

E.2

Applications of Systems of Linear Equations in Two Variables

Systems of equations are frequently used in solving applied problems. Although many problems with two unknowns can be solved with the use of a single equation with one variable, it is often easier to translate the information given in an application problem with two unknowns into two equations in two variables.

Here are some guidelines to follow when solving applied problems with two variables.

Solving Applied Problems with the Use of System of Equations

1. **Read** the problem, several times if necessary. When reading, watch the given information and what the problem asks for. Recognize the type of problem, such as geometry, total value, motion, solution, percent, investment, etc.
2. **Assign variables** for the unknown quantities. Use meaningful letters, if possible.
3. **Organize** the given information. Draw appropriate tables or diagrams; list relevant formulas.
4. **Write equations** by following a relevant formula(s) or a common sense pattern.
5. **Solve** the system of equations.
6. **Check** if the solution is reasonable in the context of the problem.
7. **State the answer** to the problem.

Below we show examples of several types of applied problems that can be solved with the aid of systems of equations.

Number Relations Problems

Example 1 ► Finding Unknown Numbers

Twice a number minus a second number equals 5. The sum of the two numbers is 16. Find the two numbers.

Solution ► Let a be the first number and b be the second number. The first sentence of the problem translates into the equation

$$2a - b = 5.$$

The second sentence translates to

$$a + b = 16.$$

Now, we can solve the system of the above equations, using the elimination method. Since

$$\begin{array}{r} 2a - b = 5 \\ + \{ a + b = 16 \\ \hline 3a = 21 \end{array} \quad / \div 3$$

$$a = 7,$$

then $b = 16 - a = 16 - 7 = 9$. Therefore, the two numbers are **7** and **9**.

Observation: A single equation in two variables gives us infinitely many solutions. For example, some of the solutions of the equation $a + b = 16$ are $(0, 16)$, $(1, 15)$, $(2, 14)$, and so on. Generally, any ordered pair of the type $(a, 16 - a)$ is the solution to this equation. So, when working with two variables, to find a specific solution we are in need of a second equation (not equivalent to the first) that relates these variables. This is why problems with two unknowns are solved with the use of systems of two equations.

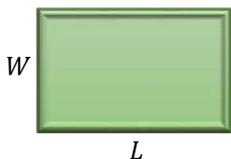
Geometry Problems

When working with geometry problems, we often use formulas for perimeter, area, or volume of basic figures. Sometimes, we rely on particular properties or theorems, such as *the sum of angles in a triangle is 180°* or *the ratios of corresponding sides of similar triangles are equal*.

Example 2 ▶ Finding Dimensions of a Rectangle

A homeowner sections off part of her backyard for a vegetable garden. The width of the garden is 4 ft shorter than the length. The perimeter of the garden is 52 ft. What are the dimensions of the garden?

Solution ▶



The problem refers to the perimeter of a rectangular garden. Suppose L and W represent the length and width of the rectangle. Then the perimeter is represented by the expression $2L + 2W$. Since the perimeter of the garden equals 52 feet, we set up the first equation

$$2L + 2W = 52 \quad (1)$$

The second equation comes directly from translating the second sentence of the problem, which tells us that the width is 4 feet shorter than the length. So, we write

$$W = L - 4 \quad (2)$$

Now, we can solve the system of the above equations, using the substitution method. After substituting equation (2) into equation (1), we obtain

$$2L + 2(L - 4) = 52$$

$$2L + 2L - 8 = 52 \quad \text{/+8}$$

$$4L = 60 \quad \text{/}\div 4$$

$$L = 15$$

So, $W = L - 4 = 15 - 4 = 11$.

Therefore, the garden is **15 feet** long and **11 feet** wide.

Number-Value Problems

Problems that refer to the number of different types of items and the value of these items are often solved by setting two equations. Either one equation compares the number of

items, and the other compares the value of these items, like in coin types of problems, or both equations compare the values of different arrangements of these items.

Example 3 ► Finding the Number of Each Type of Items

Paint Town sold 45 paintbrushes, one kind at \$8.50 each and another at \$9.75 each. In all, \$398.75 was taken in for the brushes. How many of each type were sold?

Solution ► Let x represent the number of brushes at \$8.50 each, and let y represent the number of brushes at \$9.75 each. Then the value of x brushes at \$8.50 each is $8.50x$. Similarly, the value of y brushes at \$9.75 each is $9.75y$. To organize the given information, we suggest to create and complete the following table.

	brushes at \$8.50 each	+	brushes at \$9.75 each	= Total
number of brushes	x		y	45
value of brushes (in \$)	$8.50x$		$9.75y$	398.75

Since we work with two variables, we need two different equations in these variables. The first equation comes from comparing the number of brushes, as in the middle row. The second equation comes from comparing the values of these brushes, as in the last row. So, we have the system

$$\begin{cases} x + y = 45 \\ 8.50x + 9.75y = 398.75 \end{cases}$$

to solve. This can be solved by substitution. From the first equation, we have $y = 45 - x$, which after substituting to the second equation gives us

$$\begin{aligned} 8.50x + 9.75(45 - x) &= 398.75 && / \cdot 100 \\ 850x + 975(45 - x) &= 39875 \\ 850x + 43875 - 975x &= 39875 && / -43875 \\ -125x &= -4000 && / \div (-125) \\ x &= \mathbf{32} \end{aligned}$$

Then, $y = 45 - x = 45 - 32 = \mathbf{13}$.

Therefore, there were **32** brushes sold at \$8.50 each and **13** brushes sold at \$9.75 each.

Example 4 ► Finding the Unit Cost of Each Type of Items

A carpenter purchased 60 ft of redwood and 80 ft of pine for a total cost of \$286. A second purchase, at the same prices, included 100 ft of redwood and 60 ft of pine for a total cost of \$396. Find the cost per foot of redwood and of pine.

Solution ▶ Let r be the cost per foot of redwood and let p be the cost per foot of pine. Then the value of 60 ft of redwood is represented by $60r$, and the value of 80 ft of pine is represented by $80p$. Using the total cost of \$286, we write the first equation

$$60r + 80p = 286$$

Similarly, using the total cost of \$396 for 100 ft of redwood and 60 ft of pine, we write the second equation

$$100r + 60p = 396$$

We will solve the system of the above equations via the elimination method.

$$\begin{array}{r} \left\{ \begin{array}{l} 60r + 80p = 286 \\ 100r + 60p = 396 \end{array} \right. \begin{array}{l} / \cdot (-3) \\ / \cdot 4 \end{array} \\ + \left\{ \begin{array}{l} -180r - 240p = -858 \\ 400r + 240p = 1584 \end{array} \right. \\ \hline 220r = 726 \qquad / \div 220 \\ r = \mathbf{3.30} \end{array}$$

After substituting $r = 3.30$ into the first equation, we obtain

$$60 \cdot 3.30 + 80p = 286$$

$$198 + 80p = 286$$

$$80p = 88$$

$$p = \mathbf{1.10}$$

So, redwood costs **3.30 \$/ft**, and pine costs **1.10 \$/ft**.

Investment Problems

Investment problems usually involve calculation of simple annual interest according to the formula $I = Prt$, where I represents the amount of interest, P represents the principal which is the amount of investment, r represents the annual interest rate in decimal form, and t represents the time in years.

Example 5 ▶ Finding the Amount of Each Loan

A student takes out two loans totaling \$4600 to help pay for college expenses. One loan is at 5% annual interest, and the other is at 8% annual interest. The first-year interest is \$278. Find the amount of each loan.

Solution ▶ Suppose the amount of loan taken at 5% is x , and the amount of loan taken at 8% is y . The situation can be visualized by the diagram



So, the system of equations to solve is

$$\begin{cases} x + y = 4600 \\ 0.05x + 0.08y = 278 \end{cases}$$

Using the substitution method, we solve the first equation for x ,

$$x = 4600 - y, \quad (*)$$

substitute it into the second equation,

$$0.05(4600 - y) + 0.08y = 278, \quad / \cdot 100$$

and after elimination of decimals via multiplication by 100,

$$5(4600 - y) + 8y = 27800,$$

solve it for y :

$$23000 - 5y + 8y = 27800 \quad / -23000$$

$$3y = 4800$$

$$y = \mathbf{1600}$$

Then, after plugging in the y -value to the substitution equation (*), we obtain

$$x = 4600 - 1600 = \mathbf{3000}$$

So, the amount of loan taken at 5% is **\$3000**, and the amount of loan taken at 8% is **\$1600**.

decimal
elimination is
optional

Mixture – Solution Problems

In mixture or solution problems, we typically mix two or more mixtures or solutions with different concentrations of a particular substance that we will refer to as the content. For example, if we are interested in the salt concentration in salty water, the salt is referred to as the content. When solving mixture problems, it is helpful to organize data in a table such as the one shown below.

	%	·	volume =	content
type I				
type II				
mixture/solution				

Example 6 ▶ Solving a Mixture Problem

The owner of a natural foods store made 20 lb of a dried fruit and granola blend that is 50% dried fruit by mixing a dried fruit and granola blend that is 20% dried fruit with a blend that is 60% dried fruit. How many pounds of each type were used to get the desired blend?

Solution ▶ Suppose x is the amount of the 20% blend and y is the amount of the 60% blend. We complete the table

	%	·	volume	=	dried fruit
20% blend	0.2		x		$0.2x$
60% blend	0.6		y		$0.6y$
50% blend	0.5		20		10

The last column is completed by multiplying the “%” and “volume” columns.

The first equation comes from combining the weight of blends as indicated in the “volume” column. The second equation comes from combining the weight of dried fruit, as indicated in the last column. So, we solve

$$\begin{cases} x + y = 20 \\ 0.2x + 0.6y = 10 \end{cases}$$

Using the substitution method, we solve the first equation for x ,

$$x = 20 - y, \quad (*)$$

and substitute it into the second equation and solve it for y :

$$0.2(20 - y) + 0.6y = 10 \quad / \cdot 10$$

$$2(20 - y) + 6y = 100$$

$$40 - 2y + 6y = 100 \quad / -40$$

$$4y = 60 \quad / \div 4$$

$$y = 15$$

Then, using the substitution equation (*), we find the value of x :

$$x = 20 - 15 = 5$$

So, to obtain the desired blend, **5 lb** of 20% and **15 lb** of 60% blend was used.

Motion Problems

In motion problems, we follow the formula **Rate · Time = Distance**. Drawing a diagram and completing a table based on the **$R \cdot T = D$** formula is usually helpful. In some motion problems, in addition to the rate of the moving object itself, we need to consider the rate of a moving medium such as water current or wind. The overall rate of a moving object is typically either the sum or the difference between the object’s own rate and the rate of the moving medium.

Example 7 ▶ Finding Rates in a Motion Problem

A car travels 250 km in the same time that a truck travels 225 km. If the rate of the car is 8 kilometers per hour faster than the rate of the truck, find both rates.

Solution ▶ Using meaningful letters, let c and t represent the rate of the car and the truck, respectively. Since the rate of the car, c , is 8 kph faster than the rate of the truck, t , we can write the first equation:

$$c = t + 8$$

The second equation comes from comparing the travel time of each vehicle, as indicated in the table below.

	R	\cdot	T	$=$	D
car	c		$\frac{250}{c}$		250
truck	t		$\frac{225}{t}$		225

To find the expression for time, we follow the formula $T = \frac{D}{R}$, which comes from solving $R \cdot T = D$ for T .

So, we need to solve the system

$$\begin{cases} c = t + 8 \\ \frac{250}{c} = \frac{225}{t} \end{cases}$$

Cross-multiplication can only be applied to a **proportion** (an equation with a single fraction on each side.)

Notice that multiplication by the **LCD** would give the same result.

After substituting the first equation into the second, we obtain

$$\frac{250}{t+8} = \frac{225}{t},$$

which can be solved by cross-multiplying

$$\begin{aligned} 250t &= 225(t+8) \\ 250t &= 225t + 1800 && /-225t \\ 25t &= 1800 && / \div 25 \\ t &= 72 \end{aligned}$$

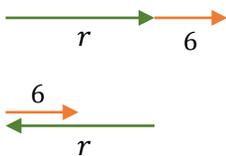
Then, we use this value to find $c = t + 8 = 72 + 8 = 64$.

So, the rate of the car is **64 kph**, and the rate of the truck is **72 kph**.

Example 8 ▶ Solving a Motion Problem with a Current

Mia's motorboat took 3 hr to make a trip downstream with a 6-mph current. The return trip against the same current took 5 hr. Find the speed of the boat in still water.

Solution ▶



Let r be the speed of the boat in still water.

Then the speed of the boat moving downstream is 6 mph faster because of the current going in the same direction as the boat. So, it is represented by $r + 6$.

The speed of the boat moving against the current is 6 mph slower. So, it is represented by $r - 6$.

Also, let d represent the distance covered by the boat going in one direction.

To organize the information, we can complete the table below.

	R	\cdot	T	$=$	D
downstream	$r + 6$		3		d
upstream	$r - 6$		5		d

The two equations come from following the formula $R \cdot T = D$, as indicated in each row.

$$\begin{cases} (r + 6) \cdot 3 = d \\ (r - 6) \cdot 5 = d \end{cases}$$

Since left sides of both equations represent the same distance d , we can equal them and solve for r :

$$\begin{aligned} (r + 6) \cdot 3 &= (r - 6) \cdot 5 \\ 3r + 18 &= 5r - 30 && /-3r, +30 \\ 48 &= 2r && /\div 2 \\ \mathbf{24} &= r \end{aligned}$$

So, the speed of the boat in still water is **24 mph**.

E.2 Exercises

Concept Check

- If a container of liquid contains 60 liters of 40% acid solution, what is the volume of pure acid?
- If \$5000 is invested in an account paying 3% simple annual interest, how much interest will be earned during the first year?
- If 1 kilogram of turkey costs \$8.49, give an expression for the cost of x kilograms. 
- If 1 ticket to the movie Avatar costs \$9 and y tickets are sold, give an expression for the amount collected from the sale.
- If the speed of a motorboat in still water is x kph and the speed of a current is c kph, give an expression for the speed of this boat travelling **a.** downstream, **b.** upstream.
- If a plane travels with its own rate of r kph and there is a wind blowing at w kph, give an expression for the rate of this plane travelling with the **a.** headwind, **b.** tailwind.

Solve each problem using two variables.

- Two angles are complementary, which means that their sum is 90° . The measure of the larger angle is 9° more than eight times the measure of the smaller angle. Find the measures of the two angles.
- Two angles are supplementary, which means that their sum is 180° . The measure of the larger angle is 40° more than three times the measure of the smaller angle. Find the measures of the two angles.
- The sum of the base and the height of a triangle is 192 inches. The height is twice the base. Find the base and the height.
- The width of a standard tennis court used for playing doubles is 42 ft less than the length. The perimeter of the court is 228 ft. Find the dimensions of the court. 



11. A marathon is a run that covers about 26 mi. Paula Radcliffe of Great Britain holds the world record time in the women's marathon. At one point in her record-setting race, she was three times as far from the end of the course as she was from the starting point. How many miles had she run at that time?

12. Jane asked her students to determine the two numbers that she placed in a sealed envelope. The smaller number is 2 more than one-third of the larger number. Three times the larger number is 1 less than eight times the smaller. Find the numbers.

13. In the 2008 Summer Olympics in Beijing, China, Russia earned 5 fewer gold medals than bronze. The number of silver medals earned was 35 less than twice the number of bronze medals. Russia earned a total of 72 medals. How many of each kind of medal did Russia earn?



14. A piece of wire that is 102 cm long is to be cut into two pieces, each to be bent to make a square. The length of a side of one square is to be twice the length of a side of the other. How should the wire be cut?



15. A coffee shop sells 12-oz and 20-oz cups of coffee. On a particular day, the shop sold a total of 508 cups of coffee. If the shop sold 3 times as many 20-oz cups of coffee as 12-oz cups of coffee, how many cups of each size did the coffee shop sell that day?

16. Mountainside Fleece sold 40 neckwarmers. Solid-color neckwarmers sold for \$9.90 each and print ones sold for \$12.75 each. In all, \$421.65 was taken in for the neckwarmers. How many of each type were sold?

17. A local performing-arts group holds its annual winter arts festival in December. The group charges \$20 for adult admission tickets and \$12 for children's admission tickets. The group raised \$13,344 on the sale of 824 tickets. How many of each type of ticket were sold?

18. The total of Megan's writing and math scores on the SAT was 1094. Her math score was 248 points higher than her writing score. What were her math and writing SAT scores?

19. Andrew McGinnis works at Arby's. During one particular day, he sold 15 Junior Roast Beef sandwiches and 10 Big Montana sandwiches, totaling \$75.25. Another day he sold 30 Junior Roast Beef sandwiches and 5 Big Montana sandwiches, totaling \$84.65. How much did each type of sandwich cost?



20. At a business meeting at Panera Bread, the bill for two cappuccinos and three house lattes was \$14.55. At another table, the bill for one cappuccino and two house lattes was \$8.77. How much did each type of beverage cost?

21. New York City and Washington, D.C., were the two most expensive cities for business travel in 2009. Based on the average total costs per day for each city (which include a hotel room, car rental, and three meals), 2 days in New York and 3 days in Washington cost \$2772, while 4 days in New York and 2 days in Washington cost \$3488. What was the average cost per day in each city?



22. A public storage facility rents small and large storage lockers. A small storage locker has 200 ft² of storage space, and a large storage locker has 800 sq ft of storage space. The facility has 54 storage lockers and a total of 22,800 ft² of storage space. How many large storage lockers does the facility have?

23. Brandt's student loans totaled \$3500. Part of this loan was borrowed at 3% interest and the rest at 2.5%. After one year, Brandt had accumulated \$97.50 in interest. What was the amount of each loan?

24. An executive nearing retirement made two investments totaling \$45,000. In one year, these investments yielded \$2430 in simple interest. Part of the money was invested at 4% and the rest at 6%. How much was invested at each rate?
25. Twice the amount of money that was invested in a low-risk fund was invested in a high-risk fund. After one year, the low-risk fund earned 6%, and the high-risk fund lost 9%. The investments had a net loss of \$210. How much was invested in each fund?
26. Mary has two accounts to choose from, one at 5.5% and the other at 7%. Mary invested \$2,000 more in the 7% account than the 5.5% account. If the total interest for one year was \$827.50, then how much was invested in each account?
27. How many ounces of 5% hydrochloric acid, 20% hydrochloric acid, and water must be combined to get 10 oz of solution that is 8.5% hydrochloric acid if the amount of water used must equal the total amount of the other two solutions?
28. A sample of dimes and quarters totals \$18. If there are 111 coins in all, how many of each type of coin are there?
29. A pharmacist has two vitamin-supplement powders. The first powder is 20% vitamin B1 and 10% vitamin B2. The second is 15% vitamin B1 and 20% vitamin B2. How many milligrams of each powder should the pharmacist use to make a mixture that contains 130 mg of vitamin B1 and 80 mg of vitamin B2?
30. Soybean meal is 16% protein, and cornmeal is 9% protein. How many pounds of each should be mixed to get a 350-kg mixture that is 12% protein?
31. Christina makes a \$9.25 purchase at a bookstore in Reno with a \$20 bill. The store has no bills and gives her the change in quarters and dollar coins. There are 19 coins in all. How many of each kind are there?



32. A motorboat traveling with the current went 36 km in 2 h. Against the current, it took 3 h to travel the same distance. Find the rate of the boat in the calm water and the rate of the current.

33. A cabin cruiser traveling with the current went 45 km in 3 h. Against the current, it took 5 h to travel the same distance. Find the rate of the cabin cruiser in calm water and the rate of the current.

34. A jet plane flying with the wind went 2200 km in 4 h. Against the wind, the plane could fly only 1840 km in the same amount of time. Find the rate of the plane in the calm air and the rate of the wind.



35. Flying with the wind, a pilot flew 1200 km between two cities in 4 h. The return trip against the wind took 5 h. Find the rate of the plane in the calm air and the rate of the wind.

36. A rowing team rowing with the current traveled 20 km in 2 h. Rowing against the current, the team rowed 12 km in the same amount of time. Find the rate of the team in calm water and the rate of the current.

37. A plane flying with a tailwind flew 1500 km in 5 h. Against the wind, the plane required 6 h to fly the same distance. Find the rate of the plane in the calm air and the rate of the wind.

Analytic Skills Solve each problem using two variables.

38. Rod is a pilot for Crossland Airways. He computes his flight time against a headwind for a trip of 2900 mi at 5 hr. The flight would take 4 hr and 50 min if the headwind were half as great. Find the headwind and the plane's airspeed (speed in still air).

39. Donna is late for a sales meeting after traveling from one town to another at a speed of 32 mph. If she had traveled 4 mph faster, she could have made the trip in 1 hr less time. How far apart are the towns?



40. A truck and a car leave a service station at the same time and travel in the same direction. The truck travels at 96 kph and the car at 108 kph. They can maintain CB radio contact within a range of 10 km. When will they lose contact?

41. The radiator in Michelle's car contains 16 L of antifreeze and water. This mixture is 25% antifreeze. How much of this mixture should she drain and replace with pure antifreeze so that there will be a mixture of 50% antifreeze?

42. Bennet Custom Flooring has 0.5 gallons of stain that is 20% brown and 80% neutral. A customer orders 1.5 gallons of a stain that is 60% brown and 40% neutral. How much of the pure brown stain and how much of the neutral stain should be added to the original 0.5 gallons to make up the order?

