T.2 Trigonometric Ratios of an Acute Angle and of Any Angle

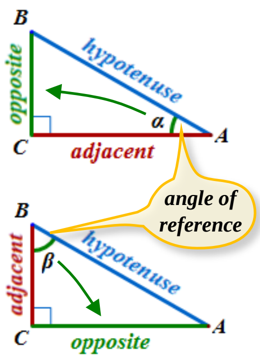


Figure 2.1

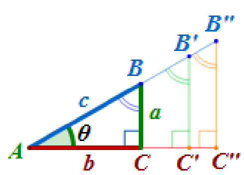


Figure 2.2

Generally, trigonometry studies ratios between sides in right angle triangles. When working with right triangles, it is convenient to refer to the side **opposite** to an angle, the side **adjacent** to (next to) an angle, and the **hypotenuse**, which is the longest side, opposite to the right angle. Notice that the opposite and adjacent sides depend on the **angle of reference** (one of the two acute angles.) However, the hypotenuse stays the same, regardless of the choice of the angle or reference. See *Figure 2.1*.

Notice that any two right triangles with the same acute angle $\boldsymbol{\theta}$ are **similar**. See *Figure 2.2*. **Similar** means that their corresponding angles are **congruent** and their corresponding sides are **proportional**. For instance, assuming notation as on *Figure 2.2*, we have

$$\frac{AB}{AB'} = \frac{AC}{AC'} = \frac{BC}{B'C'},$$

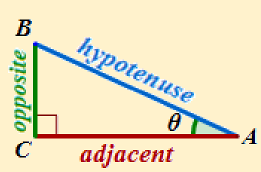
or equivalently

$$\frac{BC}{AB} = \frac{B'C'}{AB'}, \qquad \frac{AC}{AB} = \frac{AC'}{AB'}, \qquad \frac{BC}{AC} = \frac{B'C'}{AC'}.$$

Therefore, the ratios of any two sides of a right triangle does not depend on the size of the triangle but only on the size of the angle of reference. See the following demonstration. This means that we can study those **ratios** of sides as **functions** of an acute angle.

Trigonometric Functions of Acute Angles

Definition 2.1 ►

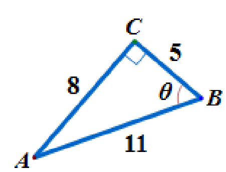


Given a **right angle triangle** with an **acute** angle $\boldsymbol{\theta}$, the three **primary trigonometric ratios** of the angle $\boldsymbol{\theta}$, called ***sine***, ***cosine***, and ***tangent*** (abbreviation: *sin*, *cos*, *tan*) are defined as follows:

$$\sin\theta = \frac{\mathit{Opposite}}{\mathit{Hypotenuse}}, \qquad \cos\theta = \frac{\mathit{Adjacent}}{\mathit{Hypotenuse}}, \qquad \tan\theta = \frac{\mathit{Opposite}}{\mathit{Adjacent}}$$

For easier memorization, we can use the acronym **SOH – CAH – TOA** (read: *so - ka - toe - ah*), formed from the first letter of the function and the corresponding ratio.

Example 1 ► **Identifying Sides of a Right Triangle to Form Trigonometric Ratios**

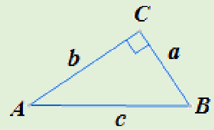


Identify the hypotenuse, opposite, and adjacent side of angle θ and state values of the three trigonometric ratios.

Solution ► Side AB is the hypotenuse, as it lies across the right angle.
Side BC is the adjacent, as it is part of the angle θ, other than hypotenuse.
Side AC is the opposite, as it lies across angle θ.
Therefore, $\sin\theta = \frac{opp.}{hyp.} = \frac{\mathbf{8}}{\mathbf{11}}$, $\cos\theta = \frac{adj.}{hyp.} = \frac{\mathbf{5}}{\mathbf{11}}$, and $\tan\theta = \frac{opp.}{adj.} = \frac{\mathbf{8}}{\mathbf{5}}$.

The three **primary trigonometric ratios** together with the **Pythagorean Theorem** allow us to **solve** any right angle triangle. That means that given the measurements of two sides, or one side and one angle, with a little help of algebra, we can find the measurements of all remaining sides and angles of any right triangle. See *section T.4*.

Pythagorean Theorem ► A **triangle** $\boldsymbol{ABC}$ **is right** with $\angle C = 90°$ if and only if $\boldsymbol{a^2 + b^2 = c^2}$.

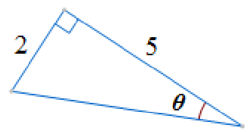


Convention: *The **side opposite** the given **vertex** (or angle) is named after the vertex, except that by a small rather than a capital letter. For example, the side opposite vertex **A** is called **a**.*

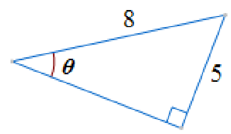
Example 2 ► **Finding Values of Trigonometric Ratios With the Aid of Pythagorean Theorem**

Given the triangle, find the exact values of the sine, cosine, and tangent ratios for angle θ.

a.



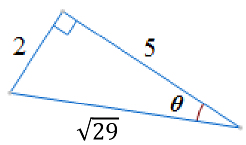
b.



Solution ► **a.** Let $\boldsymbol{h}$ denote the hypotenuse. By **Pythagorean Theorem**, we have

$$h^2 = 2^2 + 5^2$$
$$h = \sqrt{4+25} = \sqrt{29}$$

Now, we are ready to state the exact values of the three trigonometric ratios:



$$\sin\theta = \frac{2}{\sqrt{29}} \cdot \frac{\sqrt{29}}{\sqrt{29}} = \boldsymbol{\frac{2\sqrt{29}}{29}}$$

$$\cos\theta = \frac{5}{\sqrt{29}} \cdot \frac{\sqrt{29}}{\sqrt{29}} = \boldsymbol{\frac{5\sqrt{29}}{29}}$$

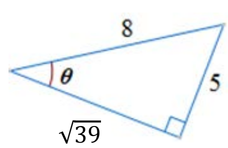
Note:
It is customary to rationalize the denominator.

$$\tan\theta = \boldsymbol{\frac{2}{5}}$$

b. Let $\boldsymbol{a}$ denote the adjacent side. By the **Pythagorean Theorem**, we have

$$a^2 + 5^2 = 8^2$$
$$a = \sqrt{8^2 - 5^2} = \sqrt{64-25} = \sqrt{39}$$

Now, we are ready to state the exact values of the three trigonometric ratios:



$$\sin\theta = \boldsymbol{\frac{5}{8}}$$

$$\cos\theta = \boldsymbol{\frac{\sqrt{39}}{8}}$$

$$\tan\theta = \frac{5}{\sqrt{39}} \cdot \frac{\sqrt{39}}{\sqrt{39}} = \boldsymbol{\frac{5\sqrt{39}}{39}}$$

Trigonometric Functions of Any Angle

Notice that any angle of a right triangle, other than the right angle, is acute. Thus, the “SOH – CAH – TOA” definition of the trigonometric ratios refers to acute angles only. However, we can extend this definition to include all angles. This can be done by observing our right triangle within the Cartesian Coordinate System.

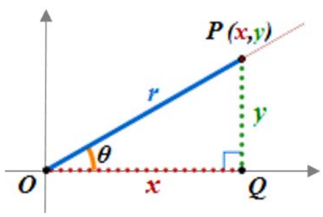


Figure 2.3

Let triangle OPQ with $\angle Q = 90°$ be placed in the coordinate system so that O coincides with the origin, Q lies on the positive part of the x-axis, and P lies in the first quadrant. See *Figure 2.3*. Let (x, y) be the coordinates of the point P, and let θ be the measurement of $\angle QOP$. This way, angle θ is in standard position and the triangle OPQ is obtained by **projecting** point P perpendicularly onto the x-axis. Thus in this setting, the position of point P actually determines both the angle θ and the ΔOPQ. Observe that the coordinates of point P ($\boldsymbol{x}$ and $\boldsymbol{y}$) really represent the length of the **adjacent** and the **opposite** side, correspondingly. Since the length of the **hypotenuse** represents the distance of the point P from the origin, it is often denoted by $\boldsymbol{r}$ (from *radius*.)

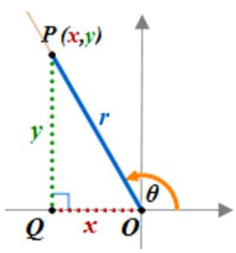


Figure 2.4

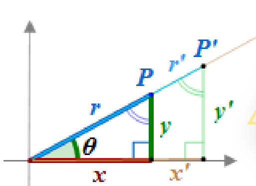
By rotating the radius $\boldsymbol{r}$ and projecting the point P perpendicularly onto x-axis (follow the green dotted line from P to Q in *Figure 2.4*), we can obtain a right triangle corresponding to any angle θ, not only an acute angle. Since the coordinates of a point in a plane can be negative, to establish a correcpondence between the coordinates x and y of the point P, and the distances OQ and QP, it is convenient to think of **directed distances** rather than just distances. Distance becomes directed if we assign a sign to it. So, lets assign a positive sign to horizontal or vertical distances that follow the directions of the corresponding number lines, and a negative sign otherwise. For example, the directed distance $OQ = x$ in *Figure 2.3* is positive because the direction from O to Q follows the order on the x-axis while the directed distance $OQ = x$ in *Figure 2.4* is negative because the direction from O to Q is against the order on the x-axis. Likewise, the directed distance $QP = y$ is positive for angles in the first and second quadrant (as in *Figure 2.3* and *2.4*), and it is negative for angles in the third and fourth quadrant (convince yourself by drawing a diagram).

Definition 2.2 ► Let $P(\boldsymbol{x}, \boldsymbol{y})$ be any point, different than the origin, on the terminal side of an angle $\boldsymbol{\theta}$ in standard position. Also, let $\boldsymbol{r} = \sqrt{\boldsymbol{x}^2 + \boldsymbol{y}^2}$ be the distance of the point P from the origin. We define

$$\sin\boldsymbol{\theta} = \frac{\boldsymbol{y}}{\boldsymbol{r}}, \qquad \cos\boldsymbol{\theta} = \frac{\boldsymbol{x}}{\boldsymbol{r}}, \qquad \tan\boldsymbol{\theta} = \frac{\boldsymbol{y}}{\boldsymbol{x}} \quad (\text{for } x \neq 0)$$

Observations:

- For acute angles, definition 2.2 agrees with the “**SOH** – **CAH** – **TOA**” definition 2.1.



$$\frac{y}{r} = \frac{y'}{r'}$$

$$\frac{x}{r} = \frac{x'}{r'}$$

$$\frac{y}{x} = \frac{y'}{x'}$$

- Proportionality of similar triangles guarantees that each point of the same terminal ray defines the same trigonometric ratio. This means that the above definition assigns a unique value to each trigonometric ratio for any given angle regardless of the point chosen on the terminal side of this angle. Thus, the above trigonometric ratios are in fact **functions of any real angle** and these functions are properly defined in terms of x, y, and r.

- Since $r > 0$, the first two trigonometric functions, **sine** $\left(\frac{y}{r}\right)$ and **cosine** $\left(\frac{x}{r}\right)$, are defined for any real angle θ.

- The third trigonometric function, **tangent** $\left(\frac{y}{x}\right)$, is defined for all real angles θ except for angles with terminal sides on the y-axis. This is because the x-coordinate of any point on the y-axis equals zero, which cannot be used to create the ratio $\frac{y}{x}$. Thus, tangent is a function of all real angles, except for $90°, 270°$, and so on (generally, except for angles of the form $\mathbf{90° + k \cdot 180°}$, where $\boldsymbol{k}$ is an integer.)

- Notice that after dividing both sides of the Pythagorean equation $\boldsymbol{x^2 + y^2 = r^2}$ by $\boldsymbol{r^2}$, we have

$$\left(\frac{\boldsymbol{x}}{\boldsymbol{r}}\right)^2 + \left(\frac{\boldsymbol{y}}{\boldsymbol{r}}\right)^2 = \mathbf{1}.$$

Since $\frac{x}{r} = \cos\theta$ and $\frac{y}{r} = \sin\theta$, we obtain the following **Pythagorean Identity**:

$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

- Also, observe that as long as $x \neq 0$, the quatient of the first two ratios gives us the third ratio:

$$\frac{\sin\theta}{\cos\theta} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{\cancel{r}} \cdot \frac{\cancel{r}}{x} = \frac{y}{x} = \tan\theta.$$

Thus, we have the identity

$$\boxed{\tan\theta = \frac{\sin\theta}{\cos\theta}}$$

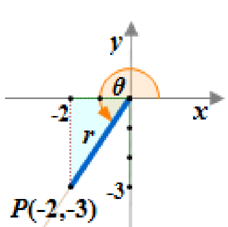
for all angles θ in the domain of the tangent.

Example 3 ► **Evaluating Trigonometric Functions of any Angle in Standard Position**

Find the exact value of the three primary trigonometric functions of an angle θ in standard position whose terminal side contains the point

a. $P(-2,-3)$ **b.** $P(0,1)$

Solution ►



a. To ilustrate the situation, lets sketch the least positive angle θ in standard position with the point $P(-2,-3)$ on its terminal side.

To find values of the three trigonometric functions, first, we will determine the length of r:

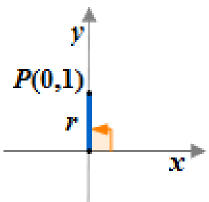
$$r = \sqrt{(-2)^2 + (-3)^2} = \sqrt{4+9} = \sqrt{13}$$

Now, we can state the exact values of the three trigonometric functions:

$$\sin\theta = \frac{y}{r} = \frac{-3}{\sqrt{13}} = \boldsymbol{\frac{-3\sqrt{13}}{13}}$$

$$\cos\theta = \frac{x}{r} = \frac{-2}{\sqrt{13}} = \boldsymbol{\frac{-2\sqrt{13}}{13}}$$

$$\tan\theta = \frac{y}{x} = \frac{-3}{-2} = \boldsymbol{\frac{3}{2}}$$



b. Since $x = 0,\ y = 1,\ r = \sqrt{0^2 + 1^2} = 1,$ then

$$\sin\theta = \frac{y}{r} = \frac{1}{1} = \mathbf{1}$$

$$\cos\theta = \frac{x}{r} = \frac{0}{1} = \mathbf{0}$$

$$\tan\theta = \frac{y}{x} = \frac{1}{0} = \textbf{undefined}$$

we can't divide by zero!

Notice that the measure of the least positive angle θ in standard position with the point $P(0,1)$ on its terminal side is 90°. Therefore, we have

$$\sin 90° = 1, \quad \cos 90° = 0, \quad \tan 90° = \text{undefined}$$

The values of trigonometric functions of other commonly used quadrantal angles, such as $0°, 180°, 270°$, and $360°$, can be found similary as in *Example 3b*. These values are summarized in the table below.

Table 2.1 Function Values of Quadrantal Angles

function \ θ =	**0°**	**90°**	**180°**	**270°**	**360°**
sin θ	0	1	0	-1	0
cos θ	1	0	-1	0	1
tan θ	0	undefined	0	undefined	0

Example 4 ► **Evaluating Trigonometric Functions Using Basic Identities**

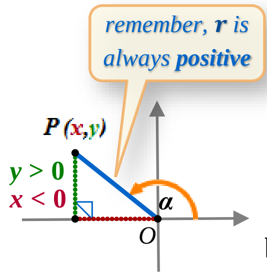
Knowing that $\cos\alpha = -\frac{3}{4}$ and the angle α is in quadrant II, find

a. $\sin\alpha$ **b.** $\tan\alpha$

Solution ► **a.** To find the value of $\sin\alpha$, we can use the Pythagorean Identity $\sin^2\alpha + \cos^2\alpha = 1$. After substituting $\cos\alpha = -\frac{3}{4}$, we have

$$\sin^2\alpha + \left(-\frac{3}{4}\right)^2 = 1$$

$$\sin^2\alpha = 1 - \frac{9}{16} = \frac{7}{16}$$



$$\sin\alpha = \pm\sqrt{\frac{7}{16}} = \pm\frac{\sqrt{7}}{4}$$

Since α is in in the second quadrant, $\sin\theta = \frac{y}{r}$ must be positive (as $y > 0$ in QII), so

$$\sin\alpha = \frac{\sqrt{\mathbf{7}}}{\mathbf{4}}.$$

b. To find the value of $\tan\alpha$, since we already know the value of $\sin\alpha$, we can use the identity $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$. After substituting values $\sin\alpha = \frac{\sqrt{7}}{4}$ and $\cos\alpha = -\frac{3}{4}$, we obtain

$$\tan\alpha = \frac{\frac{\sqrt{7}}{4}}{-\frac{3}{4}} = \frac{\sqrt{7}}{4}\cdot\left(-\frac{4}{3}\right) = -\frac{\sqrt{\mathbf{7}}}{\mathbf{3}}.$$

To confirm that the sign of $\tan\alpha = \frac{y}{x}$ in the second quadrant is indeed negative, observe that $y > 0$ and $x < 0$ in QII.

T.2 Exercises

Concept Check *Find the exact values of the three trigonometric functions for the indicated angle θ. Rationalize denominators when applicable.*

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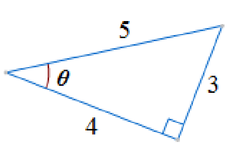

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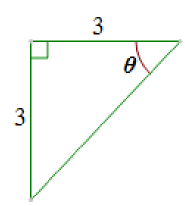

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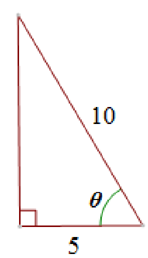

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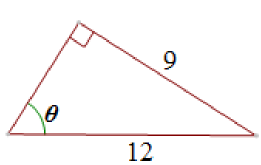

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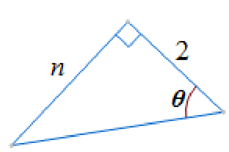

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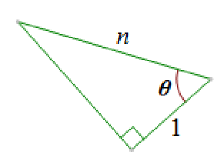

Concept Check *Sketch an angle θ in standard position such that θ has the least positive measure, and the given point is on the terminal side of θ. Then find the values of the three trigonometric functions for each angle. Rationalize denominators when applicable.*

7. $(-3,4)$ **8.** $(-4,-3)$ **9.** $(5,-12)$ **10.** $(0,3)$ **11.** $(-4,0)$

12. $(1,\sqrt{3})$ **13.** $(3,5)$ **14.** $(0,-8)$ **15.** $(-2\sqrt{3},-2)$ **16.** $(5,0)$

17. If the terminal side of an angle θ is in quadrant III, what is the sign of each of the trigonometric function values of θ?

Suppose that the point (x, y) is in the indicated quadrant. Decide whether the given ratio is ***positive*** *or* ***negative****.*

18. $Q\text{I}$, $\frac{y}{x}$ **19.** $Q\text{II}$, $\frac{y}{x}$ **20.** $Q\text{II}$, $\frac{y}{r}$ **21.** $Q\text{III}$, $\frac{x}{r}$ **22.** $Q\text{IV}$, $\frac{y}{x}$

23. $Q\text{III}$, $\frac{y}{x}$ **24.** $Q\text{IV}$, $\frac{y}{r}$ **25.** $Q\text{I}$, $\frac{y}{r}$ **26.** $Q\text{IV}$, $\frac{x}{r}$ **27.** $Q\text{II}$, $\frac{x}{r}$

Concept Check *Use the definition of trigonometric functions in terms of x, y, and r to determine each value. If it is undefined, say so.*

28. $\sin 90°$ **29.** $\cos 0°$ **30.** $\tan 180°$ **31.** $\cos 180°$ **32.** $\tan 270°$

33. $\cos 270°$ **34.** $\sin 270°$ **35.** $\cos 90°$ **36.** $\sin 0°$ **37.** $\tan 90°$

Analytic Skills *Use basic identities to determine values of the remaining two trigonometric functions of the angle satisfying given conditions. Rationalize denominators when applicable.*

39. $\sin \alpha = \frac{\sqrt{2}}{4};\ \alpha \in Q\text{II}$ **40.** $\sin \beta = -\frac{2}{3};\ \beta \in Q\text{III}$ **41.** $\cos \theta = \frac{2}{5};\ \theta \in Q\text{IV}$