Sequences and Series

In everyday life, we can observe sequences or series of events in many contexts. For instance, we line up to enter a store in a sequence, we make a sequence of mortgage payments, or we observe a series of events that lead to a particular outcome. In this section, we will consider mathematical definitions for sequences and series, and explore some applications of these concepts.

S.1 Sequences and Series

Think of a **sequence of numbers** as an **ordered list of numbers**. For example, the waiting time, in minutes, of each person standing in line to Tim Horton's to be served

$$0, 1, 2, 2, 3, 5, 5, 5, 7, 9, 10, 11$$

or the number of bacteria in a colony after each hour, if the colony starts with one bacteria and each bacteria divides into two every hour

$$1, 2, 4, 8, 16, 32, \ldots, 2^{n-1}, \ldots$$

The first example illustrates a finite sequence, while the second example shows an infinite sequence. Notice that numbers listed in a sequence, called **terms**, can repeat, like in the first example, or they can follow a certain pattern, like in the second example. If we can recognize the pattern of the listed terms, it is convenient to state it as a general rule by listing the n-th term. The sequence of numbers in our second example shows consecutive powers of two, starting with 2^0, so the n-th term of this sequence is 2^{n-1}.

Sequences as Functions

Formally, the definition of sequence can be stated by using the terminology of functions.

Definition 1.1 ► An **infinite sequence** is a function whose **domain** is the set of all **natural numbers**.
A **finite sequence** is a function whose **domain** is the first n natural numbers $\{\mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots, \boldsymbol{n}\}$.
The **terms** (or elements) of a sequence are the function values, the entries of the ordered list of numbers.
The **general term** of a sequence is its n-th term.

Notation: Customarily, sequence functions assume names such as a, b, c, rather than f, g, h.
If the name of a sequence function is a, than the function values (the **terms** of the sequence) are denoted $\boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_3}, \ldots$ rather than $a(1), a(2), a(3), \ldots$.
The **index** $\boldsymbol{k}$ in the notation $\boldsymbol{a_k}$ indicates the position of the term in the sequence.
$\boldsymbol{a_n}$ denotes the **general term** of the sequence and $\{\boldsymbol{a_n}\}$ represents the entire sequence.

Example 1 ► **Finding Terms of a Sequence When Given the General Term**

Given the sequence $a_n = \frac{n-1}{n+1}$, find the following

a. the first four terms of $\{a_n\}$

b. the 12-th term a_{12}

Solution ► **a.** To find the first four terms of the given sequence, we evaluate a_n for $n = 1,2,3,4$.

We have
$$a_1 = \frac{1-1}{1+1} = 0$$
$$a_2 = \frac{2-1}{2+1} = \frac{1}{3}$$
$$a_3 = \frac{3-1}{3+1} = \frac{2}{4} = \frac{1}{2}$$
$$a_4 = \frac{4-1}{4+1} = \frac{3}{5}$$

so the first four terms are $0, \frac{1}{3}, \frac{1}{2}, \frac{3}{5}$.

b. The twelfth term is $a_{12} = \frac{12-1}{12+1} = \frac{11}{13}$.

Example 2 ► **Finding the General Term of a Sequence**

Determine the expression for the general term a_n of the sequence

a. $3, 9, 27, 81, \ldots$ **b.** $-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \ldots$

Solution ► **a.** Observe that all terms of the given sequence are powers of 3.

We have
$$a_1 = 3 = 3^1$$
$$a_2 = 9 = 3^2$$
$$a_3 = 27 = 3^3$$
$$a_4 = 81 = 3^4\text{, and so on.}$$

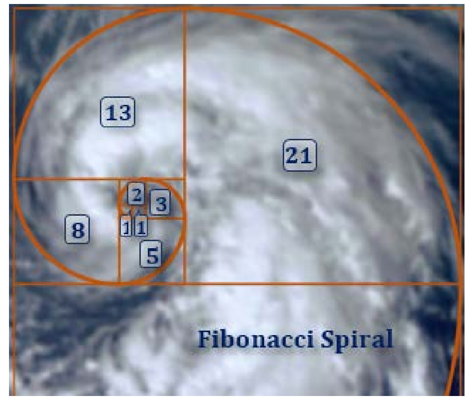
Notice that in each term, the exponent of 3 is the same as the index of the term. The above pattern suggests the candidate $a_n = 3^n$ for the general term of this sequence. To convince oneself that this is indeed the general term, one may want to generate the given terms with the aid of the developed formula $a_n = 3^n$. If the generated terms match the given ones, the formula is correct.

b. Here, observe that the signs of the given terms alter, starting with the negative sign. While building a formula for the general term, to accommodate for this change in signs, we may want to use the factor of $(-1)^n$. This is because
$$(-1)^n = \begin{cases} -1, & \text{for } n = 1, 3, 5, \ldots \\ 1\ , & \text{for } n = 2, 4, 6, \ldots \end{cases}$$
Then we observe that all terms may be seen as fractions with the numerator equal to 1 and denominator matching the index of the term,

$$a_1 = -1 = (-1)^1 \frac{1}{1}$$
$$a_2 = \frac{1}{2} = (-1)^2 \frac{1}{2}$$
$$a_3 = -\frac{1}{3} = (-1)^3 \frac{1}{3}$$
$$a_4 = \frac{1}{4} = (-1)^4 \frac{1}{4}$$
$$a_5 = -\frac{1}{5} = (-1)^5 \frac{1}{5}\text{, and so on.}$$

The above pattern suggests that the formula $a_n = (-1)^n \frac{1}{n}$ would work for the general term of this sequence. As before, please convince yourself that this is indeed the general term of the sequence by generating the given terms with the aid of the suggested formula.

Sometimes it is difficult to describe a sequence by stating the explicit formula for its general term. For example, in the case of the **Fibonacci sequence** $1, 1, 2, 3, 5, 8, 13, 21, \dots$, one can observe the rule of obtaining the next term by adding the previous two terms (for terms after the second term), but it would be very difficult to come up with an explicit formula for the general term a_n. Yet the Fibonacci sequence can be defined through the following equations $a_1 = a_2 = 1$ and $a_n = a_{n-2} + a_{n-1}$, for $n \geq 3$. Notice that the n-th term is not given explicitly but it can be found as long as the previous terms are known. In such a case we say that the sequence is **defined recursively**.



Definition 1.2 ► A sequence is defined **recursively** if

- the initial term or terms are given, and
- the n-th term is defined by a formula that refers to the *preceding* terms.

Example 3 ► **Finding Terms of a Sequence Given Recursively**

Find the first 5 terms of the sequence given by the conditions
$a_1 = 1$, $a_2 = 2$, and $a_n = 2a_{n-1} + a_{n-2}$, for $n \geq 3$.

Solution ► The first two terms are given, $a_1 = \mathbf{1}$, $a_2 = \mathbf{2}$. To find the third term, we substitute $n = 3$ into the recursive formula, to obtain

$$a_3 = 2a_{3-1} + a_{3-2} = 2a_2 + a_1 = 2 \cdot \mathbf{2} + \mathbf{1} = \mathbf{5}$$

Similarly
$$a_4 = 2a_{4-1} - a_{4-2} = 2a_3 - a_2 = 2 \cdot \mathbf{5} + \mathbf{2} = \mathbf{12}$$

and
$$a_5 = 2a_{5-1} - a_{5-2} = 2a_4 - a_3 = 2 \cdot \mathbf{12} + \mathbf{5} = \mathbf{29}$$

So the first five terms of this sequence are: **1**, **2**, **5**, **12**, and **29**.

Example 4 ► **Using Sequences in Application Problems**

Peter borrows \$5000 and agrees to pay \$500 monthly, plus interest of 1% per month on the unpaid balance. If his first payment is due one month from the date of borrowing, find

a. the total number of payments needed to pay off the debt,
b. the sequence of his first four payments,
c. the general term of the sequence of payments,
d. the last payment.

Solution ► **a.** Since Peter pays \$500 off his \$5000 principal each time, the total number of payments is $\frac{5000}{500} = 10$.

b. Let $a_1, a_2, \dots, a_{10}$ be the sequence of Peter's payments.
After the first month, Peter pays $a_1 = \$500 + 0.01 \cdot \$5000 = \mathbf{\$550}$ and the remaining balance becomes $\$5000 - \$500 = \$4500$.
Then, Peter's second payment is $a_2 = \$500 + 0.01 \cdot \$4500 = \mathbf{\$545}$ and the remaining balance becomes $\$4500 - \$500 = \$4000$.
The third payment is equal to $a_3 = \$500 + 0.01 \cdot \$4000 = \mathbf{\$540}$ and the remaining balance becomes $\$4000 - \$500 = \$3500$.
Finally, the fourth payment is $a_4 = \$500 + 0.01 \cdot \$3500 = \mathbf{\$535}$ with the remaining balance of $\$3500 - \$500 = \$3000$.

So the sequence of Peter's first four payments is $\mathbf{\$550, \$545, \$540, \$535.}$

c. Notice that the terms of the above sequence diminish by 5.
We have
$$\begin{aligned} a_1 &= 550 = 550 - \mathbf{0} \cdot 5 \\ a_2 &= 545 = 550 - \mathbf{1} \cdot 5 \\ a_3 &= 540 = 550 - \mathbf{2} \cdot 5 \\ a_4 &= 535 = 550 - \mathbf{3} \cdot 5, \text{ and so on.} \end{aligned}$$

Since the blue coefficient by "5" is one lower than the index of the term, we can write the general term as $a_n = 550 - (\boldsymbol{n-1}) \cdot 5$, which after simplifying can take the form $a_n = 550 - 5n + 5 = \mathbf{555 - 5n}$.

d. Since there are 10 payments, the last one equals to $a_{10} = 555 - 5 \cdot 10 = \mathbf{\$505}$.

Series and Summation Notation

Often, we take interest in finding sums of terms of a sequence. For instance, in *Example 4*, we might be interested in finding the total amount paid in the first four months $\$550 + \$545 + \$540 + \$535$, or the total cost of borrowing $\$550 + \$545 + \dots + \$505$. The terms of a sequence connected by the operation of addition create an expression called a **series**.

Note: *The word "**series**" is both singular and plural.*

Definition 1.3 ► A **series** is the sum of terms of a finite or infinite sequence, before evaluation.
The value of a **finite series** can always be determined because addition of a finite number of values can always be performed.
The value of an **infinite series** may not exist. For example, $\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n} + \dots = 1$ but $1 + 2 + \dots + n + \dots = DNE$ (doesn't exist).

Series involve writing sums of many terms, which is often cumbersome. To write such sums in compact form, we use summation notation referred to as **sigma notation**, where the Greek letter $\boldsymbol{\Sigma}$ (sigma) is used to represent the operation of adding all the terms of a sequence. For example, the finite series $1^2 + 2^2 + 3^2 + \dots + 10^2$ can be recorded in sigma notation as

$$\sum_{i=1}^{10} i^2 \quad \text{or} \quad \sum\nolimits_{i=1}^{10} i^2$$

Here, the letter i is called the **index of summation** and takes integral values from 1 to 10. The expression i^2 (the general term of the corresponding sequence) generates the terms being added. The number 1 is the lower limit of the summation, and the number 10 is the upper limit of the summation. We read "the sum from $i = 1$ to 10 of i^2." To find this sum, we replace the letter i in i^2 with 1, 2, 3, …, 10, and add the resulting terms.

Note: *Any letter can be used for the index of summation; however, the most commonly used letters are i,j,k,m,n.*

A finite series with an unknown number of terms, such as $1 + 2 + \cdots + n$, can be recorded as

$$\sum\nolimits_{i=1}^{n} i$$

Here, since the last term equals to n, the value of the overall sum is an expression in terms of n, rather than a specific number.

An infinite series, such as $0.3 + 0.03 + 0.003 + \cdots$ can be recorded as

$$\sum\nolimits_{i=1}^{\infty} \frac{3}{10^i}$$

In this case, the series can be evaluated and its sum equals to $0.333\ldots = \frac{1}{3}$.

Example 5 ► **Evaluating Finite Series Given in Sigma Notation**

Evaluate the sum.

a. $\sum_{i=1}^{5}(2i + 1)$ **b.** $\sum_{k=1}^{6}(-1)^k \frac{1}{k}$

Solution ► **a.** $\sum_{i=0}^{5}(2i + 1) = (2 \cdot 0 + 1) + (2 \cdot 1 + 1) + (2 \cdot 2 + 1) + (2 \cdot 3 + 1) + (2 \cdot 4 + 1)$

$= 1 + 3 + 5 + 7 + 9 = \mathbf{25}$

b. $\sum_{k=1}^{6}(-1)^k \frac{1}{k} = (-1)^1 \frac{1}{1} + (-1)^2 \frac{1}{2} + (-1)^3 \frac{1}{3} + (-1)^4 \frac{1}{4} + (-1)^5 \frac{1}{5} + (-1)^6 \frac{1}{6}$

$= -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} = \frac{-60+30-20+15-12+10}{60} = -\mathbf{\frac{37}{60}}$

Example 6 ► **Writing Series in Sigma Notation**

Write the given series using sigma notation.

a. $5 + 7 + 9 + \cdots + 47 + 49$ **b.** $\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \cdots$

Solution ► **a.** Observe that the series consists of a sequence of odd integers, from 5 to 49. An odd integer can be represented by the expression $2n + 1$ or $2n - 1$. The first expression assumes a value of 5 for $n = 2$, and a value of 49 for $n = 24$. Therefore, the general term of the series could be written as $a_n = 2n + 1$, for $n = 2,3, \ldots, 24$. Hence, the series might be written in the form

$$\sum_{i=2}^{24}(2i+1)$$

Using the second expression, $2n-1$, the general term $a_n = 2n-1$ would work for $n = 3,4, \dots ,25$. Hence, the series might also be written in the form

$$\sum_{i=3}^{25}(2i-1)$$

The sequence can also be written with the index of summation set to start with 1. Then we would have

$$\sum_{i=1}^{23}(2i+3)$$

Check that all of the above sigma expressions produce the same series.

b. In this infinite series $\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\cdots$ the signs of consecutive terms alter. To accommodate for the change of signs, we may want to use a factor of $(-1)^n$ or $(-1)^{n+1}$, depending on the sign of the first term. Since the first term is positive, we use the factor of $(-1)^{n+1}$ that equals to 1 for $n = 1$. In addition, the terms consist of fractions with constant numerators equal to 1 and denominators equal to consecutive even numbers that could be represented by $2n$. Hence, the series might be written in the form

$$\sum_{i=1}^{\infty}(-1)^{i+1}\frac{1}{2i}$$

Notice that by renaming the index of summation to, for example, $k = i-1$, the series takes the form

$$\sum_{k=0}^{\infty}(-1)^{k}\frac{1}{2(k+1)}$$

Check on your own that both of the above sigma expressions produce the same series.

Observation: *Series in sigma notation can be written in many different yet equivalent forms. This is because the starting value of the index of summation is arbitrary. Commonly, we start at 1, or 0, unless other values make the general term formula simpler.*

Example 7 ► **Adjusting the Index of Summation**

In each series, change index j to index k that starts at 1.

a. $\sum_{j=2}^{7}(-1)^{j-1}j^3$ **b.** $\sum_{j=0}^{\infty}3^{2j-1}$

Solution ► **a.** If $k = 1$ when $j = 2$, then $j-k = 1$, or equivalently $j = k+1$. In this relation, the upper limit $j = 7$ corresponds to $k = 6$. So by substitution, we obtain

$$\sum_{j=2}^{7}(-1)^{j-1}j^3 = \sum_{k=1}^{6}(-1)^{k+1-1}(k+1)^3 = \boldsymbol{\sum_{k=1}^{6}(-1)^{k}(k+1)^3}$$

b. If $k = 1$ when $j = 0$, then $k - j = 1$, or equivalently $j = k - 1$. By substitution, we have

$$\sum_{j=0}^{\infty} 3^{2j-1} = \sum_{k=1}^{\infty} 3^{2(k-1)-1} = \sum_{k=1}^{\infty} \mathbf{3^{2k-3}}$$

Example 8 ► **Using Series in Application Problems**

In reference to Example 4 of this section:

Peter borrows \$5000 and agrees to pay \$500 monthly, plus interest of 1% per month on the unpaid balance. If his first payment is due one month from the date of borrowing, find

a. the sequence $\{b_n\}$, where b_n represents the remaining balance before the n-th payment,

b. the total interest paid by Peter.

Solution ►

a. As indicated in the solution to *Example 4b*, the sequence of monthly balances before the n-th payment is $5000, 4500, 4000, \dots, 1000, 500$. Since the balance decreases each month by 500, the general term of this sequence is

$$\boldsymbol{b_n} = 5000 - (n-1)500 = \mathbf{5500 - 500}\boldsymbol{n}.$$

b. Since Peter pays 1% on the unpaid balance b_n each month and the number of payments is 10, the total interest paid can be represented by the series

$$\sum_{k=1}^{10} (0.01 \cdot b_k) = \sum_{k=1}^{10} [0.01 \cdot (5500 - 500k)] = \sum_{k=1}^{10} (55 - 5k)$$

$$= 50 + 45 + 40 + 35 + 30 + 25 + 20 + 15 + 10 + 5 = 275$$

Therefore, Peter paid the total interest of **\$275**.

Arithmetic Mean

When calculating the final mark in a course, we often take an average of a sequence of marks we received on assignments, quizzes, or tests. We do this by adding all the marks and dividing the sum into the number of marks used. This average gives us some information about the overall performance on the particular task.

We are often interested in finding averages in many other life situations. For example, we might like to know the average number of calories consumed per day, the average lifetime of particular appliances, the average salary of a particular profession, the average yield from a crop, the average height of a Christmas tree, etc.
In mathematics, the commonly used term "**average**" is called the **arithmetic mean** or just **mean**. If a mean is calculated for a large set of numbers, it is convenient to write it using summation notation.

Definition 1.4 ► The **arithmetic mean (average)** of the numbers $\boldsymbol{x_1, x_2, x_3, \ldots, x_n}$, often denoted $\boldsymbol{\bar{x}}$, is given by the formula

$$\boldsymbol{\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}}$$

Example 9 ► **Calculating the Arithmetic Mean**

a. Find the arithmetic mean of $-12, 3, 0, 4, -2, 10$.

b. Evaluate the mean $\bar{x} = \frac{\sum_{i=1}^{5} i^2}{5}$.

Solution ► **a.** The arithmetic mean equals

$$\frac{-12 + 3 + 0 + 4 + (-2) + 10}{6} = \frac{3}{6} = \boldsymbol{\frac{1}{2}}$$

b. To evaluate this mean by may want to write out all the terms first. We have

$$\bar{x} = \frac{\sum_{i=1}^{5} i^2}{5} = \frac{1^2 + 2^2 + 3^2 + 4^2 + 5^2}{5} = \frac{1 + 4 + 9 + 16 + 25}{5} = \frac{55}{5} = \boldsymbol{11}$$

S.1 Exercises

Vocabulary Check *Fill in each blank with the most appropriate term or phrase from the given list:* ***alternating, arithmetic mean, finite, function, general, infinite, inputs, limit, recursively, sequence, series, sigma, summation, terms, upper.***

1. A ________________ of numbers is an ordered list of numbers.

2. A ____________ whose domain is the set of all natural numbers is an ______________ sequence. The values of such a function are called ___________ of the sequence. The ___________ of the function are the indexes of the terms.

3. A sequence $\{a_n\}$ can be defined explicitly, by stating the ______________ term a_n, or _______________ , by stating the initial term(s) and the relation of the n-th term to the preceding terms.

4. The sum of terms of a sequence before evaluation is called a ___________. The sum of a ___________ series always exists while the sum of an _____________ series may or may not exist.

5. ____________ notation provides a way of writing a sum without writing out all of the terms.

6. The index of ______________ in sigma notation assumes integral values between the lower ___________ and the ___________ limit.

7. In an _________________ series the signs of consecutive terms alternate.

8. “Average” is a popular name of an _______________ ___________ .

Concept Check *Find the* ***first four terms*** *and the* ***10-th*** *term of each infinite sequence whose n-th term is given.*

9. $a_n = 2n - 3$

10. $a_n = \frac{n+2}{n}$

11. $a_n = (-1)^{n-1}3n$

12. $a_n = 1 - \frac{1}{n}$

13. $a_n = \frac{(-1)^n}{n^2}$

14. $a_n = (n+1)(2n-3)$

15. $a_n = (-1)^n(n-2)^2$

16. $a_n = \frac{1}{n(n+1)}$

17. $a_n = \frac{(-1)^{n+1}}{2n-1}$

Analytic Skills *Write a formula for the n-th term of each sequence.*

18. $2, 5, 8, 11, \ldots$

19. $1, -1, 1, -1, \ldots$

20. $\frac{\pi}{12}, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \ldots$

21. $3, 9, 27, 81, \ldots$

22. $15, 10, 5, 0, \ldots$

23. $6, 9, 12, 15, \ldots$

24. $-1, \frac{1}{4}, -\frac{1}{9}, \frac{1}{25}, \ldots$

25. $1, -\frac{1}{8}, \frac{1}{27}, -\frac{1}{64}, \ldots$

26. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

Concept Check *Find the first five terms of each infinite sequence given by a recursion formula.*

27. $a_n = 2a_{n-1} + 5,\ a_1 = -3$

28. $a_n = 1 - \frac{1}{a_{n-1}},\ a_1 = 2$

29. $a_n = 2a_{n-1} + a_{n-2},\ a_1 = 1, a_2 = 2$

30. $a_n = (a_{n-1})^2 - 1,\ a_1 = 2$

Analytic Skills *Solve each applied problem by writing the first few terms of a sequence.*

31. Lucy borrows $1600 and agrees to pay $200 plus interest of 2% on the unpaid balance each month. Find the payments for the first six months and the remaining debt at the end of that period.

32. Maria is offered a new modeling job with a salary of $20{,}000 + 1500n$ dollars per year at the end of the n-th year. Write a sequence showing her salary at the end of each of the first 5 years. What would her salary be at the end of the tenth year?

33. If a contractor does not complete a multimillion-dollar construction project on time, he must pay a penalty of $500 for the first day that he is late, $700 for the second day, $900 for the third day, and so on. Each day the penalty is $200 larger than the previous day. Write a formula for the penalty on the n-th day. What is the penalty for the 10-th day?

34. The MSRP (manufacturer's suggested retail price) for a 2008 Jeep Grand Cherokee was $43,440. Analysts estimated that prices increase 6% per year for the next five years. To the nearest dollar, find the estimated price of this model in the years 2009 through 2013. Write a formula for this sequence.

35. Chlorine is often added to swimming pools to control microorganisms. If the level of chlorine rises above 3 ppm (parts per million), swimmers will experience burning eyes and skin discomfort. If the level drops below 1 ppm, there is a possibility that the water will turn green because of a large algae count. Chlorine must be added to pool water at regular intervals. If no chlorine is added to a pool during a 24-hour period, approximately 20% of the chlorine will dissipate into the atmosphere and 80% will remain in the water.

a. Determine a sequence a_n that expresses the amount of chlorine present after n days if the pool has a_0 ppm of chlorine initially and no chlorine is added.

b. If a pool has 7 ppm of chlorine initially, construct a table to determine the first day on which the chlorine level is expected to drop below 3 ppm.

Concept Check *Evaluate each sum.*

36. $\sum_{i=1}^{5}(i+2)$

37. $\sum_{i=1}^{10} 5$

38. $\sum_{i=1}^{8}(-1)^i i$

39. $\sum_{i=1}^{4}(-1)^i(2i-1)$

40. $\sum_{i=1}^{6}(i^2-1)$

41. $\sum_{i=3}^{7} 2^i$

42. $\sum_{i=0}^{5}(i-1)\,i$

43. $\sum_{i=2}^{5}\frac{(-1)^i}{i}$

44. $\sum_{i=0}^{7}(i-2)\,(i-3)$

Analytic Skills *Write each series using sigma notation.*

45. $2+4+6+8+10+12$

46. $3-6+9-12+15-18$

47. $\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\cdots+\frac{1}{50}$

48. $1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\cdots-\frac{1}{100}$

49. $1+8+27+64+\cdots$

50. $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\cdots$

Concept Check *In each series change index **n** to index **m** that starts at **1**.*

51. $\sum_{n=0}^{9}(3n-1)$

52. $\sum_{n=3}^{7} 2^{n-2}$

53. $\sum_{n=2}^{6}\frac{n}{n+2}$

54. $\sum_{n=0}^{\infty}(-1)^{n+1}n$

55. $\sum_{n=2}^{\infty}(-1)^n(n-1)$

56. $\sum_{n=2}^{\infty}\frac{1}{n^2}$

Analytic Skills *Use a series to model the situation in each of the following problems.*

57. A frog with a vision problem is 1 yard away from a dead cricket. He spots the cricket and jumps halfway to the cricket. After the frog realizes that he has not reached the cricket, he again jumps halfway to the cricket. Write a series in summation notation to describe how far the frog has moved after nine such jumps.

58. Cleo deposited \$1000 at the beginning of each year for 5 years into an account paying 10% interest compounded annually. Write a series using summation notation to describe how much she has in the account at the end of the fifth year. Note that the first \$1000 will receive interest for 5 years, the second \$1000 will receive interest for 4 years, and so on.

Concept Check *Find the **arithmetic mean** of each sequence, or evaluate the mean written in sigma notation.*

59. $10, 12, 8, 0, 2, 19, 23, 6$

60. $5, -9, 8, 2, -4, 7, 5$

61. $\bar{x} = \frac{\sum_{i=-5}^{5} i}{11}$

62. $\bar{x} = \frac{\sum_{i=1}^{8} 2^i}{8}$

S.2 Arithmetic Sequences and Series

Some sequences consist of random values while other sequences follow a certain pattern that is used to arrive at the sequence's terms. In this section, we will take interest in sequences that follow the pattern of having a constant difference between their consecutive terms. Such sequences are called **arithmetic**. For example, the sequence $3, 5, 7, 9, \ldots$ is arithmetic because its consecutive terms always differ by 2.

Definition 2.1 ► A sequence $\{a_n\}$ is called **arithmetic** if the difference $d = a_{n+1} - a_n$ between any consecutive terms of the sequence is constantly the same.

The **general term** of an **arithmetic sequence** is given by the formula

$$a_n = a_1 + (n-1)d$$

The difference d is referred to as **the common difference** of the sequence.

Similarly to functions, sequences can be visualized by plotting their values in a system of coordinates. For instance, *Figure 1* presents the graph of the sequence $3, 5, 7, 9, \ldots$. Notice that the common difference of 2 makes the graph linear in nature. This is because the slope between the consecutive points of the graph is always the same. Generally, any arithmetic sequence follows a linear pattern with the **slope** being the **common difference** and the ***y*-intercept** being the **first term diminished by the common difference**, as illustrated by *Figure 1*. This means that we should be able to write the general term of an arithmetic sequence by following the slope-intercept equation of a line. Using this strategy, we obtain

$$a_n = \text{slope} \cdot n + (y\text{-intercept}) = dn + (a_1 - d) = a_1 + dn - d$$
$$= a_1 + (n-1)d$$

Particularly, the general term of the sequence $3, 5, 7, 9, \ldots$ is equal to $a_n = 3 + (n-1)2$, or equivalently to $a_n = 2n + 1$.

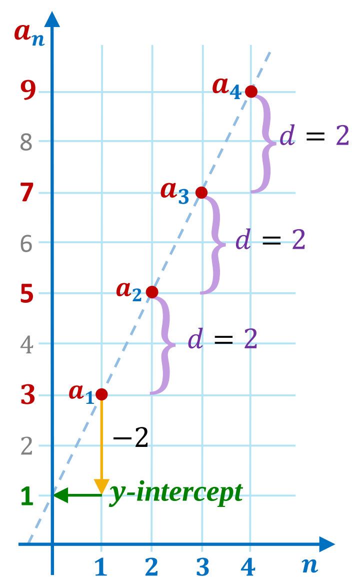


Figure 1

Example 1 ► **Identifying Arithmetic Sequences and Writing its General Term**

Determine whether the given sequence $\{a_n\}$ is arithmetic. If it is, then write a formula for the general term of the sequence.

a. $2, 4, 8, 16, \ldots$ **b.** $3, 1, -1, -3, \ldots$

Solution ►

a. Since the differences between consecutive terms, $a_2 - a_1 = 4 - 2 = 2$ and $a_3 - a_2 = 8 - 4 = 4$, are not the same, the sequence is not arithmetic.

b. Here, the differences between consecutive terms are constantly equal to -2, so the sequence is arithmetic with $a_1 = 3$, and the common difference $d = -2$. Therefore, using the formula for the general term $a_n = a_1 + (n-1)d$, we have

$$a_n = 3 + (n-1)(-2) = 3 - 2n + 2 = -2n + 5.$$

Example 2 ► **Finding Terms of an Arithmetic Sequence**

Given the information, write out the first five terms of the arithmetic sequence $\{a_n\}$. Then, find the 10-th term a_{10}.

a. $a_n = 12 - 3n$ **b.** $a_1 = 3,\ d = 5$

Solution ► **a.** To find the first five terms of this sequence, we evaluate a_n for $n = 1,2,3,4,5$.

$$a_1 = 12 - 3 \cdot 1 = 9$$
$$a_2 = 12 - 3 \cdot 2 = 6$$
$$a_3 = 12 - 3 \cdot 3 = 3$$
$$a_4 = 12 - 3 \cdot 4 = 0$$
$$a_5 = 12 - 3 \cdot 5 = -3$$

So, the first five terms are $\mathbf{9}, \mathbf{6}, \mathbf{3}, \mathbf{0}$, and $\mathbf{-3}$.

The 10-th term equals $a_{10} = 12 - 3 \cdot 10 = \mathbf{-18}$.

b. To find the first five terms of an arithmetic sequence with $a_1 = 3$, $d = 5$, we substitute these values into the general term formula

$$a_n = a_1 + (n-1)d = 3 + (n-1)5,$$

and then evaluate it for $n = 1,2,3,4,5$.

This gives us
$$a_1 = 3 + 0 \cdot 5 = 3$$
$$a_2 = 3 + 1 \cdot 5 = 8$$
$$a_3 = 3 + 2 \cdot 5 = 13$$
$$a_4 = 3 + 3 \cdot 5 = 18$$
$$a_5 = 3 + 4 \cdot 5 = 23$$

So, the first five terms are $\mathbf{3}, \mathbf{8}, \mathbf{13}, \mathbf{18}$, and $\mathbf{23}$.

The 10-th term equals $a_{10} = 3 + 9 \cdot 5 = \mathbf{48}$.

Example 3 ► **Finding the Number of Terms in a Finite Arithmetic Sequence**

Determine the number of terms in the arithmetic sequence $1,5,9,13, \dots, 45$.

Solution ► Notice that the common difference d of this sequence is $5 - 1 = 4$ and the first term $a_1 = 1$. Therefore the n-th term $a_n = 1 + (n-1)4 = 4n - 3$. Since the last term is 45, we can set up the equation

$$a_n = 4n - 3 = 45, \quad \text{and solve it for } n.$$

This gives us $$4n = 48,$$

and finally $$n = 12.$$

So, there are 12 terms in the given sequence.

Example 4 ► **Finding Missing Terms of an Arithmetic Sequence**

Given the information, determine the values of the indicated terms of an arithmetic sequence.

a. $a_5 = 2$ and $a_7 = 8$; find a_6 **b.** $a_3 = 5$ and $a_{10} = -9$; find a_1 and a_{15}

Solution ► **a.** Let d be the common difference of the given sequence. Since $a_7 = a_6 + d$ and $a_6 = a_5 + d$, then $a_7 = a_5 + 2d$. Hence, $2d = a_7 - a_5$, which gives

$$d = \frac{a_7 - a_5}{2} = \frac{8-2}{2} = 3.$$

Therefore,

$$\boldsymbol{a_6} = a_5 + d = 2 + 3 = \mathbf{5}.$$

Remark: *An **arithmetic mean** of two quantities **a** and **b** is defined as* $\frac{\boldsymbol{a}+\boldsymbol{b}}{\mathbf{2}}$.
Notice that $\boldsymbol{a_6} = 5 = \frac{2+8}{2} = \frac{\boldsymbol{a_5}+\boldsymbol{a_7}}{\mathbf{2}}$, *so* $\boldsymbol{a_6}$ *is indeed the arithmetic mean of* $\boldsymbol{a_5}$ *and* $\boldsymbol{a_7}$.
Generally, for any $n > 1$, *we have*

$$\boldsymbol{a_n} = a_{n-1} + d = \frac{2a_{n-1} + 2d}{2} = \frac{a_{n-1} + (a_{n-1} + 2d)}{2} = \frac{\boldsymbol{a_{n-1}} + \boldsymbol{a_{n+1}}}{\mathbf{2}},$$

so every term (except for the first one) of an arithmetic sequence is the arithmetic mean of its adjacent terms.

b. As before, let d be the common difference of the given sequence. Using the general term formula $a_n = a_1 + (n-1)d$ for $n = 10$ and $n = 3$, we can set up a system of two equations in two variables, d and a_1:

$$\begin{cases} -9 = a_1 + 9d & (1) \\ 5 = a_1 + 2d & (2) \end{cases}$$

To solve this system, we can subtract the two equations side by side, obtaining

$$-14 = 7d,$$

which gives

$$d = -2.$$

After substitution to equation (2), we have $5 = a_1 + 2 \cdot 2$, which allows us to find the value a_1:

$$\boldsymbol{a_1} = 5 - 4 = \mathbf{1}.$$

To find the value of a_{15}, we substitute $a_1 = 1, d = -2$, and $n = 15$ to the formula $a_n = a_1 + (n-1)d$ to obtain

$$\boldsymbol{a_{15}} = 1 + (15-1)(-2) = 2 - 28 = \mathbf{-26}.$$

Partial Sums

Sometimes, we are interested in evaluating the sum of the first n terms of a sequence. For example, we might be interested in finding a formula for the sum $S_n = 1 + 2 + \cdots n$ of the first n consecutive natural numbers. To do this, we can write this sum in increasing and decreasing order, as below.

$$\begin{array}{l} S_n = 1 + \quad 2 \quad + \cdots + (n-1) + n \\ S_n = n + (n-1) + \cdots + \quad 2 \quad + 1 \end{array}$$

Now, observe that the sum of terms in each column is always $(n+1)$, and there are n columns. Therefore, after adding the two equations side by side, we obtain:

$$2S_n = n(n+1),$$

which in turn gives us a very useful formula

$$S_n = \frac{\boldsymbol{n(n+1)}}{\mathbf{2}} \tag{1}$$

for the sum of the first n consecutive natural numbers.

Figure 2 shows us a geometrical interpretation of this formula, for $n = 6$. For example, to find the area of the shape composed of blocks of heights from 1 to 6, we cut the shape at half the height and rearrange it to obtain a rectangle of length $6 + 1 = 7$ and height $\frac{6}{2} = 3$. This way, the area of the original shape equals to the area of the 7 by 3 rectangle, which according to equation (1), is calculated as $\frac{6(6+1)}{2} = \frac{6}{2} \cdot (6+1) = 3 \cdot 7 = 21$.

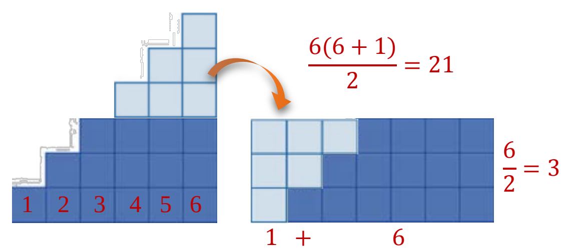


Figure 2

Formally, a partial sum of any sequence is defined as follows:

Definition 1.2 ► Let $\{a_n\}$ be a sequence and $a_1 + a_2 + \cdots + a_n + \cdots$ be its associated series.
The ***n*-th partial sum**, denoted $\boldsymbol{S_n}$, of the sequence (or the series) is the sum

$$\boldsymbol{a_1 + a_2 + \cdots + a_n}.$$

The overall sum of the entire series can be denoted by $\boldsymbol{S_\infty}$.
The **partial sums** on its own create a **sequence** $\{\boldsymbol{S_n}\}$.

Observation:

$$\boldsymbol{S_1 = a_1}$$
$$\boldsymbol{a_n = (a_1 + a_2 + \cdots + a_{n-1} + a_n) - (a_1 + a_2 + \cdots + a_{n-1}) = S_n - S_{n-1}}$$

To find the partial sum $\boldsymbol{S_n}$ of the first n terms of an **arithmetic sequence**, as before, we write it in increasing and decreasing order of terms and then add the resulting equations side by side.

$$\begin{array}{l} \boldsymbol{S_n} = \boldsymbol{a_1} + (\boldsymbol{a_1 + d}) + (\boldsymbol{a_1 + 2d}) + \cdots + (\boldsymbol{a_1 + (n-1)d}) \\ \boldsymbol{S_n} = \boldsymbol{a_n} + (\boldsymbol{a_n - d}) + (\boldsymbol{a_n - 2d}) + \cdots + (\boldsymbol{a_n - (n-1)d}) \end{array}$$

each column adds to $\boldsymbol{a_1 + a_n}$ *and there are* $\boldsymbol{n}$ *columns*

So, we obtain $\quad \boldsymbol{2S_n = n(a_1 + a_n)}$,

which gives us

$$\boldsymbol{S_n = \frac{n(a_1 + a_n)}{2}} \tag{2}$$

Notice that by substituting of the general term $a_n = a_1 + (n-1)d$ into the above formula, we can express the partial sum S_n in terms of the first term a_1 and the common difference d, as follows:

$$\boldsymbol{S_n = \frac{n(2a_1 + (n-1)d)}{2} \overset{\text{or equivalently}}{=} \frac{n}{2}(2a_1 + (n-1)d)} \qquad (3)$$

Example 5 ► **Finding a Partial Sum of an Arithmetic Sequence**

a. Find the sum of the first 100 consecutive natural numbers.
b. Find S_{20}, for the sequence $-10, -5, 0, 5, \ldots$.
c. Evaluate the sum $2 + (-1) + (-4) + \cdots + (-25)$.

Solution ► **a.** Using the formula (1) for $n = 100$, we have

$$\boldsymbol{S_{100}} = \frac{100 \cdot (100+1)}{2} = 50 \cdot 101 = \mathbf{5050}.$$

So the sum of the first 100 consecutive natural numbers is 5050.

b. To find S_{20}, we can use either formula (2) or formula (3). We are given $n = 20$ and $a_1 = -10$. To use formula (2) it is enough to calculate a_{20}. Since $d = 5$, we have

$$\boldsymbol{a_{20}} = a_1 + 19d = -10 + 19 \cdot 5 = \mathbf{85},$$

which gives us

$$\boldsymbol{S_{20}} = \frac{20(-10+85)}{2} = 10 \cdot 75 = \mathbf{750}.$$

Alternatively, using formula (3), we also have

$$\boldsymbol{S_{20}} = \frac{20}{2}(2(-10) + 19 \cdot 5) = 10(-20+95) = 10 \cdot 75 = \mathbf{750}.$$

c. This time, we are given $a_1 = 2$ and $a_n = -25$, but we need to figure out the number of terms n. To do this, we can use the n-th term formula $a_1 + (n-1)d$ and equal it to -25. Since $d = -1 - 2 = -3$, then we have

$$2 + (n-1)(-3) = -25,$$

which becomes

$$(n-1) = \frac{-27}{-3}$$

and finally

$$n = 10.$$

Now, using formula (2), we evaluate the requested sum to be

$$\boldsymbol{S_{10}} = \frac{10\big(2 + (-25)\big)}{2} = 5 \cdot (-23) = \mathbf{-115}.$$

As we saw in the beginning of this section, an **arithmetic sequence** is **linear** in nature and, as such, it can be identified by the formula $a_n = dn + b$, where $n \in \mathbb{N}$, $d, b \in \mathbb{R}$, and $b = a_1 - d$. This means that the n-th partial sum $S_n = a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_n$ of the associated arithmetic series can be written as

$$\sum_{i=1}^{n}(di + b),$$

and otherwise; each such sum represents the n-th partial sum S_n of an arithmetic series with the first term $d + b$ and the common difference d. Therefore, the above sum can be evaluated with the aid of formula (2), as shown in the next example.

Example 6 ► **Evaluating Finite Arithmetic Series Given in Sigma Notation**

Evaluate the sum $\sum_{i=1}^{16}(2i - 1)$.

Solution ► First, notice that the sum $\sum_{i=1}^{16}(2i - 1)$ represents S_{16} of an arithmetic series with the general term $a_n = 2n - 1$. Since $a_1 = 2 \cdot 1 - 1 = 1$ and $a_{16} = 2 \cdot 16 - 1 = 31$, then applying formula (2), we have

$$\sum_{i=1}^{16}(2i - 1) = \frac{16(1 + 31)}{2} = 8 \cdot 32 = \mathbf{256}.$$

Example 7 ► **Using Arithmetic Sequences and Series in Application Problems**

A worker is stacking wooden logs in layers. Each layer contains three logs less than the layer below it. There are two logs in the topmost layer, five logs in the next layer, and so on. There are 7 layers in the stack.

a. How many logs are in the bottom layer?
b. How many logs are in the entire stack?

Solution ► **a.** First, we observe that the number of logs in consecutive layers, starting from the top, can be expressed by an arithmetic sequence with $a_1 = 2$ and $d = 3$. Since we look for the number of logs in the seventh layer, we use $n = 7$ and the formula

$$a_n = 2 + (n - 1)3 = 3n - 1.$$

This gives us $a_7 = 3 \cdot 7 - 1 = \mathbf{20}$.
Therefore, there are 20 wooden logs in the bottom layer.

b. To find the total number of logs in the stack, we can evaluate the 7-th partial sum $\sum_{i=1}^{7}(3i - 1)$. Using formula (2), we have

$$\sum_{i=1}^{7}(3i - 1) = \frac{7(2 + 20)}{2} = 7 \cdot 11 = \mathbf{77}.$$

So, the entire stack consists of 77 wooden logs.

S.1 Exercises

Vocabulary Check *Fill in each blank with one of the suggested words, or the most appropriate term or phrase from the given list:* ***arithmetic, consecutive natural, difference, general, linear, partial sum, sigma.***

1. A sequence with a common difference between consecutive terms is called an ________________ sequence.

2. The sequence $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ $\underset{is\ /is\ not}{\underline{\qquad\qquad}}$ arithmetic because the _______________ between consecutive terms is not the same.

3. The _______________ term of an arithmetic sequence is given by the formula $a_n = a_1 + (n-1)d$.

4. A graph of an arithmetic sequence follows a _____________ pattern, therefore the general term of this sequence can be written in the fom $a_n = dn + b$.

5. The n-th ______________ _______ of a sequence is the sum of its first n terms. Partial sums can be written using _____________ notation.

6. The formula $\frac{n(n+1)}{2}$ allows for calculation of the sum of the first n ____________ __________ numbers.

Concept Check *True or False?*

7. The sequence $3, 1, -1, -3, \ldots$ is an arithmetic sequence.

8. The common difference for $2, 4, 2, 4, 2, 4, \ldots$ is 2.

9. The series $\sum_{i=1}^{12}(3 + 2i)$ is an arithmetic series.

10. The n-th partial sum S_n of any series can be calculated according to the formula $S_n = \frac{n(a_1+a_n)}{2}$.

Concept Check *Write a formula for the n-th term of each arithmetic sequence.*

11. $1, 3, 5, 7, 9, \ldots$

12. $0, 6, 12, 18, 24, \ldots$

13. $-4, -2, 0, 2, 4, \ldots$

14. $5, 1, -3, -7, -11, \ldots$

15. $-2, -\frac{3}{2}, -1, -\frac{1}{2}, 0, \ldots$

16. $1, \frac{5}{3}, \frac{7}{3}, 3, \frac{11}{3}, \ldots$

Concept Check *Given the information, write out the first five terms of the arithmetic sequence $\{a_n\}$. Then, find the 12-th term a_{12}.*

17. $a_n = 3 + (n-1)(-2)$

18. $a_n = 3 + 5n$

19. $a_1 = -8,\ d = 4$

20. $a_1 = 5,\ d = -2$

21. $a_1 = 10,\ a_2 = 8$

22. $a_1 = -7,\ a_2 = 3$

Concept Check *Find the number of terms in each arithmetic sequence.*

23. $3, 5, 7, 9, \ldots, 31$ **24.** $0, 5, 10, 15, \ldots, 55$ **25.** $4, 1, -2, \ldots, -32$

26. $-3, -7, -11, \ldots, -39$ **27.** $-2, -\frac{3}{2}, -1, -\frac{1}{2}, \ldots, 5$ **28.** $\frac{3}{4}, 3, \frac{21}{4}, \ldots, 12$

Given the information, find the indicated term of each arithmetic sequence.

29. $a_2 = 5,\ d = 3;\quad a_8$

30. $a_3 = -4,\ a_4 = -6;\quad a_{20}$

31. $1, 5, 9, 13, \ldots;\quad a_{50}$

32. $6, 3, 0, -3, \ldots;\quad a_{25}$

33. $a_1 = -8,\ a_9 = -64;\quad a_{10}$

34. $a_1 = 6,\ a_{18} = 74;\quad a_{20}$

35. $a_8 = 28,\ a_{12} = 40;\quad a_1$

36. $a_{10} = -37,\ a_{12} = -45;\quad a_2$

Given the arithmetic sequence, evaluate the indicated partial sum.

37. $a_n = 3n - 8;\quad S_{12}$

38. $a_n = 2 - 3n;\quad S_{16}$

39. $6, 3, 0, -3, \ldots;\quad S_9$

40. $1, 6, 11, 16, \ldots;\quad S_{15}$

41. $a_1 = 4,\ d = 3;\quad S_{10}$

42. $a_1 = 6,\ a_4 = -2;\quad S_{19}$

Use a formula for S_n to evaluate each series.

43. $1 + 2 + 3 + \cdots + 25$

44. $2 + 4 + 6 + \cdots + 50$

45. $\sum_{i=1}^{17} 3i$

46. $\sum_{i=1}^{22} (5i + 4)$

47. $\sum_{i=1}^{15} \left(\frac{1}{2}i + 1\right)$

48. $\sum_{i=1}^{20} (4i - 7)$

49. $\sum_{i=1}^{25} (-3 - 2i)$

50. $\sum_{i=1}^{13} \left(\frac{1}{4} + \frac{3}{4}i\right)$

Analytic Skills *Solve each problem.*

51. The sum of the interior angles of a triangle is 180º, of a quadrilateral is 360º and of a pentagon is 540º. Assuming this pattern continues, find the sum of the interior angles of a dodecagon (*12-sided figure*).

52. Deanna's aunt has promised to deposit \$1 in her account on the first day of her birthday month, \$2 on the second day, \$3 on the third day, and so on for 30 days. How much will this amount to over the entire month?

53. Ben is learning to drive. His first lesson is 26 minutes long, and each subsequent lesson is 4 minutes longer than the lesson before.

a. How long will his 15-th lesson be?

b. Overall, how long will Ben's training be after his 15-th lesson?

54. Suppose you visit the Grand Canyon and drop a penny off the edge of a cliff. The distance the penny will fall is 16 feet the first second, 48 feet the next second, 80 feet the third second, and so on in an arithmetic progression. What is the total distance the object will fall in 6 seconds?

55. If a contractor does not complete a multimillion-dollar construction project on time, he must pay a penalty of \$500 for the first day that he is late, \$700 for the second day, \$900 for the third day, and so on. Each day the penalty is \$200 larger than the previous day.

a. Write a formula for the penalty on the n-th day.

b. What is the penalty for the 10-th day?

c. If the contractor completes the project 14 days late, then what is the total amount of the penalties that the contractor must pay?

56. On the first day of October, an English teacher suggests to his students that they read five pages of a novel and every day thereafter increase their daily reading by two pages. If his students follow this suggestion, then how many pages will they read during October?

S.3 Geometric Sequences and Series

In the previous section, we studied sequences where each term was obtained by adding a constant number to the previous term. In this section, we will take interest in sequences where each term is obtained by multiplying the previous term by a constant number. Such sequences are called **geometric**. For example, the sequence $1, 2, 4, 8, \ldots$ is geometric because each term is multiplied by 2 to obtain the next term. Equivallently, the ratios between consecutive terms of this sequence are always 2.

Definition 2.1 ► A sequence $\{\boldsymbol{a_n}\}$ is called **geometric** if the quotient $\boldsymbol{r} = \frac{\boldsymbol{a_{n+1}}}{\boldsymbol{a_n}}$ of any consecutive terms of the sequence is constantly the same.

The **general term** of a **geometric sequence** is given by the formula

$$\boldsymbol{a_n = a_1 r^{n-1}}$$

The quotient $\boldsymbol{r}$ is referred to as **the common ratio** of the sequence.

Similarly as in the previous section, we can be visualize geometric sequences by plotting their values in a system of coordinates. For instance, *Figure 1* presents the graph of the sequence $1, 2, 4, 8, \ldots$. The common ratio of 2 causes each cosecutive point of the graph to be plotted twice as high as the previous one, and the slope between the n-th and $(n+1)$-st point to be exactly equal to the value of a_n. Generally, the **slope** between the n-th and $(n+1)$-st point of any geometric sequence **is proportional** to the value of a_n. This property characterises exponential functions. Hence, geometric sequences are **exponential** in nature. To develop the formula for the general term, we observe the pattern

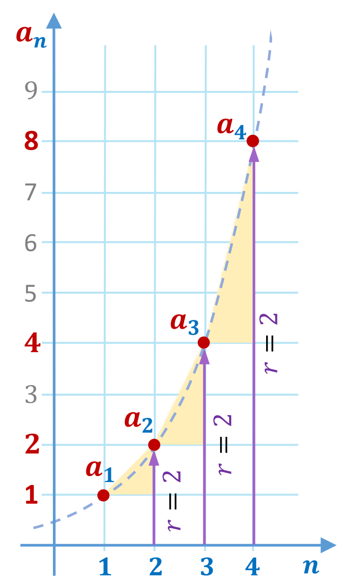


Figure 1

$$\begin{aligned} \boldsymbol{a_1} &= \boldsymbol{a_1} \\ \boldsymbol{a_2} &= \boldsymbol{a_1 r} \\ \boldsymbol{a_3} &= \boldsymbol{a_1 r^2} \\ \boldsymbol{a_4} &= \boldsymbol{a_1 r^3} \\ &\vdots \end{aligned}$$

so

$$\boxed{\boldsymbol{a_n = a_1 r^{n-1}}}$$

This is the **general term** of a geometric sequence

Particularly, the general term of the sequence $1, 2, 4, 8, \ldots$ is equal to $\boldsymbol{a_n = 2^{n-1}}$, because $a_1 = 1$ and $r = 2$ (the ratios of consecutive terms are constantly equal to 2).

Note: *To find the common ratio of a geometric sequence, divide any of its terms by the preceeding term.*

Example 1 ► **Identifying Geometric Sequences and Writing Their General Terms**

Determine whether the given sequence $\{a_n\}$ is geometric. If it is, then write a formula for the general term of the sequence.

a. $\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \ldots$

b. $\frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \frac{1}{12}, \ldots$

Solution ► **a.** After calculating ratios of terms by their peceding terms, we notice that they are always equal to $-\frac{1}{2}$. Indeed, $\frac{a_2}{a_1} = \frac{-\frac{1}{4}}{\frac{1}{3}} = -\frac{1}{2}$, $\frac{a_3}{a_2} = \frac{\frac{1}{8}}{-\frac{1}{4}} = -\frac{1}{2}$, and so on. Therefore, the given sequence is geometric with $a_1 = \frac{1}{2}$ and the common ratio $r = \frac{1}{2}$.
To find its general term, we follow the formula $a_n = a_1 r^{n-1}$. This gives us

$$\boldsymbol{a_n} = \frac{1}{2}\left(-\frac{1}{2}\right)^{n-1} = \boldsymbol{\frac{(-1)^{n-1}}{2^n}}.$$

b. Here, the ratios of terms by their peceding terms, $\frac{a_2}{a_1} = \frac{\frac{1}{6}}{\frac{1}{3}} = \frac{1}{2}$ and $\frac{a_3}{a_2} = \frac{\frac{1}{9}}{\frac{1}{6}} = \frac{2}{3}$, are not the same. So the sequence is not geometric.

Example 2 ► **Finding Terms of a Gometric Sequence**

Given the information, write out the first five terms of the geometric sequence $\{a_n\}$. Then, find the 8-th term a_8.

a. $a_n = 5(-2)^{n-1}$ **b.** $a_1 = 3,\ r = \frac{2}{3}$

Solution ► **a.** To find the first five terms of this sequence, we evaluate a_n for $n = 1,2,3,4,5$.

$$a_1 = 5(-2)^0 = 5$$
$$a_2 = 5(-2)^1 = -10$$
$$a_3 = 5(-2)^2 = 20$$
$$a_4 = 5(-2)^3 = -40$$
$$a_5 = 5(-2)^4 = 80$$

So, the first five terms are $\mathbf{5}, \mathbf{-10}, \mathbf{20}, \mathbf{-40}$, and $\mathbf{80}$.

The 8-th term equals $a_8 = 5(-2)^7 = \mathbf{-640}$.

b. To find the first five terms of a geometric sequence with $\boldsymbol{a_1} = \mathbf{3}$, $\boldsymbol{r} = \frac{\mathbf{2}}{\mathbf{3}}$, we substitute these values into the general term formula

$$\boldsymbol{a_n} = \boldsymbol{a_1 r^{n-1}} = \mathbf{3}\left(\frac{\mathbf{2}}{\mathbf{3}}\right)^{\boldsymbol{n-1}},$$

and then evaluate it for $n = 1,2,3,4,5$.

This gives us

$$a_1 = 3\left(\frac{2}{3}\right)^0 = 3$$
$$a_2 = 3\left(\frac{2}{3}\right)^1 = 2$$
$$a_3 = 3\left(\frac{2}{3}\right)^2 = \frac{4}{3}$$
$$a_4 = 3\left(\frac{2}{3}\right)^3 = \frac{8}{9}$$
$$a_5 = 3\left(\frac{2}{3}\right)^4 = \frac{16}{27}$$

So, the first five terms are $\mathbf{3}, \mathbf{2}, \frac{\mathbf{4}}{\mathbf{3}}, \frac{\mathbf{8}}{\mathbf{9}}$, and $\frac{\mathbf{16}}{\mathbf{27}}$.

The 8-th term equals $a_8 = 3\left(\frac{2}{3}\right)^7 = \frac{\mathbf{128}}{\mathbf{6561}}$.

Example 3 ► **Finding the Number of Terms in a Finite Geometric Sequence**

Determine the number of terms in the geometric sequence $1, -3, 9, -27, \dots, 729$.

Solution ► Since the common ratio r of this sequence is -3 and the first term $a_1 = 1$, then the n-th term $a_n = (-3)^{n-1}$. Since the last term is 729, we can set up the equation

$$a_n = (-3)^{n-1} = 729,$$

which can be written as

$$(-3)^{n-1} = (-3)^6$$

This equation holds if $\quad n - 1 = 6,$

which gives us $\quad n = 7.$

So, there are 7 terms in the given sequence.

Example 4 ► **Finding Missing Terms of a Geometric Sequence**

Given the information, determine the values of the indicated terms of a geometric sequence.

a. $a_3 = 5$ and $a_6 = -135$; find a_1 and a_8

b. $a_3 = 200$ and $a_5 = 50$; find a_4 if $a_4 > 0$

Solution ► **a.** As before, let r be the common ratio of the given sequence. Using the general term formula $a_n = a_1 r^{n-1}$ for $n = 6$ and $n = 3$, we can set up a system of two equations in two variables, r and a_1:

$$\begin{cases} -135 = a_1 r^5 \\ 5 = a_1 r^2 \end{cases}$$

To solve this system, let's divide the two equations side by side, obtaining

$$-27 = r^3,$$

which gives us

$$r = -3.$$

Substituting this value to equation (2), we have $5 = a_1 \cdot (-3)^2$, which gives us

$$\boldsymbol{a_1 = \frac{5}{9}}.$$

To find value a_8, we substitute $a_1 = -\frac{5}{2}, r = -3$, and $n = 8$ to the formula $a_n = a_1 r^{n-1}$. This gives us

$$\boldsymbol{a_8} = \frac{5}{9}(-3)^7 = \mathbf{-1215}.$$

b. Let r be the common ratio of the given sequence. Since $a_5 = a_4 r$ and $a_4 = a_3 r$, then $a_5 = a_3 r^2$. Hence, $r^2 = \frac{a_5}{a_3}$. Therefore,

$$r = \pm\sqrt{\frac{a_5}{a_3}} = \pm\sqrt{\frac{50}{200}} = \pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2}. \qquad (1)\ (2)$$

Since $a_3, a_4 > 0$ and $a_4 = a_3 r$, we choose the positive r-value. So we have

$$\boldsymbol{a_4} = a_3 r = 200\left(\frac{1}{2}\right) = \mathbf{100}.$$

Remark: *A **geometric mean** of two quantities $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined as $\sqrt{\boldsymbol{ab}}$. Notice that $\boldsymbol{a_4} = 100 = \sqrt{50 \cdot 200} = \sqrt{\boldsymbol{a_3} \cdot \boldsymbol{a_5}}$, so $\boldsymbol{a_4}$ is indeed the geometric mean of $\boldsymbol{a_3}$ and $\boldsymbol{a_5}$. Generally, for any $n > 1$, we have*

$$\boldsymbol{a_n} = a_{n-1} r = \sqrt{a_{n-1}^2 r^2} = \sqrt{a_{n-1} \cdot a_{n-1} r^2} = \sqrt{\boldsymbol{a_{n-1}} \cdot \boldsymbol{a_{n+1}}},$$

so every term (except for the first one) of a geometric sequence is the geometric mean of its adjacent terms.

Partial Sums

Similarly as with arithmetic sequences, we might be interested in evaluating the sum of the first n terms of a geometric sequence.

To find partial sum $\boldsymbol{S_n}$ of the first n terms of a **geometric sequence**, we line up formulas for $\boldsymbol{S_n}$ and $\boldsymbol{-rS_n}$ as shown below and then add the resulting equations side by side.

$$\begin{array}{rcl}
\boldsymbol{S_n} & = & \boldsymbol{a_1} \boxed{+\,\boldsymbol{a_1 r}} \boxed{+\,\boldsymbol{a_1 r^2}} + \cdots \boxed{+\,\boldsymbol{a_1 r^{n-1}}} \\
\boldsymbol{-rS_n} & = & \phantom{\boldsymbol{a_1}} \boxed{-\,\boldsymbol{a_1 r}} \boxed{-\,\boldsymbol{a_1 r^2}} - \cdots \boxed{-\,\boldsymbol{a_1 r^{n-1}}} - \boldsymbol{a_1 r^n} \\
\hline
(\mathbf{1} - \boldsymbol{r})\boldsymbol{S_n} & = & \boldsymbol{a_1} - \boldsymbol{a_1 r^n},
\end{array}$$

the terms of inside columns add to zero, so they cancel each other out

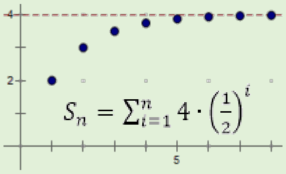
So, we obtain $(\mathbf{1} - \boldsymbol{r})\boldsymbol{S_n} = \boldsymbol{a_1} - \boldsymbol{a_1 r^n}$,

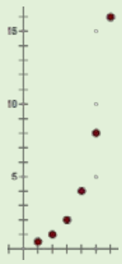
which gives us

$$\boldsymbol{S_n} = \frac{\boldsymbol{a_1}(\mathbf{1} - \boldsymbol{r^n})}{\mathbf{1} - \boldsymbol{r}}, \qquad (3)$$

as long as $r \neq 1$.

Observe that





$S_n = \sum_{i=1}^{n} \frac{1}{2} \cdot (2)^i$

If $|\boldsymbol{r}| < \mathbf{1}$, then the value of r^n gets closer and closer to zero for larger and larger *n* (we write: $r^n \to 0$ for $n \to \infty$). This means that the sum of all infinitely many terms of such a sequence **exists** and is equal to

$$\boldsymbol{S_\infty} = \frac{\boldsymbol{a_1}}{\mathbf{1} - \boldsymbol{r}} \qquad (4)$$

If $|r| > 1$, then the value of $|r^n|$ grows without bound for larger and larger *n*. Therefore, the sum S_∞ of all terms of such a sequence does not have a finite value. We say that such a sum does not exist.

If $|r| = 1$, then the sum S_∞ becomes $a_1 + a_1 + a_1 + \cdots$, or $a_1 - a_1 + a_1 - \cdots$. Neither of these sums has a finite value, unless $a_1 = 0$.

Hence overall, if $|\boldsymbol{r}| \geq \mathbf{1}$, then the sum $\boldsymbol{S_\infty}$ of a nonzero geometric sequence **does not exist**.

Example 5 ► **Finding a Partial Sum of a Geometric Sequence**

a. Find S_6, for the geometric sequence with $a_1 = 0.5$ and $r = 0.1$.

b. Evaluate the sum $1 - \left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)^2 - \cdots - \left(\frac{3}{4}\right)^9$.

Solution ►

a. Using formula (3) for $n = 6$, $a_1 = 0.5$ and $r = 0.1$, we calculate

$$\boldsymbol{S_6} = \frac{0.5(1 - 0.1^6)}{1 - 0.1} = \frac{0.5 \cdot 0.999999}{0.9} = \mathbf{0.555555}.$$

b. First, we observe that the given series is geometric with $a_1 = 1$ and $r = -\frac{3}{4}$. Equating the formula for the general term to the last term of the sum

$$a_1 r^{n-1} = \left(-\frac{3}{4}\right)^{n-1} = -\left(\frac{3}{4}\right)^9 = \left(-\frac{3}{4}\right)^9$$

and comparing the exponents,

$$n - 1 = 9$$

allows us to find the number of terms $n = \mathbf{10}$.

Now, we are ready to calculate the sum of the given series

$$\boldsymbol{S_{10}} = \frac{1\left(1 - \left(-\frac{3}{4}\right)^{10}\right)}{1 - \left(-\frac{3}{4}\right)} \cong \mathbf{0.539249}$$

Example 6 ► **Evaluating Infinite Geometric Series**

Decide wether or not the overall sum S_∞ of each geometric series exists and if it does, evaluate it.

a. $3 - \frac{9}{2} + \frac{27}{4} - \frac{81}{8} + \cdots$

b. $\sum_{i=0}^{\infty} 3 \cdot \left(\frac{2}{3}\right)^i$

Solution ►

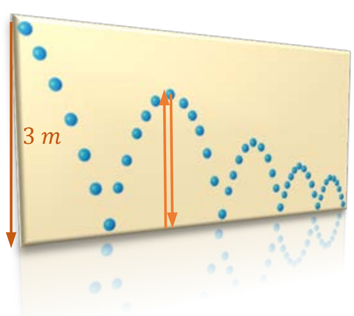
a. Since the common ratio of this series is $|r| = \left|\frac{-\frac{9}{2}}{3}\right| = \left|-\frac{3}{2}\right| = \frac{3}{2} > 1$, then the sum S_∞ does not exist.

b. This time, $|r| = \frac{2}{3} < 1$, so the sum S_∞ exists and can be calculated by following the formula (4). Using $a_1 = 3$ and $r = \frac{2}{3}$, we have

$$S_\infty = \frac{3}{1 - \frac{2}{3}} = \frac{3}{\frac{1}{3}} = \mathbf{9}$$

So, $\sum_{i=0}^{\infty} 3 \cdot \left(\frac{2}{3}\right)^i = \mathbf{9}$.

Example 4 ► **Using Geometric Sequences and Series in Application Problems**



When dropped from a certain height, a ball rebounds $\frac{3}{4}$ of the original height.

a. How high will the ball rebound after the fourth bounce if it was dropped from a height of 3 meters? *Round the answer to the nearest centimeter.*

b. Find a formula for the rebound height of the ball after its n-th bounce.

c. Assuming that the ball bounce forever, what is the total vertical distance traveled by the ball?

Solution ►

a. Let h_n represents the ball's rebound height after the n-th bounce, where $n \in \mathbb{N}$. Since the ball rebounds $\frac{3}{4}$ of the previous height, we have

$$h_1 = 3 \cdot \left(\frac{3}{4}\right)$$

$$h_2 = h_1 \cdot \left(\frac{3}{4}\right) = 3 \cdot \left(\frac{3}{4}\right)^2$$

$$h_3 = h_2 \cdot \left(\frac{3}{4}\right) = 3 \cdot \left(\frac{3}{4}\right)^3$$

$$h_4 = h_3 \cdot \left(\frac{3}{4}\right) = 3 \cdot \left(\frac{3}{4}\right)^4 \simeq .949\, m \simeq \mathbf{95\ cm}$$

After the fourth bounce, the ball will rebound approximately 95 centimeters.

b. Notice that the formulas developed in solution to *Example 4a* follow the pattern

$$h_n = 3 \cdot \left(\frac{3}{4}\right)^n$$

So this is the formula for the rebound height of the ball after its n-th bounce.

c. Let $h_0 = 3$ represents the vertical distance before the first bounce. To find the total verical distance D traveled by the ball, we add the vertical distance h_0 before the first bounce, and twice the vertical distances h_n after each bounce. So we have

$$D = h_0 + \sum_{n=1}^{\infty} h_n = 3 + \sum_{n=1}^{\infty} 3 \cdot \left(\frac{3}{4}\right)^n$$

Applying the formula $\frac{a_1}{1-r}$ for the infinite sum of a geometric series, we calculate

$$D = 3 + \frac{\frac{9}{4}}{1 - \frac{3}{4}} = 3 + \frac{\frac{9}{4}}{\frac{1}{4}} = 3 + \frac{9}{\cancel{4}} \cdot \frac{\cancel{4}}{1} = 3 + 9 = \mathbf{12}\ \boldsymbol{m}$$

Thus, the total vertical distance traveled by the ball is 12 meters.

S.3 Exercises

Vocabulary Check *Fill in each blank with appropriate formula, one of the suggested words, or with the most appropriate term or phrase from the given list:* ***consecutive, geometric mean, ratio.***

1. A geometric sequence is a sequence in which there is a constant ___________ between consecutive terms.

2. The sequence $1, 2, 6, 24, \ldots$ ________ (*is / is not*) geometric because the ratio between _______________terms is not the same.

3. The general term of a geometric sequence is given by the formula $a_n =$ __________.

4. If the absolute value of the common ratio of a geometric sequence is ____________ (*less / more*) than ____, then the sum of the associated geometric series exists and it is equal to $S_\infty =$ __________ .

5. The expression $\sqrt{ab}$ is called the ________________ ____________ of a and b.

Concept Check *True or False?*

6. The sequence $3, -1, \frac{1}{3}, -\frac{1}{9}, \ldots$ is a geometric sequence.

7. The common ratio for $0.05, 0.0505, 0.050505, \ldots$ is 0.05 .

8. The series $\sum_{i=1}^{7}(3 \cdot 2^i)$ is a geometric series.

9. The n-th partial sum S_n of any finite geometric series exists and it can be evaluated by using the formula $S_n = \frac{a_1(1-r^n)}{1-r}$.

Concept Check *Identify whether or not the given sequence is geometric. If it is, write a formula for its n-th term.*

10. $0, 3, 9, 27, \ldots$

11. $1, 5, 25, 125, \ldots$

12. $-9, 3, -1, \frac{1}{3}, \ldots$

13. $\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}, \ldots$

14. $1, -1, 1, -1, \ldots$

15. $0.9, 0.09, 0.009, 0.0009, \ldots$

16. $81, -27, 9, -3, \ldots$

17. $\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{8}{3}, \ldots$

18. $-\frac{1}{4}, -\frac{1}{5}, -\frac{4}{25}, -\frac{16}{125}, \ldots$

Concept Check *Given the information, write out the first four terms of the geometyric sequence $\{a_n\}$. Then, find the 8-th term a_8.*

19. $a_n = 3 \cdot 2^{n-1}$

20. $a_n = (-2)^{-n}$

21. $a_1 = 6,\ r = \frac{1}{3}$

22. $a_1 = 5,\ r = -1$

23. $a_1 = \frac{1}{3},\ a_2 = -\frac{1}{6}$

24. $a_1 = 100,\ a_2 = 10$

Concept Check *Find the number of terms in each geometric sequence.*

25. $1, 2, 4, \ldots, 1024$

26. $20, 10, 5, \ldots, \frac{5}{128}$

27. $-4, 2, -1, \ldots, \frac{1}{32}$

28. $3, -1, \frac{1}{3}, \ldots, \frac{1}{243}$

29. $6, -2, \frac{2}{3}, \ldots, -\frac{2}{81}$

30. $-24, 12, -6, \ldots, -\frac{3}{32}$

Given the information, find the indicated term of each geometric sequence.

31. $a_2 = 40,\ r = 0.1;\quad a_5$

32. $a_3 = 4,\ a_4 = -8;\quad a_{10}$

33. $2, -2, 2, -2, \ldots;\quad a_{50}$

34. $-4, 2, -1, \ldots;\quad a_{12}$

35. $a_1 = 6,\ a_4 = -\frac{2}{9};\quad a_8$

36. $a_1 = \frac{1}{9},\ a_6 = 27;\quad a_9$

37. $a_3 = \frac{1}{2},\ a_7 = \frac{1}{32};\quad a_4$ if $a_4 > 0$

38. $a_5 = 48,\ a_8 = -384;\quad a_{10}$

Given the geometric sequence, evaluate the indicated partial sum. Round your answer to three decimal places, if needed.

39. $a_n = 5\left(\frac{2}{3}\right)^{n-1};\quad S_6$

40. $a_n = -2\left(\frac{1}{4}\right)^{n-1};\quad S_{10}$

41. $2, 6, 18, \ldots;\quad S_8$

42. $6, 3, \frac{3}{2}, \ \ldots;\quad S_{12}$

43. $1 + \left(\frac{1}{5}\right) + \left(\frac{1}{5}\right)^2 + \cdots + \left(\frac{1}{5}\right)^5$

44. $1 - 3 + 3^2 - \cdots - 3^9$

45. $\sum_{i=1}^{7} 2(1.05)^{i-1}$

46. $\sum_{i=1}^{10} 3(2)^{i-1}$

Decide wether or not the infinite sum S_∞ of each geometric series exists and if it does, evaluate it.

47. $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$

48. $1 - \frac{5}{4} + \frac{25}{16} - \frac{125}{64} + \cdots$

49. $1 + 1.02 + 1.02^2 + 1.02^3 + \cdots$

50. $1 + 0.8 + 0.8^2 + 0.8^3 + \cdots$

51. $\sum_{i=1}^{\infty} (0.6)^{i-1}$

52. $\sum_{i=1}^{\infty} \frac{2}{5} (1.1)^{i-1}$

53. $\sum_{i=1}^{\infty} 2 \left(\frac{4}{3}\right)^i$

54. $\sum_{i=1}^{\infty} 2 \left(-\frac{3}{4}\right)^i$

Analytic Skills *Solve each problem.*

55. A company is offering a job with a salary of $30,000 for the first year and a 5% raise each year after that. If that 5% raise continues every year, find the amount of money you would earn in the 10th year of your career.

56. Suppose you go to work for a company that pays one penny on the first day, 2 cents on the second day, 4 cents on the third day and so on. If the daily wage keeps doubling, what will your wage be on the 30th day? What will your total income be for working 30 days?

57. Suppose a deposit of $2000 is made at the beginning of each year for 45 years into an account paying 12% compounded annually. What is the amount in the account at the end of the forty-fifth year?

58. A father opened a savings account for his daughter on her first birthday, depositing $1000. Each year on her birthday he deposits another $1000, making the last deposit on her 21st birthday. If the account pays 4.4% interest compounded annually, how much is in the account at the end of the day on the daughter's 21st birthday?

59. A ball is dropped from a height of 2 m and bounces 90% of its original height on each bounce.

a. How high off the floor is the ball at the top of the eighth bounce?

b. Asuming that the ball moves only vertically, how far has it traveled when it hits the ground for the eighth time?

60. Suppose that a ball always rebounds $\frac{2}{3}$ of the distance from which it falls. If this ball is dropped from a height of 9 ft, then approximately how far does it travel before coming to rest? Assime that the ball moves only vertically.

61. Suppose the midpoints of a **unit square** s_1 (*with the length of each side equal to one*) are connected to form another square, s_2, as in the accompanying figure. Suppose we continue indefinitely the process of creating a new square, s_{n+1}, by connecting the midpoints of the previous square, s_n. Calculate the sum of the areas of the infinite sequence of squares $\{s_n\}$.