = \frac{n(a_1 + a_n)}{2}} \tag{2}$$

Notice that by substituting of the general term $a_n = a_1 + (n-1)d$ into the above formula, we can express the partial sum S_n in terms of the first term a_1 and the common difference d, as follows:

$$\boldsymbol{S_n = \frac{n(2a_1 + (n-1)d)}{2} \overset{\textit{or equivalently}}{=} \frac{n}{2}(2a_1 + (n-1)d)} \qquad (3)$$

Example 5 ► **Finding a Partial Sum of an Arithmetic Sequence**

a. Find the sum of the first 100 consecutive natural numbers.
b. Find S_{20}, for the sequence $-10, -5, 0, 5, \dots$.
c. Evaluate the sum $2 + (-1) + (-4) + \cdots + (-25)$.

Solution ► **a.** Using the formula (1) for $n = 100$, we have

$$\boldsymbol{S_{100}} = \frac{100 \cdot (100+1)}{2} = 50 \cdot 101 = \mathbf{5050}.$$

So the sum of the first 100 consecutive natural numbers is 5050.

b. To find S_{20}, we can use either formula (2) or formula (3). We are given $n = 20$ and $a_1 = -10$. To use formula (2) it is enough to calculate a_{20}. Since $d = 5$, we have

$$\boldsymbol{a_{20}} = a_1 + 19d = -10 + 19 \cdot 5 = \mathbf{85},$$

which gives us

$$\boldsymbol{S_{20}} = \frac{20(-10+85)}{2} = 10 \cdot 75 = \mathbf{750}.$$

Alternatively, using formula (3), we also have

$$\boldsymbol{S_{20}} = \frac{20}{2}(2(-10) + 19 \cdot 5) = 10(-20+95) = 10 \cdot 75 = \mathbf{750}.$$

c. This time, we are given $a_1 = 2$ and $a_n = -25$, but we need to figure out the number of terms n. To do this, we can use the n-th term formula $a_1 + (n-1)d$ and equal it to -25. Since $d = -1 - 2 = -3$, then we have

$$2 + (n-1)(-3) = -25,$$

which becomes

$$(n-1) = \frac{-27}{-3}$$

and finally

$$n = 10.$$

Now, using formula (2), we evaluate the requested sum to be

$$\boldsymbol{S_{10}} = \frac{10\big(2 + (-25)\big)}{2} = 5 \cdot (-23) = \mathbf{-115}.$$

As we saw in the beginning of this section, an **arithmetic sequence** is **linear** in nature and, as such, it can be identified by the formula $a_n = dn + b$, where $n \in \mathbb{N}$, $d, b \in \mathbb{R}$, and $b = a_1 - d$. This means that the n-th partial sum $S_n = a_1 + a_2 + \dots + a_n = \sum_{i=1}^{n} a_n$ of the associated arithmetic series can be written as

$$\sum_{i=1}^{n}(di + b),$$

and otherwise; each such sum represents the n-th partial sum S_n of an arithmetic series with the first term $d + b$ and the common difference d. Therefore, the above sum can be evaluated with the aid of formula (2), as shown in the next example.

Example 6 ► **Evaluating Finite Arithmetic Series Given in Sigma Notation**

Evaluate the sum $\sum_{i=1}^{16}(2i - 1)$.

Solution ► First, notice that the sum $\sum_{i=1}^{16}(2i - 1)$ represents S_{16} of an arithmetic series with the general term $a_n = 2n - 1$. Since $a_1 = 2 \cdot 1 - 1 = 1$ and $a_{16} = 2 \cdot 16 - 1 = 31$, then applying formula (2), we have

$$\sum_{i=1}^{16}(2i - 1) = \frac{16(1 + 31)}{2} = 8 \cdot 32 = \mathbf{256}.$$

Example 7 ► **Using Arithmetic Sequences and Series in Application Problems**

A worker is stacking wooden logs in layers. Each layer contains three logs less than the layer below it. There are two logs in the top layer, five logs in the layer below, and so on. If there are 7 layers in the stack, determine

a. the number of logs in the bottom layer;
b. the number of logs in the entire stack.

Solution ►

a. First, we observe that the number of logs in consecutive layers, starting from the top, can be expressed by an arithmetic sequence with $a_1 = 2$ and $d = 3$. Since we look for the number of logs in the seventh layer, we use $n = 7$ and the formula

$$a_n = 2 + (n - 1)3 = 3n - 1.$$

This gives us $a_7 = 3 \cdot 7 - 1 = \mathbf{20}$.
Therefore, there are 20 wooden logs in the bottom layer.

b. To find the total number of logs in the stack, we can evaluate the 7-th partial sum $\sum_{i=1}^{7}(3i - 1)$. Using formula (2), we have

$$\sum_{i=1}^{7}(3i - 1) = \frac{7(2 + 20)}{2} = 7 \cdot 11 = \mathbf{77}.$$

So, the entire stack consists of 77 wooden logs.

S.2 Exercises

True or False?

1. The sequence $3, 1, -1, -3, \ldots$ is an arithmetic sequence.

2. The common difference for $2, 4, 2, 4, 2, 4, \ldots$ is 2.

3. The series $\sum_{i=1}^{12}(3 + 2i)$ is an arithmetic series.

4. The n-th partial sum S_n of any series can be calculated according to the formula $S_n = \frac{n(a_1+a_n)}{2}$.

Write a formula for the n-th term of each arithmetic sequence.

5. $1, 3, 5, 7, 9, \ldots$

6. $0, 6, 12, 18, 24, \ldots$

7. $-4, -2, 0, 2, 4, \ldots$

8. $5, 1, -3, -7, -11, \ldots$

9. $-2, -\frac{3}{2}, -1, -\frac{1}{2}, 0, \ldots$

10. $1, \frac{5}{3}, \frac{7}{3}, 3, \frac{11}{3}, \ldots$

Given the information, write out the first five terms of the arithmetic sequence $\{a_n\}$. Then, find the 12-th term a_{12}.

11. $a_n = 3 + (n-1)(-2)$

12. $a_n = 3 + 5n$

13. $a_1 = -8,\ d = 4$

14. $a_1 = 5,\ d = -2$

15. $a_1 = 10,\ a_2 = 8$

16. $a_1 = -7,\ a_2 = 3$

Find the number of terms in each arithmetic sequence.

17. $3, 5, 7, 9, \ldots, 31$

18. $0, 5, 10, 15, \ldots, 55$

19. $4, 1, -2, \ldots, -32$

20. $-3, -7, -11, \ldots, -39$

21. $-2, -\frac{3}{2}, -1, -\frac{1}{2}, \ldots, 5$

22. $\frac{3}{4}, 3, \frac{21}{4}, \ldots, 12$

Given the information, find the indicated term of each arithmetic sequence.

23. $a_2 = 5,\ d = 3;\quad a_8$

24. $a_3 = -4,\ a_4 = -6;\quad a_{20}$

25. $1, 5, 9, 13, \ldots;\quad a_{50}$

26. $6, 3, 0, -3, \ldots;\quad a_{25}$

27. $a_1 = -8,\ a_9 = -64;\quad a_{10}$

28. $a_1 = 6,\ a_{18} = 74;\quad a_{20}$

29. $a_8 = 28,\ a_{12} = 40;\quad a_1$

30. $a_{10} = -37,\ a_{12} = -45;\quad a_2$

Given the arithmetic sequence, evaluate the indicated partial sum.

31. $a_n = 3n - 8;\quad S_{12}$

32. $a_n = 2 - 3n;\quad S_{16}$

33. $6, 3, 0, -3, \ldots;\quad S_9$

34. $1, 6, 11, 16, \ldots;\quad S_{15}$

35. $a_1 = 4,\ d = 3;\quad S_{10}$

36. $a_1 = 6,\ a_4 = -2;\quad S_{19}$

Use a formula for S_n to evaluate each series.

37. $1+2+3+\cdots+25$

38. $2+4+6+\cdots+50$

39. $\sum_{i=1}^{17} 3i$

40. $\sum_{i=1}^{22}(5i+4)$

41. $\sum_{i=1}^{15}\left(\frac{1}{2}i+1\right)$

42. $\sum_{i=1}^{20}(4i-7)$

43. $\sum_{i=1}^{25}(-3-2i)$

44. $\sum_{i=1}^{13}\left(\frac{1}{4}+\frac{3}{4}i\right)$

Solve each problem.

45. The sum of the interior angles of a triangle is 180º, of a quadrilateral is 360º and of a pentagon is 540º. Assuming this pattern continues, find the sum of the interior angles of a dodecagon (*12-sided figure*).

46. Parents of a newborn made a resolution to open an account and save some money for the child's education. They plan to deposit \$500 on the first birthday, \$600 on the second birthday, \$700 on the third birthday, and so on until the child's 18^{th} birthday. According to this plan, how much money (disregarding the interest) would be gathered in this account just after the child's 18^{th} birthday?

47. Donovan signed up to a 14-lesson driving course. His first lesson was 15 minutes long, and each subsequent lesson was 5 minutes longer than the lesson before.

a. How long was his 14^{th} lesson?

b. Overall, how long was Donovan's driving training?

48. Suppose the seating in an auditorium is arranged in rows that are increasing in length. If the first row consists of 12 chairs and each consecutive row has 2 more chairs than the previous one, how many seats are in 12th row? What is the total number of seats in all 12 rows?

49. Karissa plans to do her math homework in the Math Centre, where she can get help if stuck with a question. It took her 40 minutes to do the first-week homework. She needed 7 minutes longer during the second week, and she predicted that, on average, she would need about 7 minutes longer each subsequent week than the week before.

a. How much time should she reserve for doing her math homework during the 13^{th} week of the semester?

b. What would be her overall amount of time spent in the Math Centre doing the math homework during the whole 13-week semester?

S.3 Geometric Sequences and Series

In the previous section, we studied sequences where each term was obtained by adding a constant number to the previous term. In this section, we will take interest in sequences where each term is obtained by multiplying the previous term by a constant number. Such sequences are called **geometric**. For example, the sequence $1, 2, 4, 8, \dots$ is geometric because each term is multiplied by 2 to obtain the next term. Equivallently, the ratios between consecutive terms of this sequence are always 2.

Definition 2.1 ► A sequence $\{a_n\}$ is called **geometric** if the quotient $r = \frac{a_{n+1}}{a_n}$ of any consecutive terms of the sequence is constantly the same.

The **general term** of a **geometric sequence** is given by the formula

$$a_n = a_1 r^{n-1}$$

The quotient r is referred to as **the common ratio** of the sequence.

Similarly as in the previous section, we can be visualize geometric sequences by plotting their values in a system of coordinates. For instance, *Figure 1* presents the graph of the sequence $1,2,4,8, \dots$. The common ratio of 2 causes each cosecutive point of the graph to be plotted twice as high as the previous one, and the slope between the n-th and $(n+1)$-st point to be exactly equal to the value of a_n. Generally, the **slope** between the n-th and $(n+1)$-st point of any geometric sequence **is proportional** to the value of a_n. This property characterises exponential functions. Hence, geometric sequences are **exponential** in nature. To develop the formula for the general term, we observe the pattern

$$\begin{aligned} a_1 &= a_1 \\ a_2 &= a_1 r \\ a_3 &= a_1 r^2 \\ a_4 &= a_1 r^3 \\ &\vdots \end{aligned}$$

so

$$\boxed{a_n = a_1 r^{n-1}}$$

This is the **general term** of a geometric sequence

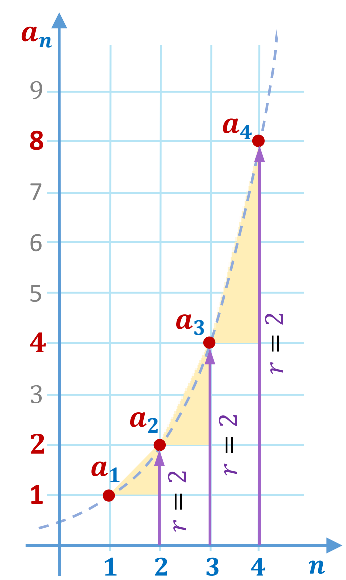


Figure 1

Particularly, the general term of the sequence $1, 2, 4, 8, \dots$ is equal to $a_n = 2^{n-1}$, because $a_1 = 1$ and $r = 2$ (the ratios of consecutive terms are constantly equal to 2).

Note: *To find the common ratio of a geometric sequence, divide any of its terms by the preceeding term.*

Example 1 ► **Identifying Geometric Sequences and Writing Their General Terms**

Determine whether the given sequence $\{a_n\}$ is geometric. If it is, then write a formula for the general term of the sequence.

a. $\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \dots$

b. $\frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \frac{1}{12}, \dots$

Solution ► **a.** After calculating ratios of terms by their peceding terms, we notice that they are always equal to $-\frac{1}{2}$. Indeed, $\frac{a_2}{a_1} = \frac{-\frac{1}{4}}{\frac{1}{3}} = -\frac{1}{2}$, $\frac{a_3}{a_2} = \frac{\frac{1}{8}}{-\frac{1}{4}} = -\frac{1}{2}$, and so on. Therefore, the given sequence is geometric with $a_1 = \frac{1}{2}$ and the common ratio $r = \frac{1}{2}$.

To find its general term, we follow the formula $a_n = a_1 r^{n-1}$. This gives us

$$\boldsymbol{a_n} = \frac{1}{2}\left(-\frac{1}{2}\right)^{n-1} = \frac{\boldsymbol{(-1)^{n-1}}}{\boldsymbol{2^n}}.$$

b. Here, the ratios of terms by their peceding terms, $\frac{a_2}{a_1} = \frac{\frac{1}{6}}{\frac{1}{3}} = \frac{1}{2}$ and $\frac{a_3}{a_2} = \frac{\frac{1}{9}}{\frac{1}{6}} = \frac{2}{3}$, are not the same. So the sequence is not geometric.

Example 2 ► **Finding Terms of a Gometric Sequence**

Given the information, write out the first five terms of the geometric sequence $\{a_n\}$. Then, find the 8-th term a_8.

a. $a_n = 5(-2)^{n-1}$ **b.** $a_1 = 3,\ r = \frac{2}{3}$

Solution ► **a.** To find the first five terms of this sequence, we evaluate a_n for $n = 1,2,3,4,5$.

$$a_1 = 5(-2)^0 = 5$$
$$a_2 = 5(-2)^1 = -10$$
$$a_3 = 5(-2)^2 = 20$$
$$a_4 = 5(-2)^3 = -40$$
$$a_5 = 5(-2)^4 = 80$$

So, the first five terms are $\mathbf{5}, \mathbf{-10}, \mathbf{20}, \mathbf{-40}$, and $\mathbf{80}$.

The 8-th term equals $a_8 = 5(-2)^7 = \mathbf{-640}$.

b. To find the first five terms of a geometric sequence with $\boldsymbol{a_1} = \mathbf{3}$, $\boldsymbol{r} = \frac{\mathbf{2}}{\mathbf{3}}$, we substitute these values into the general term formula

$$\boldsymbol{a_n} = \boldsymbol{a_1 r^{n-1}} = \mathbf{3}\left(\frac{2}{3}\right)^{\boldsymbol{n-1}},$$

and then evaluate it for $n = 1,2,3,4,5$.

This gives us $a_1 = 3\left(\frac{2}{3}\right)^0 = 3$

$$a_2 = 3\left(\frac{2}{3}\right)^1 = 2$$
$$a_3 = 3\left(\frac{2}{3}\right)^2 = \frac{4}{3}$$
$$a_4 = 3\left(\frac{2}{3}\right)^3 = \frac{8}{9}$$
$$a_5 = 3\left(\frac{2}{3}\right)^4 = \frac{16}{27}$$

So, the first five terms are $\mathbf{3}, \mathbf{2}, \frac{\mathbf{4}}{\mathbf{3}}, \frac{\mathbf{8}}{\mathbf{9}}$, and $\frac{\mathbf{16}}{\mathbf{27}}$.

The 8-th term equals $a_8 = 3\left(\frac{2}{3}\right)^7 = \frac{\mathbf{128}}{\mathbf{6561}}$.

Example 3 ► **Finding the Number of Terms in a Finite Geometric Sequence**

Determine the number of terms in the geometric sequence $1, -3, 9, -27, \ldots, 729$.

Solution ► Since the common ratio r of this sequence is -3 and the first term $a_1 = 1$, then the n-th term $a_n = (-3)^{n-1}$. Since the last term is 729, we can set up the equation

$$a_n = (-3)^{n-1} = 729,$$

which can be written as

$$(-3)^{n-1} = (-3)^6$$

This equation holds if $\quad n - 1 = 6,$

which gives us $\quad n = 7.$

So, there are 7 terms in the given sequence.

Example 4 ► **Finding Missing Terms of a Geometric Sequence**

Given the information, determine the values of the indicated terms of a geometric sequence.

a. $a_3 = 5$ and $a_6 = -135$; find a_1 and a_8

b. $a_3 = 200$ and $a_5 = 50$; find a_4 if $a_4 > 0$

Solution ► **a.** As before, let r be the common ratio of the given sequence. Using the general term formula $a_n = a_1 r^{n-1}$ for $n = 6$ and $n = 3$, we can set up a system of two equations in two variables, r and a_1:

$$\begin{cases} -135 = a_1 r^5 \\ 5 = a_1 r^2 \end{cases}$$

To solve this system, let's divide the two equations side by side, obtaining

$$-27 = r^3,$$

which gives us

$$r = -3.$$

Substituting this value to equation (2), we have $5 = a_1 \cdot (-3)^2$, which gives us

$$\boldsymbol{a_1 = \frac{5}{9}}.$$

To find value a_8, we substitute $a_1 = -\frac{5}{2}, r = -3$, and $n = 8$ to the formula $a_n = a_1 r^{n-1}$. This gives us

$$\boldsymbol{a_8} = \frac{5}{9}(-3)^7 = -\mathbf{1215}.$$

b. Let r be the common ratio of the given sequence. Since $a_5 = a_4 r$ and $a_4 = a_3 r$, then $a_5 = a_3 r^2$. Hence, $r^2 = \frac{a_5}{a_3}$. Therefore,

$$r = \pm\sqrt{\frac{a_5}{a_3}} = \pm\sqrt{\frac{50}{200}} = \pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2}. \qquad (1)\ (2)$$

Since $a_3, a_4 > 0$ and $a_4 = a_3 r$, we choose the positive r-value. So we have

$$\boldsymbol{a_4} = a_3 r = 200\left(\frac{1}{2}\right) = \mathbf{100}.$$

Remark: *A **geometric mean** of two quantities $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined as $\sqrt{\boldsymbol{ab}}$. Notice that $\boldsymbol{a_4} = 100 = \sqrt{50 \cdot 200} = \sqrt{\boldsymbol{a_3 \cdot a_5}}$, so $\boldsymbol{a_4}$ is indeed the geometric mean of $\boldsymbol{a_3}$ and $\boldsymbol{a_5}$. Generally, for any $n > 1$, we have*

$$\boldsymbol{a_n} = a_{n-1} r = \sqrt{a_{n-1}^2 r^2} = \sqrt{a_{n-1} \cdot a_{n-1} r^2} = \sqrt{\boldsymbol{a_{n-1} \cdot a_{n+1}}},$$

so every term (except for the first one) of a geometric sequence is the geometric mean of its adjacent terms.

Partial Sums

Similarly as with arithmetic sequences, we might be interested in evaluating the sum of the first n terms of a geometric sequence.

To find partial sum $\boldsymbol{S_n}$ of the first n terms of a **geometric sequence**, we line up formulas for $\boldsymbol{S_n}$ and $-\boldsymbol{rS_n}$ as shown below and then add the resulting equations side by side.

$$\begin{array}{rcl} \boldsymbol{S_n} & = & \boldsymbol{a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-1}} \\ -\boldsymbol{rS_n} & = & \quad\ \ \boldsymbol{- a_1 r - a_1 r^2 - \cdots - a_1 r^{n-1} - a_1 r^n} \\ \hline \end{array}$$

the terms of inside columns add to zero, so they cancel each other out

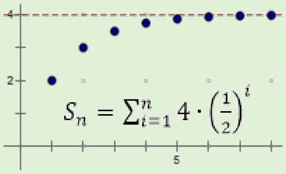
So, we obtain $\quad (\mathbf{1} - \boldsymbol{r})\boldsymbol{S_n} = \boldsymbol{a_1 - a_1 r^n},$

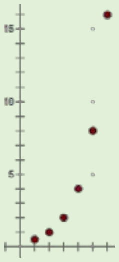
which gives us

$$\boldsymbol{S_n} = \frac{\boldsymbol{a_1(1 - r^n)}}{\boldsymbol{1 - r}}, \qquad (3)$$

as long as $r \neq 1$.

Observe that





$S_n = \sum_{i=1}^{n} \frac{1}{2} \cdot (2)^i$

If $|\boldsymbol{r}| < \mathbf{1}$, then the value of r^n gets closer and closer to zero for larger and larger n (we write: $r^n \to 0$ for $n \to \infty$). This means that the sum of all infinitely many terms of such a sequence **exists** and is equal to

$$\boldsymbol{S_\infty = \frac{a_1}{1 - r}} \qquad (4)$$

If $|r| > 1$, then the value of $|r^n|$ grows without bound for larger and larger n. Therefore, the sum S_∞ of all terms of such a sequence does not have a finite value. We say that such a sum does not exist.

If $|r| = 1$, then the sum S_∞ becomes $a_1 + a_1 + a_1 + \cdots$, or $a_1 - a_1 + a_1 - \cdots$. Neither of these sums has a finite value, unless $a_1 = 0$.

Hence overall, if $|\boldsymbol{r}| \geq \mathbf{1}$, then the sum $\boldsymbol{S_\infty}$ of a nonzero geometric sequence **does not exist**.

Example 5 ► **Finding a Partial Sum of a Geometric Sequence**

a. Find S_6, for the geometric sequence with $a_1 = 0.5$ and $r = 0.1$.

b. Evaluate the sum $1 - \left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)^2 - \cdots - \left(\frac{3}{4}\right)^9$.

Solution ►

a. Using formula (3) for $n = 6$, $a_1 = 0.5$ and $r = 0.1$, we calculate

$$\boldsymbol{S_6} = \frac{0.5(1 - 0.1^6)}{1 - 0.1} = \frac{0.5 \cdot 0.999999}{0.9} = \mathbf{0.555555}.$$

b. First, we observe that the given series is geometric with $a_1 = 1$ and $r = -\frac{3}{4}$. Equating the formula for the general term to the last term of the sum

$$a_1 r^{n-1} = \left(-\frac{3}{4}\right)^{n-1} = -\left(\frac{3}{4}\right)^9 = \left(-\frac{3}{4}\right)^9$$

and comparing the exponents,

$$n - 1 = 9$$

allows us to find the number of terms $n = \mathbf{10}$.

Now, we are ready to calculate the sum of the given series

$$\boldsymbol{S_{10}} = \frac{1\left(1 - \left(-\frac{3}{4}\right)^{10}\right)}{1 - \left(-\frac{3}{4}\right)} \cong \mathbf{0.539249}$$

Example 6 ► **Evaluating Infinite Geometric Series**

Decide wether or not the overall sum S_∞ of each geometric series exists and if it does, evaluate it.

a. $3 - \frac{9}{2} + \frac{27}{4} - \frac{81}{8} + \cdots$

b. $\sum_{i=0}^{\infty} 3 \cdot \left(\frac{2}{3}\right)^i$

Solution ►

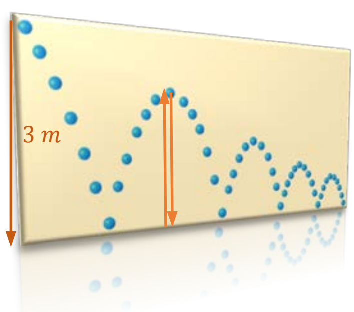
a. Since the common ratio of this series is $|r| = \left|\frac{-\frac{9}{2}}{3}\right| = \left|-\frac{3}{2}\right| = \frac{3}{2} > 1$, then the sum S_∞ does not exist.

b. This time, $|r| = \frac{2}{3} < 1$, so the sum S_∞ exists and can be calculated by following the formula (4). Using $a_1 = 3$ and $r = \frac{2}{3}$, we have

$$S_\infty = \frac{3}{1 - \frac{2}{3}} = \frac{3}{\frac{1}{3}} = \mathbf{9}$$

So, $\sum_{i=0}^{\infty} 3 \cdot \left(\frac{2}{3}\right)^i = \mathbf{9}$.

Example 7 ► **Using Geometric Sequences and Series in Application Problems**



A ball is dropped from a window that is 3 meters above the ground. Suppose the ball always rebounds to $\frac{3}{4}$ of its previous height.

a. To the nearest centimeter, determine the hight that the ball can attain (the rebound height) after the third bounce.
b. Find a formula for the n-th rebound height of the ball.
c. Assuming that the ball bounces forever, what is the total vertical distance travelled by the ball?

Solution ►

a. Let h_n represents the ball's rebound height after the n-th bounce, where $n \in \mathbb{N}$. Since the ball rebounds $\frac{3}{4}$ of the previous height, we have

$$h_1 = 3 \cdot \left(\frac{3}{4}\right)$$

$$h_2 = h_1 \cdot \left(\frac{3}{4}\right) = 3 \cdot \left(\frac{3}{4}\right)^2$$

$$h_3 = h_2 \cdot \left(\frac{3}{4}\right) = 3 \cdot \left(\frac{3}{4}\right)^3 \simeq 1.266\ m \simeq 127\ cm$$

After the third bounce, the ball will rebound approximately **127 centimeters**.

b. Notice that the formulas developed in solution to *Example 4a* follow the pattern

$$\boldsymbol{h_n = 3 \cdot \left(\frac{3}{4}\right)^n}$$

So, this is the formula for the rebound height of the ball after its n-th bounce.

c. Let $h_0 = 3$ represent the vertical distance before the first bounce. To find the total verical distance D travelled by the ball, we add the vertical distance h_0 before the first bounce and twice the vertical distances h_n after each bounce. So, we have

$$D = h_0 + \sum_{n=1}^{\infty} h_n = 3 + \sum_{n=1}^{\infty} 3 \cdot \left(\frac{3}{4}\right)^n$$

Applying the formula $\frac{\boldsymbol{a_1}}{\boldsymbol{1-r}}$ for the infinite sum of a geometric series, we calculate

$$D = 3 + \frac{\frac{9}{4}}{1 - \frac{3}{4}} = 3 + \frac{\frac{9}{4}}{\frac{1}{4}} = 3 + \frac{9}{\cancel{4}} \cdot \frac{\cancel{4}}{1} = 3 + 9 = 12\, m$$

Thus, the total vertical distance travelled by the ball is **12 meters**.

S.3 Exercises

True or False?

1. The sequence $3, -1, \frac{1}{3}, -\frac{1}{9}, \dots$ is a geometric sequence.

2. The common ratio for $0.05, 0.0505, 0.050505, \dots$ is 0.05 .

3. The series $\sum_{i=1}^{7}(3 \cdot 2^i)$ is a geometric series.

4. The n-th partial sum S_n of any finite geometric series exists and it can be evaluated by using the formula $S_n = \frac{a_1(1-r^n)}{1-r}$.

Identify whether or not the given sequence is geometric. If it is, write a formula for its n-th term.

5. $0, 3, 9, 27, \dots$

6. $1, 5, 25, 125, \dots$

7. $-9, 3, -1, \frac{1}{3}, \dots$

8. $\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}, \dots$

9. $1, -1, 1, -1, \dots$

10. $0.9, 0.09, 0.009, 0.0009, \dots$

11. $81, -27, 9, -3, \dots$

12. $\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{8}{3}, \dots$

13. $-\frac{1}{4}, -\frac{1}{5}, -\frac{4}{25}, -\frac{16}{125}, \dots$

Given the information, write out the first four terms of the geometyric sequence $\{a_n\}$. Then, find the 8-th term a_8.

14. $a_n = 3 \cdot 2^{n-1}$

15. $a_n = (-2)^{-n}$

16. $a_1 = 6,\ r = \frac{1}{3}$

17. $a_1 = 5,\ r = -1$

18. $a_1 = \frac{1}{3},\ a_2 = -\frac{1}{6}$

19. $a_1 = 100,\ a_2 = 10$

Find the number of terms in each geometric sequence.

20. $1, 2, 4, \dots, 1024$

21. $20, 10, 5, \dots, \frac{5}{128}$

22. $-4, 2, -1, \dots, \frac{1}{32}$

23. $3, -1, \frac{1}{3}, \dots, \frac{1}{243}$

24. $6, -2, \frac{2}{3}, \dots, -\frac{2}{81}$

25. $-24, 12, -6, \dots, -\frac{3}{32}$

Given the information, find the indicated term of each geometric sequence.

26. $a_2 = 40,\ r = 0.1;\quad a_5$

27. $a_3 = 4,\ a_4 = -8;\quad a_{10}$

28. $2, -2, 2, -2, \dots;\quad a_{50}$

29. $-4, 2, -1, \dots;\quad a_{12}$

30. $a_1 = 6,\ a_4 = -\frac{2}{9};\quad a_8$

31. $a_1 = \frac{1}{9},\ a_6 = 27;\quad a_9$

32. $a_3 = \frac{1}{2},\ a_7 = \frac{1}{32};\quad a_4$ if $a_4 > 0$

33. $a_5 = 48,\ a_8 = -384;\quad a_{10}$

Given the geometric sequence, evaluate the indicated partial sum. Round your answer to three decimal places, if needed.

34. $a_n = 5\left(\frac{2}{3}\right)^{n-1};\quad S_6$

35. $a_n = -2\left(\frac{1}{4}\right)^{n-1};\quad S_{10}$

36. $2, 6, 18, \dots;\quad S_8$

37. $6, 3, \frac{3}{2}, \dots;\quad S_{12}$

38. $1 + \left(\frac{1}{5}\right) + \left(\frac{1}{5}\right)^2 + \cdots + \left(\frac{1}{5}\right)^5$

39. $1 - 3 + 3^2 - \cdots - 3^9$

40. $\sum_{i=1}^{7} 2(1.05)^{i-1}$

41. $\sum_{i=1}^{10} 3(2)^{i-1}$

Decide wether or not the infinite sum S_∞ of each geometric series exists and if it does, evaluate it.

42. $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$

43. $1 - \frac{5}{4} + \frac{25}{16} - \frac{125}{64} + \cdots$

44. $1 + 1.02 + 1.02^2 + 1.02^3 + \cdots$

45. $1 + 0.8 + 0.8^2 + 0.8^3 + \cdots$

46. $\sum_{i=1}^{\infty} (0.6)^{i-1}$

47. $\sum_{i=1}^{\infty} \frac{2}{5}(1.1)^{i-1}$

48. $\sum_{i=1}^{\infty} 2\left(\frac{4}{3}\right)^{i}$

49. $\sum_{i=1}^{\infty} 2\left(-\frac{3}{4}\right)^{i}$

Solve each problem.

50. Suppose that you got an offer to work for a company that pays \$42,000 for the first year of work and a 3% increase in the previous year salary for each consecutive year of employment. What would your salary be in the 15th year of your career?

51. Johny asked his parents to pay him for helping them with yard work for the next two weeks. He asked to be paid 10 cents for the first day and double the daily wage for each consecutive day of his work. If the parents agree to this arrangement, how much will Johny be paid on the 14^{th} day of work? How much will he earn in total for the two-week work?

52. Suppose you deposit \$2000 at the beginning of each year for 35 years into an account that pays the interest of 6% compounded annually. How much would you have in this account at the end of the thirty-fifth year?

53. Matilda's parents decided to save some money for her future education. Each year on Matilda's birthday, they are going to deposit \$1000 into an account that pays 3.5% interest compounded annually. If the first deposit is made on her first birthday, determine the amount of money in the account on her 18th birthday, just before the 18th deposit is made.

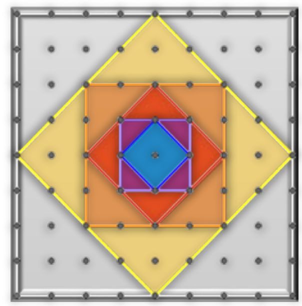
54. A ball is dropped from 2 meters above the ground. With each bounce, the ball comes back to 80% of its previous height.

a. To the nearest millimeter, how high does the ball bounce just after the fifth bounce?

b. To the nearest centimeter, what is the total vertical distance that the ball has travelled when it hits the ground for the tenth time?

55. Suppose that a ball always rebounds $\frac{3}{5}$ of its previous height. The ball was dropped from a height of 3 meters. Approximate the total vertical distance travelled by the ball before it comes to rest.

56. Suppose an infinite sequence of squares is constructed as follows: The first square, s_1, is a **unit square** (*a square whose sides are one unit in length*). The second square, s_2, is constructed by connecting the midpoints of the sides of the first square (*see the yellow square in the accompanying figure*). Generally, the $(n+1)^{st}$ square, s_{n+1}, is constructed by connecting the midpoints of the previous square, s_n. What is the sum of areas of the infinite sequence of squares $\{s_n\}$?



Attributions

p.**445** Asian Buddhism by Sadaham Yathra / Pexels Licence; Coli Bacteria by geralt / Pixabay Licence
p.**447** 20080929-laura-full by NASA / NASA (Public Domain); Fibonacci spiral 34 by Dicklyon / Public Domain
p.**451** Nursery Christmas Trees on Max Pixel / CC0 1.0
p.**453** Parked Red Honda by Javon Swaby / Pexels Licence
p.**454** Rabbit by cristty / Pixabay Licence; Carrot by majacvetojevic / Pixabay Licence
p.**460** Logs of Wood on pxhere / CC0 1.0
p.**462** Adult Automotive Blur by JESHOOTS.com / Pexels Licence; Auditorium by user:12019 / Pixabay Licence